УДК 517.9

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ON THE WEAKLY-* DENSE SUBSETS IN $L^{\infty}(\Omega)$

Abstract. In this paper we study the density property of the compactly supported smooth functions in the space $L^{\infty}(\Omega)$. We show that this set is dense with respect to the weak-* convergence in variable spaces.

Let Ω be an open bounded domain in R^2 with a Lipschitz boundary $\partial\Omega$. Throughout the paper we suppose that Ω is a measurable set in the sense of Jordan. Let $C_0^{\infty}(\Omega)$ be the set of smooth functions with a compact support in Ω. It is well known that the set $C_0^{\infty}(\Omega)$ is not dense in $L^{\infty}(\Omega)$, that is, the assertion

«... for any $f \in L^{\infty}(\Omega)$ can be found a sequence $\{u_k \in C_0^{\infty}(\Omega)\}_{k=1}^{\infty}$ such that $u_k \to f$ strongly in $L^{\infty}(\Omega)$ as $k \to \infty$...»

is not true, in general. So, the main question we are going to study in this paper is the following: how can the density concept of the locally convex space C_0^{∞} be interpreted in $L^{\infty}(\Omega)$? As we will see later it can be done through the concept of the weak- * convergence in the variable spaces.

To begin with, we define the so-called graph-like structure on the domain Ω. Let Y be the following set $Y = [0;1)^2 = [0;1) \times [0;1)$.

Definition 1. *We say that the set Y is the cell of periodicity for some graph F on R^2 if Y contains a «star»-structure such that:*

(i) all edges of this structure have a common point $M \in \operatorname{int} Y$; each edge is a line-segment and all end-points of these edges belong to the boundary of Y;

(ii) in the set of end-points (vertices) there exist pairs $(M_i; M_k)$ such that $x_1^{M_i} = x_1^{M_k}$ or $x_2^{M_i} = x_2^{M_k}$.

As follows from the condition (ii) we admit the existence of isolated vertices in the Y-periodic graph F on R^2. Let $\varepsilon \in E = (0, \varepsilon^0]$ be a small parameter. We assume that ε varies in a strictly decreasing sequence of positive numbers which converge to 0.

Definition 2. *We say that F_ε is an ε-periodic graph on R^2 if*

$$F_\varepsilon = \varepsilon F = \{\varepsilon x : x \in F\}.$$

It is clear that the cell of periodicity for Ω_ε is εY. Let

$$I^{ed} = \{I_j, j = 1,2...,K\} \tag{1}$$

be the set of all edges on Y. Let Ω be an open bounded domain in R^2 with a Lipschitz boundary such that

$$\Omega = \{(x_1, x_2) : x_1 \in \Gamma_l, 0 < x_2 < \gamma(x_1)\}, \tag{2}$$

where $\Gamma_1 = (0,a)$, $\gamma \in C^1([0,a])$, *and* $0 < \gamma_0 = inf_{x_1 \in [0,a]} \gamma(x_1)$. *Then* $\partial\Omega = \Gamma_1 \cup \Gamma_2$, *where* $\Gamma_2 = \partial\Omega \setminus \Gamma_1$.

Definition 3. *We say that* Ω_ε *has an* ε*-periodic graph-like structure if* $\Omega_\varepsilon = \Omega \cap F_\varepsilon$.

Our next step is to describe the geometry of the set Ω_ε in terms of so-called singular measures in R^2. To do so, we will follow the Zhikov's approach ([3]-[5]).

For every segment $I_i \in I^{ed}$, $i = 1,2,\dots,K$ we denote by μ_i its corresponding Lebesgue measure. Now we define the Y-periodic Borel measure μ in R^2 as follows

$$\mu = \sum_{i=1}^{K} g_i \cdot \mu_i \text{ on } Y; \tag{3}$$

where $g_1, g_2, \dots, g_K$ are non-negative weights such that $\int_Y d\mu = 1$.

Thus the support of the measure μ is the union of all edges $I_i \in I^{ed}$, each of which is a 1-dimensional manifold in R^2. Since the homothetic contraction of the plane at ε^{-1} takes the grid F to $F_\varepsilon = \varepsilon F$, we introduce a «scaling» ε-periodic measure μ_ε as follows

$$\mu_\varepsilon(B) = \varepsilon^2 \mu(\varepsilon^{-1} B) \text{ for every Borel set } B \subset R^2. \tag{4}$$

Then

$$\int_{\varepsilon Y} d\mu_\varepsilon = \varepsilon^2 \int_Y d\mu = \varepsilon^2.$$

Hence the measure μ_ε is weakly convergent to the Lebesgue measure L^2, that is

$$d\mu_\varepsilon \xrightarrow{w} dx \Leftrightarrow \lim_{\varepsilon \to 0} \int_{R^2} \varphi \, d\mu_\varepsilon = \int_{R^2} \varphi \, dx \tag{5}$$

for every $\varphi \in C_0^\infty(R^2)$ (see Zhikov [3] for a proof).

We define the space $L^\infty(\Omega, d\mu_\varepsilon)$ in the way: $y_\varepsilon \in L^\infty(\Omega, d\mu_\varepsilon)$ if and only if y_ε is a μ_ε-measurable function on Ω and there exists a constant $M > 0$ such that $|y_\varepsilon(x)| \le M$ μ_ε-every where in Ω.

Definition 4. *We say that a sequence* $\{y_\varepsilon \in L^\infty(\Omega, d\mu_\varepsilon)\}_{\varepsilon \to 0}$ *is uniformly bounded if* $sup_{\varepsilon > 0} \|y_\varepsilon\|_{L^\infty(\Omega, d\mu_\varepsilon)} < +\infty$.

Definition 5. *A uniformly bounded sequence* $\{y_\varepsilon \in L^\infty(\Omega, d\mu_\varepsilon)\}_{\varepsilon \to 0}$ *is said to be weakly-* convergent in the variable space* $L^\infty(\Omega, d\mu_\varepsilon)$ *to* $y \in L^\infty(\Omega)$ *if*

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi \, y_\varepsilon \, d\mu_\varepsilon = \int_\Omega \varphi \, y \, dx \textit{ for every } \varphi \in C_0^\infty(\Omega)$$

(in the symbols $y_\varepsilon \xrightarrow{w^*} y$*).*

We begin with the following result:

Theorem 6. *Let* $\{y_\varepsilon\}_{\varepsilon\to 0}$ *be any bounded sequence in the variable space* $L^\infty(\Omega, d\mu_\varepsilon)$. *Then this sequence is relatively compact with respect to the weak-* * *convergence in* $L^\infty(\Omega, d\mu_\varepsilon)$.

Proof. Let us set

$$l_\varepsilon(\varphi) = \int_\Omega y_\varepsilon \varphi d\mu_\varepsilon \quad \varphi \in C_0^\infty(\Omega).$$

Then, by the Hölder inequality, we have

$$|l_\varepsilon(\Omega)| \le \int_\Omega |y_\varepsilon||\varphi| d\mu_\varepsilon \le \|y_\varepsilon\|_{L^\infty(\Omega, d\mu_\varepsilon)} \int_\Omega |\varphi| d\mu_\varepsilon . \tag{6}$$

Hence

$$|l_\varepsilon(\varphi)| \le \|y_\varepsilon\|_{L^\infty(\Omega, d\mu_\varepsilon)} \|\varphi\|_{C(\Omega)} \mu_\varepsilon(K),$$

where by K we denote a support of φ in Ω. Since $d\mu_\varepsilon \xrightarrow{w} dx = dL^2$ in the space of Radone measures and

$$\limsup_{\varepsilon\to 0} \mu_\varepsilon(K) \le L^2(K) \text{ for every compact subset of } \Omega$$

(see Zhikov [3]), it follows that

$$|l_\varepsilon(\varphi)| \le 2 \|\varphi\|_{C(\Omega)} \mu(K) \sup_{\varepsilon>0} \|y_\varepsilon\|_{L^\infty(\Omega, d\mu_\varepsilon)}$$

for $\varepsilon > 0$ small enough. On the other hand, the set

$$T(K) = \{\varphi \in C_0^\infty(\Omega), sup\varphi \subseteq K\}$$

is separable with respect to the norm $\|\varphi\|_{C(\Omega)}$. Then, due to the Cantor diagonal method, it can be easy proved that the sequence $\{l_\varepsilon(\cdot)\}_{\varepsilon\to 0}$ consists a subsequence which is pointwise convergent on *T(K)*. As a result, there exists a subsequence of values $\varepsilon_j \to 0$ such that

$$\lim_{j\to\infty} l_{\varepsilon_j}(\varphi) = l(\varphi) \ \forall \varphi \in C_0^\infty(\Omega). \tag{7}$$

Taking into account the inequality (6), we conclude

$$|l(\varphi)| \le \sup_{\varepsilon>0} \|y_\varepsilon\|_{L^\infty(\Omega, d\mu_\varepsilon)} \lim_{\varepsilon\to 0} \int_\Omega |\varphi| d\mu_\varepsilon = \sup_{\varepsilon>0} \|y_\varepsilon\|_{L^\infty(\Omega, d\mu_\varepsilon)} \int_\Omega |\varphi| dx.$$

So, $l(\cdot)$ is the linear continuous functional on $L^1(\Omega)$. Hence, the following representation holds true

$$l(\varphi) = \int_\Omega \upsilon \varphi dx$$

where υ is some element of $L^\infty(\Omega)$. Thus, in view of (7), υ is a weak-* limit of the subsequence $\{y_{\varepsilon_j}\}_{j=1}^\infty$ in the variable space $L^\infty(\Omega, d\mu_\varepsilon)$.

Now we are in a position to state the main result of our paper.

Theorem 7. *For any element* $y \in L^{\infty}(\Omega)$ *there can be found a sequence of smooth functions* $\{y_{\varepsilon} \in C_0^{\infty}(\Omega)\}_{\varepsilon>0}$ *satisfying the conditions:*

$$|y_{\varepsilon}| \le \| y \|_{L^{\infty}(\Omega)} \text{ for every } \varepsilon \in E; \quad y_{\varepsilon} \xrightarrow{w^*} y \text{ in } L^{\infty}(\Omega, d\mu_{\varepsilon}) \text{ as } \varepsilon \to 0 \tag{8}$$

Proof. Let y be any element of $L^{\infty}(\Omega)$. We set $c = \| y \|_{L^{\infty}(\Omega)}$. Since $L^{\infty}(\Omega) \subset L^2(\Omega)$ and the space of smooth functions $C^{\infty}(\Omega)$ is dense in $L^2(\Omega)$ it follows that there is a sequence $\{y_{\varepsilon} \in C^{\infty}(\Omega)\}$ satisfying the conditions:

$$|y_{\varepsilon}| \le c \text{ for every } \varepsilon \in E; \ \| y_{\varepsilon} - y \|_{L^2(\Omega)} \to 0 \text{ as } \varepsilon \to 0.$$

Therefore

$$\lim_{\varepsilon \to 0} \int_{\Omega} \varphi y_{\varepsilon} dx = \int_{\Omega} \varphi y dx \text{ for every } \varphi \in C_0(\overline{\Omega}) \tag{9}$$

Further we note that $y_{\varepsilon} \in L^{\infty}(\Omega, d\mu_{\varepsilon})$ (as a smooth function) and hence $|y_{\varepsilon}| \le c\, \mu_{\varepsilon}$-almost everywhere. We have to show that $y_{\varepsilon} \to y$ weakly-* in $L^{\infty}(\Omega, d\mu_{\varepsilon})$, i.e.

$$\int_{\Omega} \varphi y_{\varepsilon} d\mu_{\varepsilon} \to \int_{\Omega} \varphi y dx \text{ for every } \varphi \in C_0^{\infty}(R^2). \tag{10}$$

We partition the domain Ω into the sets εY_j, where Y_j is periodic covering of R^2 by the cell Y. Then

$$\int_{\Omega} \varphi\, y_{\varepsilon} d\mu_{\varepsilon} = \sum_j \int_{\varepsilon Y_j} \varphi\, y_{\varepsilon}\, d\mu_{\varepsilon} + \sum \int_{\Omega \cap \varepsilon Y_j} \varphi\, y_{\varepsilon}\, d\mu_{\varepsilon} \tag{11}$$

where the second sum is calculated over the set of the 'boundary' squares such that $\varepsilon Y_j \cap \partial\Omega \ne 0$. By Mean Value Theorem, for each index j there exist points x_j in the cells εY_j such that

$$\int_{\varepsilon Y_j} \varphi\, y_{\varepsilon} d\mu_{\varepsilon} = \varphi(x_j) y_{\varepsilon}(x_j) \int_{\varepsilon Y_j} d\mu_{\varepsilon} = \varphi(x_j) y_{\varepsilon}(x_j) \varepsilon^2 \int_Y d\mu = \varphi(x_j) y_{\varepsilon}(x_j) \varepsilon^2 \ \forall j.$$

Then in view of (11), we get

$$\int_{\Omega} \varphi y_{\varepsilon} d\mu_{\varepsilon} = \left(\sum_j \varphi(x_j) y_{\varepsilon}(x_j) \varepsilon^2 - \int_{\Omega} \varphi y_{\varepsilon} dx \right) + \sum \int_{\Omega \cap \varepsilon Y_j} \varphi y_{\varepsilon} d\mu_{\varepsilon} + \int_{\Omega} \varphi y_{\varepsilon} dx = \tag{12}$$

$$= I_1 + I_2 + \int_{\Omega} \varphi y_{\varepsilon} dx.$$

Note that

$$|I_2| = \left| \sum \int_{\Omega \cap \varepsilon Y_j} \varphi\, y_{\varepsilon} d\mu_{\varepsilon} \right| \le \sup_{j \in D(\varepsilon)} \left(\sup_{x \in \Omega \cap \varepsilon Y_j} |\varphi| |y_{\varepsilon}| \right) \varepsilon^2 D(\varepsilon) \le c \| \varphi \|_{C(\Omega)} \varepsilon^2 D(\varepsilon),$$

where $D(\varepsilon)$ is the quantity of the 'boundary' squares, and $\varepsilon^2 D(\varepsilon) \to 0$ by Jordan's measurability property of the set $\partial\Omega$. Hence $I_2 \to 0$ as ε tends to zero.

Now we show that $I_1 \to 0$. To do so, we note that

$$|I_1| = \left|\sum_j \varphi(x_j)y_\varepsilon(x_j)\varepsilon^2 - \int_\Omega \varphi y_\varepsilon dx\right| \le \left|\sum_j \left(\varphi(x_j)y_\varepsilon(x_j) - \frac{1}{\varepsilon^2}\int_{\varepsilon Y_j} \varphi\, y_\varepsilon dx\right)\varepsilon^2\right| + \left|\sum \int_{\Omega \cap \varepsilon Y_j} \varphi\, y_\varepsilon dx\right| \le$$

$$\le \sum_j \left|\varphi(x_j)y_\varepsilon(x_j) - \frac{1}{\varepsilon^2}\int_{\varepsilon Y_j} \varphi\, y_\varepsilon dx\right|\varepsilon^2 + c\,\|\varphi\|_{C(\Omega)}\,\varepsilon^2 D(\varepsilon).$$

Let us suppose the converse, that is,

$$\lim_{\varepsilon \to 0} \sum_j \left|\varphi(x_j)y_\varepsilon(x_j) - \varepsilon^{-2}\int_{\varepsilon Y_j} \varphi\, y_\varepsilon dx\right|\varepsilon^2 > 0.$$

Since Ω is bounded, it is contained in a number of squares εY_j smaller than C/ε^2, where C does not depend on ε. So, there exist a constant $C^* > 0$ and a value $\varepsilon^* > 0$ such that

$$\left|\varphi(x_j)y_\varepsilon(x_j) - \varepsilon^{-2}\int_{\varepsilon Y_j} \varphi\, y_\varepsilon dx\right| \ge C^* \tag{13}$$

(for an infinite number of indices j for every fixed ε). Hence the extremely wild oscillations is present in the sequence $\{\varphi y_\varepsilon\}$. However ([1],[2]), if we have the very rapid fluctuations in the functions $\{\varphi y_\varepsilon\}$, then the convergence $\varphi y_\varepsilon \to \varphi y$ almost everywhere in Ω is excluded.

This fact immediately reflects the failure of the strong convergence $\varphi y_\varepsilon \to \varphi y$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Indeed, by the initial assumptions, we have

$$|y_\varepsilon| \le c \text{ for every } \varepsilon \in E,\ \varphi y_\varepsilon \to \varphi y \text{ in } L^1(\Omega),$$

$$\text{and } \|\varphi y_\varepsilon - \varphi y\|_{L^1(\Omega)} \to 0 \text{ as } \varepsilon \to 0 \text{ for any } \varphi \in C_0^\infty(R^2).$$

Let A be any subset of Ω with $|A| \ne 0$. Then, by Valadier's Theorem [2], $\varphi y_\varepsilon \to \varphi y$ strongly if and only if the following criterion is satisfied: $\forall \delta > 0\ \exists \varepsilon^0 > 0, \exists B \subset A$ with $|B| \ne 0$ such that

$$|B|^{-1}\int_B \left|\varphi y_\varepsilon - |B|^{-1}\int_B \varphi y_\varepsilon dx\right| dx < \delta \quad \forall \varepsilon < \varepsilon^0.$$

Hence, for any $\varepsilon < \varepsilon^0$ there is a square $\varepsilon Y_j \subset B$ such that

$$\varepsilon^{-2}\int_{\varepsilon Y_j} \left|\varphi y_\varepsilon - \varepsilon^{-2}\int_{\varepsilon Y_j} \varphi y_\varepsilon dx\right| dx < \delta.$$

Since the functions φy_ε are continuous and uniformly bounded it follows that for any point x_j of εY_j satisfying the condition

$$\varphi(x_j)y_\varepsilon(x_j)-\varepsilon^{-2}\int\limits_{\varepsilon Y_j}\varphi y_\varepsilon dx\neq 0.$$

there can be found a constant $A_*>0$ satisfying

$$\varepsilon^{-2}\int\limits_{\varepsilon Y_j}\left|\varphi y_\varepsilon-\varepsilon^{-2}\int\limits_{\varepsilon Y_j}\varphi y_\varepsilon dx\right|dx=A_*\left|\varphi(x_j)y_\varepsilon(x_j)-\varepsilon^{-2}\int\limits_{\varepsilon Y_j}\varphi y_\varepsilon dx\right|.$$

Hence

$$\left|\varphi(x_j)y_\varepsilon(x_j)-\varepsilon^{-2}\int\limits_{\varepsilon Y_j}\varphi y_\varepsilon dx\right|<A_*^{-1}\delta$$

and we come into conflict with (13). So, our supposition was wrong and we get

$$\lim_{\varepsilon\to 0}\sum_j\left|\varphi(x_j)y_\varepsilon(x_j)-\varepsilon^{-2}\int\limits_{\varepsilon Y_j}\varphi y_\varepsilon dx\right|\varepsilon^2=0.$$

As a result, we have $I_1\to 0$. Thus, summing up the results obtained above and the relations (12), (9), we come to the desired identity (10).

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Надійшла до редколегії 14.01.2008