Researches in Mathematics
Res. Math. **28**(1), 2020, p. 3–11
doi:10.15421/242001
https://vestnmath.dnu.dp.ua
ISSN (Print) 2664-4991
ISSN (Online) 2664-5009

UDK 517.5

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Sharp Nagy type inequalities for the classes of functions with given quotient of the uniform norms of positive and negative parts of a function

Для довільних $p \in (0, \infty]$, $\omega > 0$, $d \geq 2\omega$, отримана точна нерівність типу Надя

$$\|x_{\pm}\|_{\infty} \leq \frac{\|(\varphi + c)_{\pm}\|_{\infty}}{\|\varphi + c\|_{L_p(I_{2\omega})}} \|x\|_{L_p(I_d)} \tag{1}$$

на класах $S_{\varphi}(\omega)$ d-періодичних функцій x, що мають нулі, із заданою синусоподібною 2ω-періодичною функцією порівняння φ, де $c \in [-\|\varphi\|_{\infty}, \|\varphi\|_{\infty}]$ задовольняє умову

$$\|x_+\|_{\infty} \cdot \|x_-\|_{\infty}^{-1} = \|(\varphi + c)_+\|_{\infty} \cdot \|(\varphi + c)_-\|_{\infty}^{-1}.$$

Як наслідок доведена точна нерівнсть

$$\|x_{\pm}\|_{\infty} \leq \frac{\|(\varphi_r + c)_{\pm}\|_{\infty}}{\|\varphi_r + c\|_{L_p(I_{2\pi})}^{\alpha}} \|x\|_{L_p(I_{2\pi})}^{\alpha} \left\|x^{(r)}\right\|_{\infty}^{1-\alpha}, \quad \alpha = r(r + 1/p)^{-1},$$

на соболєвських класах диференційовних періодичних функцій із заданим відношенням рівномірних норм додатної і від'ємної частин функції, де φ_r- ідеальний сплайн Ейлера порядку r. Крім того, з нерівності (1) виведена нерівність типу Нікольського

$$\|T_{\pm}\|_{\infty} \leq \left(\frac{n}{m}\right)^{1/p} \frac{\|(\sin(\cdot) + c)_{\pm}\|_{\infty}}{\|\sin(\cdot) + c\|_{L_p(I_{2\pi})}} \|T\|_{L_p(I_{2\pi})}$$

на просторах тригонометричних поліномів T порядку $\leq n$ з періодом $2\pi/m, m \leq n$, і заданим відношенням рівномірних норм додатної і від'ємної частин, і аналогічна нерівність типу Нікольського

$$\|s_{\pm}\|_{\infty} \leq \left(\frac{n}{m}\right)^{1/p} \frac{\|(\varphi_r + c)_{\pm}\|_{\infty}}{\|\varphi_r + c\|_{L_p(I_{2\pi})}} \|s\|_{L_p(I_{2\pi})}$$

на просторах поліноміальних сплайнів s порядку r мінімального дефекту з вузлами в точках $k\pi/n$, $k \in \mathbf{Z}$, та періодом $2\pi/m, m \leq n$, і заданим відношенням рівномірних норм додатної і від'ємної частин.

Ключові слова: Нерівність типу Надя, клас функцій із заданою функцією порівняння, соболєвський клас, поліном, сплайн.

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For any $p \in (0,\infty]$, $\omega > 0$, $d \geq 2\omega$, we obtain the sharp inequality of Nagy type

$$\|x_{\pm}\|_\infty \leq \frac{\|(\varphi+c)_{\pm}\|_\infty}{\|\varphi+c\|_{L_p(I_{2\omega})}} \|x\|_{L_p(I_d)}$$

on the set $S_\varphi(\omega)$ of d-periodic functions x having zeros with given the sine-shaped 2ω-periodic comparison function φ, where $c \in [-\|\varphi\|_\infty, \|\varphi\|_\infty]$ is such that

$$\|x_+\|_\infty \cdot \|x_-\|_\infty^{-1} = \|(\varphi+c)_+\|_\infty \cdot \|(\varphi+c)_-\|_\infty^{-1}.$$

In particular, we obtain such type inequalities on the Sobolev sets of periodic functions and on the spaces of trigonometric polynomials and polynomial splines with given quotient of the norms $\|x_+\|_\infty / \|x_-\|_\infty$.

Key words: Nagy type inequality, a class of functions with given comparison function, Sobolev class of functions, polynomial, spline.

MSC2010: Pri 41A17, Sec 41A44, 42A05, 41A15

1. Introduction. Let $G \subset \mathbf{R}$. We will consider the spaces $L_p(G)$, $0 < p \leq \infty$, of all measurable functions $x : G \to \mathbf{R}$ such that $\|x\|_p = \|x\|_{L_p(G)} < \infty$, where

$$\|x\|_p := \left(\int_G |x(t)|^p \, dt \right)^{1/p}, \quad \text{if} \quad 0 < p < \infty,$$

$$\|x\|_\infty := \operatorname{vrai\,sup}_{t \in G} |x(t)|.$$

Let $d > 0$ and I_d denote the circle which is realized as the interval $[0, d]$ with coincident endpoints. For $r \in \mathbf{N}$, $G = \mathbf{R}$ or $G = I_d$, denote by $L^r_\infty(G)$ the space of all functions $x \in L_\infty(G)$ for which $x^{(r-1)}$ is locally absolutely continuous and $x^{(r)} \in L_\infty(G)$.

A function $f \in L^1_\infty(\mathbf{R})$ is called a comparison function for $x \in L^1_\infty(\mathbf{R})$ if there exists a constant $c \in \mathbf{R}$ satisfying

$$\max_{t \in \mathbf{R}} x(t) = \max_{t \in \mathbf{R}} f(t) + c, \qquad \min_{t \in \mathbf{R}} x(t) = \min_{t \in \mathbf{R}} f(t) + c.$$

and from $x(\xi) = f(\eta) + c$, $\xi, \eta \in \mathbf{R}$, the inequality $|x'(\xi)| \leq |f'(\eta)|$ follows (if corresponding derivatives exist).

Let $\omega > 0$. By definition, S-function is a 2ω-periodic function $\varphi \in L^1_\infty(I_{2\omega})$ that has the following properties: vanishes at 0, is odd about 0, is even about $\omega/2$, is positive and concave on $(0, \omega)$, and strictly increasing on $[0, \omega/2]$.

For 2ω-periodic S-function φ denote by $S_\varphi(\omega)$ the class of functions $x \in L^1_\infty(\mathbf{R})$ for which φ is the comparison function. Note that the classes $S_\varphi(\omega)$ were considered in [1], [2]. Examples of such classes $S_\varphi(\omega)$ are the Sobolev classes $\{x \in L^r_\infty(I_d) : \|x^{(r)}\|_\infty \leq 1\}$, the bounded subsets of the space T_n of all trigonometric polynomials of order at most n, and the same subsets of the space $S_{n,r}$ of polynomial splines of order r having defect 1 with knots at the points $k\pi/n, k \in \mathbf{Z}$.

It is shown in [3] that for $p \in [1,\infty]$ and $x \in L^r_\infty(I_{2\pi})$ there holds the following sharp inequality of Nagy type

$$E_0(x)_\infty \le \frac{\|\varphi_r\|_\infty}{\|\varphi_r\|^\alpha_{L_p(I_{2\pi})}} \|x\|^\alpha_{L_p(I_{2\pi})} \|x^{(r)}\|^{1-\alpha}_\infty, \tag{1.1}$$

where $\alpha = \frac{r}{r+1/p}$, φ_r is the perfect Euler spline of order r and $E_0(x)_\infty$ is the best uniform approximation of the function x by constants.

In this paper we generalize the inequality (1.1) on the classes $S_\varphi(\omega)$ of a function with given quotient positive and negative parts of a function (Theorem 1). In particular, we obtain such type inequalities for a function $x \in L^r_\infty(I_{2\pi})$ (Theorem 2) and for functions in spaces T_n and $S_{n,r}$ (Theorem 3 and Theorem 4) with given quotient $\|x_+\|_\infty/\|x_-\|_\infty$.

2. The inequalities of various metrics on the classes of the functions with given comparison function. For a function $f \in L_1(I_d)$ denote by $m(f,y)$, $y>0$, the distribution function defined below

$$m(f,y) := \mu\{t \in I_d : |f(t)| > y\}, \tag{2.1}$$

and let $r(f,t)$ be decreasing rearrangement (see, for example, [4, §1.3]) of the restriction of the function $|f|$ on $[0,d]$. Set $r(f,t)=0$ for $t>d$.

Theorem 1. *Let $p \in (0,\infty]$ and φ is 2ω-periodic S-function. For any d-periodic function $x \in S_\varphi(\omega)$ having zeros the inequality*

$$\|x_\pm\|_\infty \le \frac{\|(\varphi+c)_\pm\|_\infty}{\|\varphi+c\|_{L_p(I_{2\omega})}} \|x\|_{L_p(I_d)} \tag{2.2}$$

holds true, where the point $c = c(x) \in [-\|\varphi\|_\infty, \|\varphi\|_\infty]$ satisfies the condition

$$\frac{\|x_+\|_\infty}{\|x_-\|_\infty} = \frac{\|(\varphi+c)_+\|_\infty}{\|(\varphi+c)_-\|_\infty}.$$

The inequality (2.2) is sharp on the classes of functions $x \in S_\varphi(\omega)$ having zeros with given quotient $\|x_+\|_\infty/\|x_-\|_\infty$ and becomes equality for the function $x(t) = \varphi(t)+c$.

Proof. Fix any d-periodic function $x \in S_\varphi(\omega)$ having zeros. Since φ is comparison function for x, then there exists a constant $c \in \mathbf{R}$ satisfying

$$\|x_+\|_\infty = \|(\varphi+c)_+\|_\infty, \qquad \|x_-\|_\infty = \|(\varphi+c)_-\|_\infty. \tag{2.3}$$

By definition, the function φ is strictly increasing on $[-\frac{\omega}{2}, \frac{\omega}{2}]$. For $\tau \in \mathbf{R}$ set $x_\tau(t) := x(\tau+t),\ t \in \mathbf{R}$. Choose $\tau_1, \tau_2 \in \mathbf{R}$ such that

$$x_{\tau_1}\left(\frac{\omega}{2}\right) = \|x_+\|_\infty, \qquad x_{\tau_2}\left(-\frac{\omega}{2}\right) = \|x_-\|_\infty.$$

Since φ is comparison function for x, then

$$(x_{\tau_1}(t))_+ \ge (\varphi(t)+c)_+ \ , \qquad \left|t-\frac{\omega}{2}\right| \le \omega, \tag{2.4}$$

and

$$(x_{\tau_2}(t))_- \ge (\varphi(t)+c)_- \ , \qquad \left|t+\frac{\omega}{2}\right| \le \omega, \tag{2.5}$$

where $u_\pm := \max\{\pm\, u, 0\}$. Observe that from (2.4) and (2.5) follows the inequalities $d \ge 2\omega$ and

$$m(x_\pm, y) \ge m((\varphi(\cdot)+c)_\pm, y), \quad y \ge 0,$$

where the function $m(f,y)$ is defined by (2.1). It follows immediately that

$$r(x,t) \ge r(\varphi(\cdot)+c,t), \quad t \ge 0. \tag{2.6}$$

Combining the inequalities (2.6) and $d \ge 2\omega$ we have

$$\|x\|^p_{L_p(I_d)} = \int_0^{2d} r^p(x,t)dt \ge \int_0^{2\omega} r^p(\varphi(\cdot)+c,t)dt = \int_{I_{2\omega}} |\varphi(t)+c|^p dt.$$

Therefore,

$$\|x\|_{L_p(I_d)} \ge \|\varphi+c\|_{L_p(I_{2\omega}}.$$

It yields (2.2) in view of (2.3). Theorem 1 is proved.

Denote by $E_0(f)_{L_p(G)}$ the best approximation of the function f by constants in the space $L_p(G)$ and let

$$E_0^\pm(x)_{L_p(G)} := \inf_{c\in\mathbf{R}} \{\|x-c\|_{L_p(G)} : \forall t \ \pm(x(t)-c)_\pm \ge 0\} \tag{2.7}$$

be the best one-sided approximations by constants of the function f in that space.

Corollary 1. *Under the assumptions of Theorem 1 for any $x \in S_\varphi(\omega)$*

$$E_0(x)_\infty \le \frac{\|\varphi\|_\infty}{E_0(\varphi)_{L_p(I_{2\omega})}} \|x\|_{L_p(I_d)}$$

and

$$E_0^\pm(x)_\infty \le \frac{2\|\varphi\|_\infty}{\|\varphi+\bar{c}\|_{L_p(I_{2\omega})}} E_0^\pm(x)_{L_p(I_d)}, \quad \bar{c} := \|\varphi\|_\infty \, .$$

Besides, for a function $x \in S_\varphi(\omega)$ having zeros the inequality

$$\|x\|_\infty \le \sup_{c\colon |c| \le \|\varphi\|_\infty} \frac{\|\varphi+c\|_\infty}{\|\varphi+c\|_{L_p(I_{2\omega})}} \|x\|_{L_p(I_d)}$$

holds true.

3. Nagy type inequalities for differentiable periodic functions. We denote be $\varphi_r(t)$, $r \in \mathbf{N}$, the shift of the r^{th} 2π-periodic integral with zero mean value on a period of the function $\varphi_0(t) = \operatorname{sgn}\sin t$ satisfying $\varphi_r(0) = 0$ and $\varphi_r'(0) > 0$. Let $K_r := \|\varphi_r\|_\infty$ be the Favard constant. For $\lambda > 0$ set $\varphi_{\lambda,r}(t) := \lambda^{-r}\varphi_r(\lambda t)$. Obviously the spline $\varphi_{\lambda,r}(t)$ is $2\pi/\lambda$-periodic S-function.

Theorem 2. *Let $r \in \mathbf{N}$, $p \in (0,\infty]$. Then for any function $x \in L^r_\infty(I_{2\pi})$ having zeros we have*

$$\|x_\pm\|_\infty \le \frac{\|(\varphi_r + c)_\pm\|_\infty}{\|\varphi_r + c\|^\alpha_{L_p(I_{2\pi})}} \|x\|^\alpha_{L_p(I_{2\pi})} \left\|x^{(r)}\right\|^{1-\alpha}_\infty, \tag{3.1}$$

where $\alpha = \frac{r}{r+1/p}$ and $c \in [-K_r, K_r]$ is such that

$$\frac{\|x_+\|_\infty}{\|x_-\|_\infty} = \frac{\|(\varphi_r + c)_+\|_\infty}{\|(\varphi_r + c)_-\|_\infty}.$$

The inequality (3.1) is sharp on the class of functions $x \in L^r_\infty(I_{2\pi})$ having zeros with given quotient $\|x_+\|_\infty/\|x_-\|_\infty$ and becomes equality for the function $x(t) = \varphi_r(t) + c$.

Proof. Fix a function $x \in L^r_\infty(\mathbf{R})$ having zeros. In view of homogeneity of the inequality (3.1) we can assume that

$$\left\|x^{(r)}\right\|_\infty = 1. \tag{3.2}$$

Choose λ satisfying

$$E_0(x)_\infty = \|\varphi_{\lambda,r}\|_\infty. \tag{3.3}$$

Then by the Kolmogorov comparison theorem [5] the spline $\varphi := \varphi_{\lambda,r}$ is the comparison function for the function x. Consequently $x \in S_\varphi(\frac{\pi}{\lambda})$, and by Theorem 1 we have

$$\|x_\pm\|_\infty \le \frac{\|(\varphi_{\lambda,r} + \lambda^{-r}c)_\pm\|_\infty}{\|\varphi_{\lambda,r} + \lambda^{-r}c\|_{L_p\left(I_{\frac{2\pi}{\lambda}}\right)}} \|x\|_{L_p(I_{2\pi})}. \tag{3.4}$$

Besides, it follows from (3.3) in view of condition of Theorem 2 for constant c that

$$\|x_\pm\|_\infty = \left\|(\varphi_{\lambda,r} + \lambda^{-r}c)_\pm\right\|_\infty = \lambda^{-r}\|(\varphi_r + c)_\pm\|_\infty. \tag{3.5}$$

Combining (3.4) and (3.5) we get

$$\|x\|_{L_p(I_{2\pi})} \ge \|\varphi_{\lambda,r} + \lambda^{-r}c\|_{L_p\left(I_{\frac{2\pi}{\lambda}}\right)} = \lambda^{-(r+1/p)}\|\varphi_r + c\|_{L_p(I_{2\pi})}. \tag{3.6}$$

Applying (3.5), (3.6) and taking into account the definition of α we obtain

$$\frac{\|x_\pm\|_\infty}{\|x\|^\alpha_{L_p(I_{2\pi})}} \le \frac{\lambda^{-r}\|(\varphi_r + c)_\pm\|_\infty}{\lambda^{-(r+1/p)\alpha}\|\varphi_r + c\|^\alpha_{L_p(I_{2\pi})}} = \frac{\|(\varphi_r + c)_\pm\|_\infty}{\|\varphi_r + c\|^\alpha_{L_p(I_{2\pi})}}.$$

It follows (3.1) in view of (3.2). Theorem 2 is proved.

Corollary 2. *Under the assumptions of Theorem 1 for any* $x \in L^r_\infty(I_{2\pi})$

$$E_0(x)_\infty \le \frac{K_r}{E_0(\varphi_r)^\alpha_{L_p(I_{2\pi})}} \|x\|^\alpha_{L_p(I_{2\pi})} \left\|x^{(r)}\right\|^{1-\alpha}_\infty$$

and

$$E_0^\pm(x)_\infty \le \frac{2K_r}{\|\varphi_r + K_r\|^\alpha_{L_p(I_{2\pi})}} E_0^\pm(x)^\alpha_{L_p(I_{2\pi})} \left\|x^{(r)}\right\|^{1-\alpha}_\infty,$$

Besides, for a function $x \in L^r_\infty(I_{2\pi})$ *having zeros the inequality*

$$\|x\|_\infty \le \sup_{c\colon |c|\le K_r} \frac{\|\varphi_r + c\|_\infty}{\|\varphi_r + c\|^\alpha_{L_p(I_{2\pi})}} \|x\|^\alpha_{L_p(I_{2\pi})} \left\|x^{(r)}\right\|^{1-\alpha}_\infty.$$

holds true.

Remark 2. The first inequality is proved in [3] and the rest ones are proved in [6].

4. Nikolskii type inequalities for trigonometric polynomials. Let us recall that T_n is the space of all trigonometric polynomials of degree at most n.

Theorem 3. *Let* $p \in (0,\infty]$, $n, m \in \mathbf{N}$, $m \le n$. *For any trigonometric polynomial* $T \in T_n$ *with minimal period* $2\pi/m$ *having zeros the inequality*

$$\|T_\pm\|_\infty \le \left(\frac{n}{m}\right)^{1/p} \frac{\|(\sin(\cdot) + c)_\pm\|_\infty}{\|\sin(\cdot) + c\|_{L_p(I_{2\pi})}} \|T\|_{L_p(I_{2\pi})} \tag{4.1}$$

holds true, where $c \in [-1,1]$ *is the constant satisfying*

$$\frac{\|T_+\|_\infty}{\|T_-\|_\infty} = \frac{\|(\sin(\cdot) + c)_+\|_\infty}{\|(\sin(\cdot) + c)_-\|_\infty}. \tag{4.2}$$

The inequality (4.1) is sharp for $m = 1$ *in the sense*

$$\sup_{n\in\mathbf{N}} \sup_{T\in T_n(c)} \frac{\|T_\pm\|_\infty}{n^{1/p}\|T\|_{L_p(I_{2\pi})}} = \frac{\|(\sin(\cdot) + c)_\pm\|_\infty}{\|\sin(\cdot) + c\|_{L_p(I_{2\pi})}},$$

where $T_n(c)$ *is the set of all trigonometric polynomials* $T \in T_n$ *having zeros with given quotient* $\|T_+\|_\infty/\|T_-\|_\infty$ *satisfying (4.2).*

Proof. Fix a polynomial $T \in T_n$ with minimal period $\frac{2\pi}{m}$ having zeros. In view of homogeneity of the inequality (4.1) we can assume that

$$E_0(T)_\infty = 1.$$

Then the polynomial $\varphi(t) := \sin nt$ is comparison function for the polynomial $T(t)$ (see, for example, the proof of Theorem 8.1.1 [7]). It is clear that φ is $\frac{2\pi}{n}$-periodic S-function. Hence $T \in S_\varphi(\frac{\pi}{n})$. Then by Theorem 1

$$\|T_\pm\|_\infty \le \frac{\|(\sin n(\cdot) + c)_\pm\|_\infty}{\|\sin n(\cdot) + c\|_{L_p(I_{\frac{2\pi}{n}})}} \|T\|_{L_p(I_{\frac{2\pi}{m}})}.$$

Inequality (4.1) follows in view of the evident equalities

$$\|T\|_{L_p(I_{\frac{2\pi}{m}})} = m^{-1/p}\,\|T\|_{L_p(I_{2\pi})}\,,$$

$$\|\sin n(\cdot) + c\|_{L_p(I_{\frac{2\pi}{n}})} = n^{-1/p}\,\|\sin(\cdot) + c\|_{L_p(I_{2\pi})}$$

and

$$\|(\sin n(\cdot) + c)_{\pm}\|_{\infty} = \|(\sin(\cdot) + c)_{\pm}\|_{\infty}.$$

Theorem 3 is proved.

Corollary 3. *Under the assumptions of Theorem 3 for a polynomial $T \in T_n$ with minimal period $2\pi/m$ we have*

$$E_0(T)_{\infty} \le \left(\frac{n}{m}\right)^{1/p} \frac{\|T\|_{L_p(I_{2\pi})}}{E_0(\sin(\cdot))_{L_p(I_{2\pi})}}$$

and

$$E_0^{\pm}(T)_{\infty} \le 2\left(\frac{n}{m}\right)^{1/p} \frac{E_0^{\pm}(T)_{L_p(I_{2\pi})}}{\|\sin(\cdot) + 1\|_{L_p(I_{2\pi})}}.$$

Besides, for a polynomial $T \in T_n$ with minimal period $2\pi/m$ having zeros

$$\|T\|_{\infty} \le \left(\frac{n}{m}\right)^{1/p} \sup_{c\colon |c|\le 1} \frac{\|\sin(\cdot) + c\|_{\infty}}{\|\sin(\cdot) + c\|_{L_p(I_{2\pi})}}\,\|T\|_{L_p(I_{2\pi})}\,.$$

Remark 3. For $m = 1$ the first inequality is proved in [3] and the rest ones are proved in [6].

5. Nikolskii type inequalities for periodic polynomial splines. Let $r, n \in \mathbf{N}$. Recall that $S_{n,r}$ stands for the space of polynomial splines of order r having defect 1 with knots at the points $k\pi/n, k \in \mathbf{Z}$. It is clear that $S_{n,r} \subset L^r_{\infty}(\mathbf{R})$.

Theorem 4. *Let $p \in (0,\infty]$; $n, m \in \mathbf{N}$, $m \le n$. For a spline $s \in S_{n,r}$ with minimal period $2\pi/m$ having zeros the inequality*

$$\|s_{\pm}\|_{\infty} \le \left(\frac{n}{m}\right)^{1/p} \frac{\|(\varphi_r + c)_{\pm}\|_{\infty}}{\|\varphi_r + c\|_{L_p(I_{2\pi})}}\,\|s\|_{L_p(I_{2\pi})} \tag{5.1}$$

holds true, where $c \in [-K_r, K_r]$ is the constant satisfying

$$\frac{\|s_+\|_{\infty}}{\|s_-\|_{\infty}} = \frac{\|(\varphi_r + c)_+\|_{\infty}}{\|(\varphi_r + c)_-\|_{\infty}}. \tag{5.2}$$

The inequality (5.1) is sharp for $m = 1$ in the sense

$$\sup_{n\in\mathbf{N}}\ \sup_{s\in S_{n,r}(c)} \frac{\|s_{\pm}\|_{\infty}}{n^{1/p}\,\|s\|_{L_p(I_{2\pi})}} = \frac{\|(\varphi_r + c)_{\pm}\|_{\infty}}{\|\varphi_r + c\|_{L_p(I_{2\pi})}},$$

where $S_{n,r}(c)$ is the set of all splines $s \in S_{n,r}$ having zeros with given quotient $\|s_+\|_{\infty}/\|s_-\|_{\infty}$ satisfying (5.2).

Proof. Fix a spline $s \in S_{n,r}$ with minimal period $2\pi/m$ having zeros. In view of homogeneity of the inequality (5.1) we can assume that

$$E_0(s)_\infty = \|\varphi_{n,r}\|_\infty.$$

Then by the Tikhomirov inequality [8]

$$\left\|s^{(r)}\right\|_\infty \le \frac{E_0(s)_\infty}{\|\varphi_{n,r}\|_\infty} = 1.$$

Hence all conditions of Kolmogorov comparison theorem [5] are fulfilled. By this Theorem the function $\varphi(t) := \varphi_{n,r}(t)$ is comparison function for the spline s. It is clear that φ is S-function with period $2\pi/n$. So $s \in S_\varphi(\frac{\pi}{n})$. Then by Theorem 1

$$\|s_\pm\|_\infty \le \frac{\|(\varphi_{n,r} + n^{-r}c)_\pm\|_\infty}{\|\varphi_{n,r} + n^{-r}c\|_{L_p(I_{\frac{2\pi}{n}})}} \|s\|_{L_p(I_{\frac{2\pi}{m}})}.$$

(5.1) follows in view of the evident equalities

$$\|s\|_{L_p(I_{\frac{2\pi}{m}})} = m^{-1/p} \|s\|_{L_p(I_{2\pi})},$$

$$\left\|\varphi_{n,r} + n^{-r}c\right\|_{L_p(I_{\frac{2\pi}{n}})} = n^{-(r+1/p)} \|\varphi_r + c\|_{L_p(I_{2\pi})}$$

and

$$\|(\varphi_{n,r} + n^{-r}c)_\pm\|_\infty = n^{-r}\|(\varphi_r + c)_\pm\|_\infty.$$

Theorem 4 is proved.

Corollary 4. *By conditions of Theorem 4 for a spline $s \in S_{n,r}$ with minimal period $2\pi/m$ we have*

$$E_0(s)_\infty \le \left(\frac{n}{m}\right)^{1/p} \frac{K_r}{E_0(\varphi_r)_{L_p(I_{2\pi})}} \|s\|_{L_p(I_{2\pi})}$$

and

$$E_0^\pm(s)_\infty \le \left(\frac{n}{m}\right)^{1/p} \frac{2K_r}{\|\varphi_r + K_r\|_{L_p(I_{2\pi})}} E_0^\pm(s)_{L_p(I_{2\pi})}.$$

Besides, for a spline $s \in S_{n,r}$ with minimal period $2\pi/m$ having zeros

$$\|s\|_\infty \le \left(\frac{n}{m}\right)^{1/p} \sup_{c:\ |c| \le K_r} \frac{\|\varphi_r + c\|_\infty}{\|\varphi_r + c\|_{L_p(I_{2\pi})}} \|s\|_{L_p(I_{2\pi})}.$$

Remark 4. For $m = 1$ the first inequality is proved in [3] and the rest ones are proved in [6].

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Received: 14.02.2020. *Accepted:* 02.05.2020