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Strengthening the Comparison Theorem and Kolmogorov Inequality in the Asymmetric Case

Abstract. We obtain an strengthening the Kolmogorov comparison theorem. In particular, it gives us the opportunity to obtain such strengthening Kolmogorov inequality in the asymmetric case:

$$\|x_{\pm}^{(k)}\|_\infty \leq \frac{\|\varphi_{r-k}(\,\cdot\,;\alpha,\beta)_\pm\|_\infty}{E_0(\varphi_r(\,\cdot\,;\alpha,\beta))_\infty^{1-k/r}} \||x|\|_\infty^{1-k/r} \|\alpha^{-1}x_+^{(r)} + \beta^{-1}x_-^{(r)}\|_\infty^{k/r}$$

for functions $x \in L_\infty^r(\mathbf{R})$, where

$$\||x|\|_\infty := \frac{1}{2}\sup_{\alpha,\beta}\{|x(\beta) - x(\alpha)| : x'(t) \neq 0 \ \forall t \in (\alpha,\beta)\},$$

$k, r \in \mathbf{N}$, $k < r$, $\alpha, \beta > 0$, $\varphi_r(\,\cdot\,;\alpha,\beta)_r$ is the asymmetric perfect spline of Euler of order r and $E_0(x)_\infty$ is the best uniform approximation of the function x by constants.

Key words: Kolmogorov comparision theorem, Kolmogorov inequality, asymmetric case, strengthening.

Анотація. В роботі отриано посилений варіант теореми порівняння Колмогорова в несиметричному випадку. Це дозволило, зокрема, одержати підсилення нерівності Колмогорова в несиметричному випадку:

$$\|x_{\pm}^{(k)}\|_\infty \leq \frac{\|\varphi_{r-k}(\,\cdot\,;\alpha,\beta)_\pm\|_\infty}{E_0(\varphi_r(\,\cdot\,;\alpha,\beta))_\infty^{1-k/r}} \||x|\|_\infty^{1-k/r} \|\alpha^{-1}x_+^{(r)} + \beta^{-1}x_-^{(r)}\|_\infty^{k/r}$$

для функцій $x \in L_\infty^r(\mathbf{R})$, де

$$\||x|\|_\infty := \frac{1}{2}\sup_{\alpha,\beta}\{|x(\beta) - x(\alpha)| : x'(t) \neq 0 \ \forall t \in (\alpha,\beta)\},$$

$k, r \in \mathbf{N}$, $k < r$, $\alpha, \beta > 0$, $\varphi_r(\,\cdot\,;\alpha,\beta)_r -$ несиметричний ідеальний сплайн Ейлера порядку r, а $E_0(x)_\infty -$ найкраще рівномірне наближення функції x константами.

Ключові слова: Теорема порівняння Колмогорова, нерівність Колмогорова, несиметричний випадок, підсилення.

1. Introduction. Let G be the real line $\mathbf{R}$ or the unit circle $\mathbf{T}$ which is realized as the interval $[0, 2\pi]$ with coincident endpoints. We will consider the spaces $L_p(G)$, $1 \leq p \leq \infty$, of all measurable functions $x : G \to \mathbf{R}$ such that $\|x\|_p = \|x\|_{L_p(G)} < \infty$, where

$$\|x\|_p := \left(\int_G |x(t)|^p \, dt \right)^{1/p}, \quad \text{if} \quad 1 \leq p < \infty,$$

$$\|x\|_\infty := \operatorname*{vrai\,sup}_{t \in G} |x(t)|.$$

For $\alpha, \beta > 0$ and $x \in L_\infty(G)$ set

$$\|x\|_{\infty, \alpha, \beta} := \|\alpha x_+ + \beta x_-\|_\infty,$$

where $x_\pm(t) := \max\{x_\pm(t), 0\}$.

For $r \in \mathbf{N}$ denote by $L_\infty^r(G)$ the space of all functions $x \in L_\infty(G)$ for which $x^{(r-1)}$ is locally absolutely continuous and $x^{(r)} \in L_\infty(G)$.

Let $\varphi_r(\,\cdot\,; \alpha, \beta)$, $r \in \mathbf{N}$, be the 2π-periodic integral with zero mean value on a period of the 2π-periodic function $\varphi_0(\,\cdot\,; \alpha, \beta)$ defined on $[0, 2\pi]$ in the following way $\varphi_0(0\,; \alpha, \beta) = \varphi_0(2\pi\,; \alpha, \beta) := 0$ and

$$\varphi_0(\,\cdot\,; \alpha, \beta) := \alpha, \quad \text{if} \quad t \in (0, 2\pi\beta/(\alpha + \beta)),$$

$$\varphi_0(\,\cdot\,; \alpha, \beta) := -\beta, \quad \text{if} \quad t \in (2\pi\beta/(\alpha + \beta), 2\pi).$$

Notice that $\varphi_r(\,\cdot\,; 1, 1)$ is the spline of Euler of order r.

Hörmander [1] proved the following theorem.

Theorem A. *Let $k, r \in \mathbf{N}$, $k < r$, $G = \mathbf{R}$ or $G = \mathbf{T}$. Then for any function $x \in L_\infty^r(G)$ and for any $\alpha, \beta > 0$ there is the sharp inequality*

$$\|x_\pm^{(k)}\|_\infty \leq \frac{\|\varphi_{r-k}(\,\cdot\,; \alpha, \beta)_\pm\|_\infty}{E_0(\varphi_r(\,\cdot\,; \alpha, \beta))_\infty^{1-k/r}} E_0(x)_\infty^{1-k/r} \|x^{(r)}\|_{\infty; \alpha^{-1}, \beta^{-1}}^{k/r}, \tag{1}$$

where $E_0(x)_\infty$ is the best uniform approximation of the function x by constants.

The equality in (1) is achieved for the functions $x(t) = a\varphi_{\lambda, r}(t; \alpha, \beta) + b$, $a, b \in \mathbf{R}$, $\lambda > 0$ if $G = \mathbf{R}$ and $\lambda \in \mathbf{N}$ if $G = \mathbf{T}$.

The proof of Theorem A in [1] is based on the comparison theorem. In view of the importance of this theorem for further exposition, we present its formulation.

For $r \in \mathbf{N}$; $\alpha, \beta > 0$; $G = \mathbf{R}$ or $G = \mathbf{T}$ set

$$W_{\infty, \alpha, \beta}^r(G) := \left\{ x \in L_\infty^r(G) : \left\| x^{(r)} \right\|_{\infty, \alpha^{-1}, \beta^{-1}} \leq 1 \right\},$$

and let $\varphi_{\lambda, r}(t; \alpha, \beta) := \lambda^{-r} \varphi_r(\lambda t; \alpha, \beta)$ for $\lambda > 0$.

Theorem B. *Let $r \in \mathbf{N}$; $\alpha, \beta > 0$; $x \in W^r_{\infty;\alpha,\beta}(\mathbf{R})$ and the number λ is such that*

$$\|x_\pm\|_\infty \le \|\varphi_{\lambda,r}(\cdot; \alpha, \beta)_\pm\|_\infty. \tag{2}$$

If the points ξ and η satisfying conditions

$$x(\xi) = \varphi_{\lambda,r}(\eta; \alpha, \beta),$$

and

$$x'(\xi) \cdot \varphi'_{\lambda,r}(\eta; \alpha, \beta) \ge 0,$$

than

$$|x'(\xi)| \le |\varphi'_{\lambda,r}(\eta; \alpha, \beta)|.$$

In the symmetric case $\alpha = \beta$ Theorems A and B are due to Kolmogorov [2].

In this paper, we obtain (Theorem 1) a strengthening of Theorem B in which condition (2) ($\|x_\pm\|_\infty \le \|\varphi_{\lambda,r}(\cdot; \alpha, \beta)_\pm\|_\infty$) теоремы B replaced by a weaker condition $\||x\||_\infty \le E_0(\varphi_{\lambda,r}(t; \alpha, \beta))_\infty$, where

$$\||x\||_\infty := \frac{1}{2} \sup_{\alpha,\beta}\{|x(\beta) - x(\alpha)| : x'(t) \neq 0 \ \forall t \in (\alpha, \beta)\}. \tag{3}$$

At the same time, the conclusion of Theorem 1 is stronger than the conclusion of Theorem B.

It is clear that $\||x\||_\infty \le E_0(x)_\infty$, and it is easy to give examples of infinitely differentiable functions x, for which the ratio

$$\frac{\||x\||_\infty}{E_0(x)_\infty}$$

is arbitrarily little.

Using Theorem 1, we obtain (Theorem 2) a strengthening of inequality (1), in which the quantity $E_0(x)_\infty$ is replaced by a more delicate characteristic $\||x\||_\infty$. As an application, we obtain (Theorem 3) a strengthening of Ligun's inequality [3] and Babenko inequality [4].

In the symmetric case $\alpha = \beta$ the results of this paper (Theorems 1−3) are proved in [5], [6].

2. Strengthening the comparison theorem and Kolmogorov's inequality. Note that if the function $x \in L^r_\infty(\mathbf{R})$ is monotonic on one of infinite intervals $(-\infty, b]$ or $[a, +\infty)$, then there are finite limits $\lim_{t\to-\infty} x(t)$ and $\lim_{t\to+\infty} x(t)$ respectively. We will assume for such functions

$$x(-\infty) := \lim_{t\to-\infty} x(t), \quad x(+\infty) := \lim_{t\to+\infty} x(t).$$

Symbols a, b in Lemma 1 and Theorem 1 can take any values from the extended number line. In doing so, we will assume $+\infty + t = +\infty$, $-\infty + t = -\infty$, $\forall t \in \mathbf{R}$.

Lemma 1. *Let $r \in \mathbf{N}$; $\alpha, \beta > 0$; $x \in W^r_{\infty,\alpha,\beta}(\mathbf{R})$ and the number λ is chosen from the condition*

$$\|x'_\pm\|_\infty \leq \|\varphi_{\lambda,r-1}(\cdot; \alpha, \beta)_\pm\|_\infty. \tag{4}$$

Let further $[a, b]$ be an interval of increasing (decreasing) function x such that $x'(t) \neq 0$, $t \in (a, b)$, and $x'(a) = 0$ if $a \neq -\infty$, $x'(b) = 0$, if $b \neq +\infty$, and $[\xi, \eta]$ is the interval of increase (decrease) of the function $\varphi_{\lambda,r}(\cdot; \alpha, \beta)$, where $\varphi_{\lambda,r-1}(\xi; \alpha, \beta) = \varphi_{\lambda,r-1}(\eta; \alpha, \beta) = 0$, and let γ be a local extremum point of the function $\varphi_{\lambda,r-1}(\cdot; \alpha, \beta)$ on the interval (ξ, η).

If t is an arbitrary point of the segment $[a, b]$ for which there exists a point $y \in [\gamma, \eta]$ such that

$$|x(b) - x(t)| = |\varphi_{\lambda,r}(\eta; \alpha, \beta) - \varphi_{\lambda,r}(y; \alpha, \beta)| \tag{5}$$

or a point $y \in [\xi, \gamma]$ such that

$$|x(t) - x(a)| = |\varphi_{\lambda,r}(y) - \varphi_{\lambda,r}(\xi; \alpha, \beta)|, \tag{6}$$

then

$$|x'(t)| \leq |\varphi_{\lambda,r-1}(y; \alpha, \beta)|. \tag{7}$$

Proof. For $r = 1$ the assertion of the lemma is obvious. Therefore, we will assume that $r \geq 2$.

Consider the case when the function x is increasing on the segment $[\alpha, \beta]$, while the function $\varphi_{\lambda,r}(\cdot; \alpha, \beta)$ increases on the segment $[\xi, \eta]$. The case of decreasing functions x and $\varphi_{\lambda,r}(\cdot; \alpha, \beta)$ is treated similarly.

Let us prove (7) under assumption (5).

Let us assume that inequality (7) is not satisfied, i.e.,

$$|x'(t)| > |\varphi_{\lambda,r-1}(y; \alpha, \beta)|.$$

Then $t < b$ and hence $y < \eta$. Using condition (4) and applying Theorem B to the derivative x', we obtain

$$x'(t + u) > \varphi_{\lambda,r-1}(y + u; \alpha, \beta), \quad u \in (0, \eta - y),$$

while $b - t \geq \eta - y$. But then

$$x(\beta) - x(t) = \int_t^b x'(u)du = \int_0^{b-t} x'(t+u)du > \int_0^{\eta-y} \varphi_{\lambda,r-1}(y + u; \alpha, \beta)du =$$

$$= \int_y^\eta \varphi_{\lambda,r-1}(u; \alpha, \beta)du = \varphi_{\lambda,r}(\eta; \alpha, \beta) - \varphi_{\lambda,r}(y; \alpha, \beta),$$

which contradicts condition (5).

Inequality (7) is proved similarly under assumption (6).

Lemma 1 is proved.

The following theorem is a strengthening of Kolmogorov's comparison theorem in the nonsymmetric case.

Theorem 1. *. Let $r \in \mathbf{N}$; $\alpha, \beta > 0$; $x \in W^r_{\infty,\alpha,\beta}(\mathbf{K})$ and the number λ is chosen from the condition*

$$|||x|||_\infty = E_0(\varphi_{\lambda,r}(\cdot;\alpha,\beta))_\infty, \tag{8}$$

where the quantity $|||x|||_\infty$ is defined by equality (3). Let further $[a,b]$ be an interval of increasing (decreasing) function x such that $x'(t) \neq 0$, $t \in (a,b)$, and $x'(a) = 0$ if $a \neq -\infty$, $x'(b) = 0$, if $b \neq +\infty$, and $[\xi, \eta]$ is the interval of increase (decrease) of the function $\varphi_{\lambda,r}(\cdot;\alpha,\beta)$, where $\varphi_{\lambda,r-1}(\xi;\alpha,\beta) = \varphi_{\lambda,r-1}(\eta;\alpha,\beta) = 0$, and let γ be a local extremum point of the function $\varphi_{\lambda,r-1}(\cdot;\alpha,\beta)$ on the interval (ξ,η).

Then if for the point $t \in [a,b]$ the point $y \in [\xi,\eta]$ is chosen so that

$$|x(b) - x(t)| = |\varphi_{\lambda,r}(\eta;\alpha,\beta) - \varphi_{\lambda,r}(y;\alpha,\beta)| \tag{9}$$

or so that

$$|x(t) - x(a)| = |\varphi_{\lambda,r}(y;\alpha,\beta) - \varphi_{\lambda,r}(\xi;\alpha,\beta)|, \tag{10}$$

then

$$|x'(t)| \leq |\varphi_{\lambda,r-1}(y;\alpha,\beta)|. \tag{11}$$

Proof.. Note, first of all, that in view of condition (8) for an arbitrary point $t \in [a,b]$ there exists a point $y = y_1 \in [\xi,\eta]$ satisfying equality (9) and there exists a point $y = y_2 \in [\xi,\eta]$ satisfying equality (10).

Let us now prove that condition (8) implies the inequality

$$\|x'_\pm\|_\infty \leq \|\varphi_{\lambda,r-1}(\cdot;\alpha,\beta)_\pm\|_\infty, \tag{12}$$

those. condition (4) of Lemma 1 is satisfied.

Let's assume the opposite. Let, for example.

$$\|x'_+\|_\infty > \|\varphi_{\lambda,r-1}(\cdot;\alpha,\beta)_+\|_\infty, \;\; ; \;\; \|x'_-\|_\infty \leq \|\varphi_{\lambda,r-1}(\cdot;\alpha,\beta)_-\|_\infty. \tag{13}$$

Other options for not fulfilling (12) are considered similarly.

Then, considering the equality $\|\varphi_{\lambda,r}(\cdot;\alpha,\beta)_\pm\|_\infty = \lambda^{-r}\|\varphi_r(\cdot;\alpha,\beta)_\pm\|_\infty$, we conclude that there exists a number $\omega \in (0,\lambda)$ such that

$$\|x'_+\|_\infty = \|\varphi_{\omega,r-1}(\cdot;\alpha,\beta)_+\|_\infty. \tag{14}$$

It follows from condition (8) and the inequality $\omega < \lambda$ that

$$|||x|||_\infty < E_0(\varphi_{\omega,r}(\cdot;\alpha,\beta))_\infty. \tag{15}$$

In addition, in view of the second relation in (13)

$$\|x'_-\|_\infty < \|\varphi_{\omega,r-1}(\cdot;\alpha,\beta)_-\|_\infty. \tag{16}$$

We choose a point $t_0 \in \mathbf{R}$ so that

$$\|x'_+\|_\infty = x'(t_0), \tag{17}$$

and let the number $\tau \in \mathbf{R}$ satisfy the condition

$$\|\varphi_{\lambda,r-1}(\cdot;\alpha,\beta)_+\|_\infty = \varphi_{\lambda,r-1}(t_0+\tau;\alpha,\beta)_+. \qquad (18)$$

Denote by t_1 and t_2 the zeros closest to the left and right of the point t_0 functions $\varphi_{\gamma,r-1}(\cdot+\tau)$. Taking into account (14), (16)−(18) and applying Theorem B to the function x', we obtain

$$x'(t) \geq \varphi_{\omega,r-1}(t+\tau;\alpha,\beta), \quad t \in (t_1,t_2).$$

Hence we conclude that $x'(t) > 0, \quad t \in (t_1,t_2)$ and

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} x'(t)dt \geq \int_{t_1}^{t_2} \varphi_{\omega,r-1}(t+\tau;\alpha,\beta)dt =$$

$$= \varphi_{\omega,r}(t_2+\tau;\alpha,\beta) - \varphi_{\omega,r}(t_1+\tau;\alpha,\beta) = 2E_0(\varphi_{\omega,r}(\cdot;\alpha,\beta))_\infty.$$

In this way,

$$|||x|||_\infty \geq \frac{1}{2}|x(t_2) - x(t_1)| \geq E_0(\varphi_{\omega,r}(\cdot;\alpha,\beta))_\infty,$$

which contradicts inequality (15). Thus (12) is proved.

Let us now prove (11) under assumption (9).

Consider the case when the function x is increasing on the interval $[a,b]$ and the function $\varphi_{\lambda,r}(\cdot;\alpha,\beta)$ increases on the interval $[\xi,\eta]$. The case of decreasing functions x and $\varphi_{\lambda,r}(\cdot;\alpha,\beta)$ is treated similarly.

If the point y, chosen from condition (9) is such that $y \in [\gamma,\eta]$, then taking into account (12) and applying Lemma 1, we obtain (11).

Now let the inclusion $y \in [\xi,\gamma]$ take place for the point y chosen from condition (9). From (8) and (9) it follows that

$$x(t) - x(a) \leq \varphi_{\lambda,r}(y;\alpha,\beta) - \varphi_{\lambda,r}(\xi;\alpha,\beta).$$

Therefore, there exists a point $y_1 \in [\xi,\gamma], y_1 \leq y$ such that

$$x(t) - x(a) = \varphi_{\lambda,r}(y_1;\alpha,\beta) - \varphi_{\lambda,r}(\xi;\alpha,\beta).$$

Taking into account inequality (12) and applying Lemma 1, we obtain

$$|x'(t)| \leq |\varphi_{\lambda,r-1}(y_1)| \leq |\varphi_{\lambda,r-1}(y)|.$$

Thus, (11) under assumption (9) is proved.

One can prove (11) similarly under assumption (10).

Theorem 1 is proved.

The following theorem is a strengthened version of inequality (1).

Theorem 2. *Let $k, r \in \mathbf{N}$, $k < r$, $G = \mathbf{R}$ or $G = \mathbf{T}$. Then for any function $x \in L_\infty^r(G)$ and for any $\alpha, \beta > 0$ has place of sharp inequalities*

$$\|x_\pm^{(k)}\|_\infty \leq \frac{\|\varphi_{r-k}(\,\cdot\,; \alpha, \beta)_\pm\|_\infty}{E_0(\varphi_r(\,\cdot\,; \alpha, \beta))_\infty^{1-k/r}} \||x\||_\infty^{1-k/r} \|x^{(r)}\|_{\infty; \alpha^{-1}, \beta^{-1}}^{k/r}, \qquad (19)$$

where the quantity $\||x\||_\infty$ is defined by relation (3).

Equality in (19) is achieved for the functions $x(t) = a\varphi_{\lambda,r}(t; \alpha, \beta) + b$, $a, b \in \mathbf{R}$, $\lambda > 0$ for $G = \mathbf{R}$ and $\lambda \in \mathbf{N}$ for $G = \mathbf{T}$.

Proof. Let us fix any $x \in L_\infty^r(G)$. Because of the homogeneity inequality (19), we can assume that

$$\|x^{(r)}\|_{\infty; \alpha^{-1}, \beta^{-1}} = 1. \qquad (20)$$

Then $x \in W_{\infty, \alpha, \beta}^r(G)$ and Theorem 1 applies to the function x. Let's choose $\lambda > 0$ from the condition

$$\||x\||_\infty = E_0(\varphi_{\lambda,r}(\cdot; \alpha, \beta))_\infty. \qquad (21)$$

When proving Theorem 1, it was established that condition (21) implies (see (12)) the inequality

$$\|x_\pm'\|_\infty \leq \|\varphi_{\lambda, r-1}(\cdot; \alpha, \beta)_\pm\|_\infty, \qquad (22)$$

and hence

$$E_0(x')_\infty \leq E_0(\varphi_{\lambda, r-1}(\cdot; \alpha, \beta))_\infty. \qquad (23)$$

By virtue of Theorem A applied to the function x', we have

$$\|x_\pm^{(k)}\|_\infty \leq \frac{\|\varphi_{r-k}(\,\cdot\,; \alpha, \beta)_\pm\|_\infty}{E_0(\varphi_{r-1}(\,\cdot\,; \alpha, \beta))_\infty^{1-(k-1)/(r-1)}} E_0(x')_\infty^{1-(k-1)/(r-1)} \|x^{(r)}\|_{\infty; \alpha^{-1}, \beta^{-1}}^{(k-1)/(r-1)}.$$

Hence, taking into account (20), (23) and the equality $\|\varphi_{\lambda,r}(\cdot; \alpha, \beta)_\pm\|_\infty = \lambda^{-r}\|\varphi_r(\cdot; \alpha, \beta)_\pm\|_\infty$, derive the estimate

$$\|x_\pm^{(k)}\|_\infty \leq \|\varphi_{\lambda, r-k}(\cdot; \alpha, \beta)_\pm\|_\infty, \qquad (24)$$

From (21) and (24) we obtain

$$\frac{\|x_\pm^{(k)}\|_\infty}{\||x\||_\infty^{1-k/r}} \leq \frac{\|\varphi_{\lambda, r-k}(\,\cdot\,; \alpha, \beta)_\pm\|_\infty}{E_0(\varphi_{\lambda,r}(\,\cdot\,; \alpha, \beta))_\infty^{1-k/r}} =$$

$$= \frac{\lambda^{-(r-k)}\|\varphi_{r-k}(\,\cdot\,; \alpha, \beta)_\pm\|_\infty}{(\lambda^{-r}E_0(\varphi_r(\,\cdot\,; \alpha, \beta))_\infty)^{1-k/r}} = \frac{\|\varphi_{r-k}(\,\cdot\,; \alpha, \beta)_\pm\|_\infty}{E_0(\varphi_r(\,\cdot\,; \alpha, \beta))_\infty^{1-k/r}}. \qquad (25)$$

From (25) and (20) follows (19).

Theorem 2 is proved.

3. Strengthening Ligun's and Babenko's inequalities. In [4] Babenko proved the following inequality.

Theorem C. *Let $q \geq 1$, $k, r \in \mathbf{N}$, $k < r$,. Then for any function $x \in L_\infty^r(\mathbf{T})$ and for any $\alpha, \beta > 0$ the exact inequality holds*

$$\|x_\pm^{(k)}\|_{L_q(\mathbf{T})} \leq \frac{\|\varphi_{r-k}(\,\cdot\,;\alpha,\beta)_\pm\|_{L_q(\mathbf{T})}}{E_0(\varphi_r(\,\cdot\,;\alpha,\beta))_\infty^{1-k/r}} E_0(x)_\infty^{1-k/r}\|x^{(r)}\|_{\infty;\alpha^{-1},\beta^{-1}}^{k/r}. \tag{26}$$

Equality in (26) is achieved for the functions $x(t) = a\varphi_{n,r}(t;\alpha,\beta)+b$, $a, b \in \mathbf{R}$, $n \in \mathbf{N}$.

In the symmetric case $\alpha = \beta$ inequality (26) is due to Ligun [3].

The following theorem is a strengthening of inequality (26).

Theorem 3. *Let $q \geq 1$, $k, r \in \mathbf{N}$, $k < r$. Then for any function $x \in L_\infty^r(\mathbf{T})$ and for any $\alpha, \beta > 0$ the exact inequality holds*

$$\|x_\pm^{(k)}\|_{L_q(\mathbf{T})} \leq \frac{\|\varphi_{r-k}(\,\cdot\,;\alpha,\beta)_\pm\|_{L_q(\mathbf{T})}}{E_0(\varphi_r(\,\cdot\,;\alpha,\beta))_\infty^{1-k/r}} \||x|\|_\infty^{1-k/r}\|x^{(r)}\|_{\infty;\alpha^{-1},\beta^{-1}}^{k/r}. \tag{27}$$

Equality in (27) is achieved for the functions $x(t) = a\varphi_{n,r}(t;\alpha,\beta)+b$, $a, b \in \mathbf{R}$, $n \in \mathbf{N}$.

Proof. Let us fix a function $x \in L_\infty^r(\mathbf{T})$. In the case of $k \geq 2$, inequalities (27) can be obtained by compiling inequalities (26) and (19) as follows.

Note that inequality (26) applied to the function x' implies the inequality

$$\|x_\pm^{(k)}\|_{L_q(\mathbf{T})} \leq$$

$$\leq \frac{\|\varphi_{r-k}(\,\cdot\,;\alpha,\beta)_\pm\|_{L_q(\mathbf{T})}}{E_0(\varphi_{r-1}(\,\cdot\,;\alpha,\beta))_\infty^{1-(k-1)/(r-1)}} E_0(x')_\infty^{1-(k-1)/(r-1)}\|x^{(r)}\|_{\infty;\alpha^{-1},\beta^{-1}}^{(k-1)/(r-1)}. \tag{28}$$

On the other hand, inequality (19) for $k = 1$ implies the inequality

$$E_0(x')_\infty \leq \frac{E_0(\varphi_{r-1}(\,\cdot\,;\alpha,\beta))_\infty}{E_0(\varphi_r(\,\cdot\,;\alpha,\beta))_\infty^{1-1/r}} \||x|\|_\infty^{1-1/r}\|x^{(r)}\|_{\infty;\alpha^{-1},\beta^{-1}}^{1/r}, \tag{29}$$

Estimating $E_0(x')_\infty$ in inequality (28) using inequality (29) and taking into account that

$$\left(1 - \frac{k-1}{r-1}\right)\left(1 - \frac{1}{r}\right) = 1 - \frac{k}{r},$$

we obtain inequality (27) for $k \geq 2$.

In the case of $k = 1$ (as in the general case), the proof of inequality (27) can be obtained by repeating the arguments from the proof of inequality (26) in [4], using Theorem 1 instead of Theorem B in these arguments.

Theorem 3 is proved.

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