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K. Nantomah*, **İ. Ege****
* Department of Mathematics, School of Mathematical Sciences,
C. K. Tedam University of Technology and Applied Sciences,
P. O. Box 24, Navrongo, Upper-East Region, Ghana. *E-mail: knantomah@cktutas.edu.gh*
** Department of Mathematics, Aydin Adnan Menderes University,
Aydin, Turkey. *E-mail: iege@adu.edu.tr*

A Lambda Analogue of the Gamma Function and its Properties

Abstract. We consider a generalization of the gamma function which we term as lambda analogue of the gamma function or λ-gamma function and further, we establish some of its accompanying properties. For the particular case when $\lambda = 1$, the results established reduce to results involving the classical gamma function. The techniques employed in proving our results are analytical in nature.
Key words: Gamma function, lambda analogue, λ-gamma function, λ-beta function, Bohr-Mollerup theorem, inequality

Анотація. Ми розглядаємо узагальнення гамма-функції, яку називаємо лямбда-аналогом гамма-функції або λ-гамма-функцією, а також встановлюємо деякі з її супутніх властивостей. Для окремого випадку, коли $\lambda = 1$, встановлені результати зводяться до результатів, що включають класичну гамма-функцію. Методи, які використовуються для підтвердження наших результатів, мають аналітичний характер.
Ключові слова: Гамма-функція, лямбда-аналог, λ-гамма-функція, λ-бета-функція, теорема Бора-Моллерапа, нерівність

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1. Introduction

The gamma function, which is an extension of the factorial notation to non-integer values, is encountered in every aspect of mathematics. Arguably, it is one of the most studied special functions. It has numerous applications in mathematical analysis, number theory, combinatorics, mathematical modeling, statistics, probability theory, engineering, and physics, just to mention a few. It is normally defined by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

for $x > 0$. The upper incomplete gamma function is defined as

$$\Gamma(x,s) = \int_s^\infty t^{x-1}e^{-t}dt$$

for $s > 0$ and $x \in (-\infty, \infty)$ whereas the lower incomplete gamma function is defined as

$$\gamma(x,s) = \int_0^s t^{x-1}e^{-t}dt$$

for $x > 0$.

As a result of the important roles of this special function, it has been investigated in various ways by several renowned researchers. A particular focus has been on developing generalizations or extensions of the function. In the recent past, some new generalizations have been defined and investigated. For example see [3], [4], [5], [6], [10], [11], [13]. In this work, we continue the investigation in this direction. Specifically, we consider a generalization of the gamma function which we term as lambda analogue of the gamma function or λ-gamma function, and further study some of its properties.

2. Main Results

In this section, we define the lambda analogue of the gamma function and further study some of its accompanying properties.

Definition 1. Let $x > 0$ and $\lambda > 0$. Then the λ-gamma function is defined as

$$\Gamma_\lambda(x) = \int_0^\infty t^{x-1}e^{-\lambda t}dt \tag{1}$$

$$= \lambda^{-x}\Gamma(x) \tag{2}$$

$$= \lim_{k\to\infty} \frac{\lambda^{-x}k!k^x}{x(x+1)(x+2)\dots(x+k)}. \tag{3}$$

In view of this definition, the upper incomplete λ-gamma function is given as

$$\Gamma_\lambda(x,a) = \int_a^\infty t^{x-1}e^{-\lambda t}dt \tag{4}$$

whilst the lower incomplete λ-gamma function is given as

$$\gamma_\lambda(x,a) = \int_0^a t^{x-1}e^{-\lambda t}dt. \tag{5}$$

Obviously,

$$\gamma_\lambda(x,a) + \Gamma_\lambda(x,a) = \Gamma_\lambda(x). \tag{6}$$

The λ-gamma function may be interpreted as the Laplace transform of the function $f(t) = t^{x-1}$ or the Mellin transform of the function $f(t) = e^{-\lambda t}$. When $\lambda = 1$, the generalized functions $\Gamma_\lambda(x)$, $\Gamma_\lambda(x,a)$ and $\gamma_\lambda(x,a)$ respectively reduces to the classical functions $\Gamma(x)$, $\Gamma(x,a)$ and $\gamma(x,a)$. The function (5) has been studied in the recent works [7] and [12].

The λ-gamma function satisfies the following properties.

$$\Gamma_\lambda(1) = \frac{1}{\lambda}, \tag{7}$$

$$\Gamma_\lambda(x+1) = \frac{x}{\lambda}\Gamma_\lambda(x), \quad x > 0, \tag{8}$$

$$\Gamma_\lambda(k+1) = \frac{k!}{\lambda^{k+1}}, \quad k \in \mathbb{N}_0, \tag{9}$$

$$\Gamma_\lambda(x)\Gamma_\lambda(1-x) = \frac{\pi}{\lambda\sin(\pi x)}, \quad x \in (0,1), \tag{10}$$

$$\Gamma_\lambda(1+x)\Gamma_\lambda(1-x) = \frac{\pi x}{\lambda^2\sin(\pi x)}, \quad x \in (0,1), \tag{11}$$

$$\Gamma_\lambda(x)\Gamma_\lambda\left(x+\frac{1}{2}\right) = 2^{1-2x}\sqrt{\frac{\pi}{\lambda}}\Gamma_\lambda(2x), \quad x > 0, \tag{12}$$

$$\frac{\Gamma_\lambda(x+k)}{\Gamma_\lambda(x)} = \frac{(x)_k}{\lambda^k}, \quad x > 0, k \in \mathbb{N}_0, \tag{13}$$

$$\Gamma_\lambda\left(k+\frac{1}{2}\right) = \frac{(2k-1)!!}{2^k\lambda^k}.\sqrt{\frac{\pi}{\lambda}}, \quad k \in \mathbb{N}_0, \tag{14}$$

where $(x)_k = x(x+1)(x+2)\dots(x+k-1)$ is the Pochhammer symbol and $m!!$ is the double factorial of m.

Remark 1. Let the λ-beta function be defined as

$$\beta_\lambda(x,y) = \frac{\Gamma_\lambda(x)\Gamma_\lambda(y)}{\Gamma_\lambda(x+y)}$$

for $x > 0$ and $y > 0$. Then this function coincides with the ordinary beta function $\beta(x,y)$ since

$$\beta_\lambda(x,y) = \frac{\Gamma_\lambda(x)\Gamma_\lambda(y)}{\Gamma_\lambda(x+y)} = \frac{\lambda^{-x}\Gamma(x)\lambda^{-y}\Gamma(y)}{\lambda^{-(x+y)}\Gamma(x+y)} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \beta(x,y). \tag{15}$$

Definition 2. Let the λ-psi function be defined as

$$\psi_\lambda(x) = \frac{d}{dx}\ln\Gamma_\lambda(x)$$

for $x > 0$. Then

$$\psi_\lambda(x) = \frac{\Gamma'_\lambda(x)}{\Gamma_\lambda(x)} = -\ln\lambda + \psi(x) \tag{16}$$

$$= -(\ln\lambda + \gamma) + \int_0^1 \frac{1-t^{x-1}}{1-t}dt \tag{17}$$

$$= -(\ln\lambda + \gamma) + \int_0^\infty \frac{e^{-t}-e^{-xt}}{1-e^{-t}}dt \tag{18}$$

$$= -(\ln\lambda + \gamma) + \sum_{k=0}^{\infty}\frac{x-1}{(k+1)(k+x)}, \tag{19}$$

where $\gamma = \lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{r} - \ln n\right] = 0.5772...$ is the Euler-Mascheroni constant and $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ is the classical psi or digamma function.

Remark 2. It follows from (8) that

$$\psi_\lambda(x+1) = \frac{1}{x} + \psi_\lambda(x) \tag{20}$$

and since

$$\psi_\lambda(x+1) - \psi_\lambda(x) = \frac{1}{x} > 0,$$

then $\psi_\lambda(x)$ is increasing. The increasing property of $\psi_\lambda(x)$ implies that $\Gamma_\lambda(x)$ is logarithmically convex and hence convex. Also, (16) implies that

$$\psi'_\lambda(x) = \psi'(x)$$

and thus, the derivatives of $\psi_\lambda(x)$ and $\psi_\lambda(x)$ coincides. In particular,

$$\psi_\lambda(1) = -(\ln\lambda + \gamma), \quad \psi'_\lambda(1) = \frac{\pi^2}{6}, \quad \text{and} \quad \Gamma'_\lambda(1) = -\left(\frac{\ln\lambda+\gamma}{\lambda}\right).$$

Theorem 1. *For $x > 0$, $\lambda > 0$ and $u \in (0,1)$, the inequality*

$$\left(\frac{x}{x+u}\right)^{1-u} \leq \frac{\Gamma_\lambda(x+u)}{\left(\frac{x}{\lambda}\right)^u \Gamma_\lambda(x)} \leq 1 \tag{21}$$

holds.

Proof. By making use of Hölder's inequality, we obtain

$$\begin{aligned}
\Gamma_\lambda(x+u) &= \int_0^\infty t^{x+u-1}e^{-\lambda t}dt \\
&= \int_0^\infty t^{ux}e^{-u\lambda t}t^{(1-u)(x-1)}e^{-(1-u)\lambda t}dt \\
&\leq \left(\int_0^\infty t^x e^{-\lambda t}dt\right)^u \left(\int_0^\infty t^{x-1}e^{-\lambda t}dt\right)^{1-u} \\
&= \left[\Gamma_\lambda(x+1)\right]^u \left[\Gamma_\lambda(x)\right]^{1-u}.
\end{aligned} \tag{22}$$

By applying (8) to (22) we obtain

$$\Gamma_\lambda(x+u) \leq \left(\frac{x}{\lambda}\right)^u \Gamma_\lambda(x) \tag{23}$$

and replacing u by $1-u$ in (23) gives

$$\Gamma_\lambda(x+1-u) \leq \left(\frac{x}{\lambda}\right)^{1-u} \Gamma_\lambda(x). \tag{24}$$

Further, replacing x by $x+u$ in (24) gives

$$\Gamma_\lambda(x+1) \leq \left(\frac{x+u}{\lambda}\right)^{1-u} \Gamma_\lambda(x+u). \tag{25}$$

Now combining (23) and (25) gives

$$\frac{\Gamma_\lambda(x+1)}{\left(\frac{x+u}{\lambda}\right)^{1-u}} \leq \Gamma_\lambda(x+u) \leq \left(\frac{x}{\lambda}\right)^u \Gamma_\lambda(x). \tag{26}$$

Finally, applying (8) to (26) and rearranging gives the desired result (21).

Remark 3. Inequality (21) is equivalent to the inequality

$$\left(\frac{x}{\lambda}\right)^{1-u} \leq \frac{\Gamma_\lambda(x+1)}{\Gamma_\lambda(x+u)} \leq \left(\frac{x+u}{\lambda}\right)^{1-u}, \tag{27}$$

and in particular, when $u=\frac{1}{2}$, we obtain

$$\sqrt{\frac{x}{\lambda}} \leq \frac{\Gamma_\lambda(x+1)}{\Gamma_\lambda(x+\frac{1}{2})} \leq \sqrt{\frac{x}{\lambda}+\frac{1}{2\lambda}}. \tag{28}$$

When $\lambda = 1$, inequalities (27) and (28) respectively reduce to inequality (2.8) of [14] and inequality (3) of [15] for $s=\frac{1}{2}$.

Theorem 2. *The following limits are valid.*

$$\lim_{x\to\infty} \frac{\Gamma_\lambda(x+u)}{x^u\Gamma_\lambda(x)} = \frac{1}{\lambda^u}, \quad x>0, u\in(0,1), \tag{29}$$

$$\lim_{x\to\infty} x^{v-u}\frac{\Gamma_\lambda(x+u)}{\Gamma_\lambda(x+v)} = \lambda^{v-u}, \quad x>0, u,v\in(0,1), \tag{30}$$

$$\lim_{x\to 0} x\Gamma_\lambda(x) = 1, \quad x>0, \tag{31}$$

$$\lim_{x\to 0} \left[\frac{1}{x} - \Gamma_\lambda(x)\right] = \ln\lambda + \gamma, \quad x>0, \tag{32}$$

$$\lim_{x\to 0} \frac{1}{x}\left[\frac{1}{\Gamma_\lambda(1-x)} - \frac{1}{\Gamma_\lambda(1+x)}\right] = -2\lambda(\ln\lambda+\gamma), \quad x\in(0,1), \tag{33}$$

$$\lim_{x\to 0} x\psi_\lambda(x) = -1, \quad x>0, \tag{34}$$

$$\lim_{x\to 0} \left[\frac{1}{x} + \psi_\lambda(x)\right] = -(\ln\lambda + \gamma), \quad x>0, \tag{35}$$

$$\lim_{x\to 0} \frac{1}{x}\left[\frac{1}{\psi_\lambda(1-x)} - \frac{1}{\psi_\lambda(1+x)}\right] = \frac{\pi^2}{3(\ln\lambda+\gamma)^2}, \quad x\in(0,1). \tag{36}$$

Proof. By applying Squeezes theorem on (21), we obtain the limit (29). Next, by using (29), we obtain

$$\lim_{x\to\infty} x^{v-u}\frac{\Gamma_\lambda(x+u)}{\Gamma_\lambda(x+v)} = \lim_{x\to\infty} \frac{\Gamma_\lambda(x+u)}{x^u\Gamma_\lambda(x)}.\lim_{x\to\infty} \frac{x^v\Gamma_\lambda(x)}{\Gamma_\lambda(x+v)} = \lambda^{v-u}$$

which proves (30). Next, by using (8), we obtain

$$\lim_{x\to 0} x\Gamma_\lambda(x) = \lambda \lim_{x\to 0} \Gamma_\lambda(x+1) = \lambda.\frac{1}{\lambda} = 1$$

which proves (31). Next, we have

$$\begin{aligned}
\lim_{x\to 0}\left[\frac{1}{x} - \Gamma_\lambda(x)\right] &= -\lim_{x\to 0}\left[\Gamma_\lambda(x) - \frac{1}{x}\right] \\
&= -\lim_{x\to 0}\left[\frac{\lambda\Gamma_\lambda(x+1)}{x} - \frac{1}{x}\right] \\
&= -\lambda \lim_{x\to 0}\left[\frac{\Gamma_\lambda(x+1) - \frac{1}{\lambda}}{x}\right] \\
&= -\lambda \lim_{x\to 0} \Gamma'_\lambda(x+1) \\
&= \ln\lambda + \gamma
\end{aligned}$$

which proves (32). Next, we have

$$\begin{aligned}
\lim_{x\to 0}\frac{1}{x}\left[\frac{1}{\Gamma_\lambda(1-x)} - \frac{1}{\Gamma_\lambda(1+x)}\right] &= \lim_{x\to 0}\left[\frac{\Gamma'_\lambda(1-x)}{\Gamma_\lambda(1-x)^2} + \frac{\Gamma'_\lambda(1+x)}{\Gamma_\lambda(1+x)^2}\right] \\
&= -2\lambda(\ln\lambda + \gamma)
\end{aligned}$$

which proves (33). Next, by using the functional equation (20), we have

$$\lim_{x\to 0} x\psi_\lambda(x) = \lim_{x\to 0} (x\psi_\lambda(x+1) - 1) = -1$$

which proves (34). Moreover, the functional equation (20) also implies that

$$\lim_{x\to 0}\left[\frac{1}{x} + \psi_\lambda(x)\right] = \psi_\lambda(1) = -(\ln\lambda + \gamma)$$

which proves (35). Finally, we have

$$\begin{aligned}
\lim_{x\to 0}\frac{1}{x}\left[\frac{1}{\psi_\lambda(1-x)} - \frac{1}{\psi_\lambda(1+x)}\right] &= \lim_{x\to 0}\left[\frac{\psi'_\lambda(1-x)}{\psi_\lambda(1-x)^2} + \frac{\psi'_\lambda(1+x)}{\psi_\lambda(1+x)^2}\right] \\
&= \frac{\pi^2}{3(\ln\lambda + \gamma)^2}
\end{aligned}$$

which proves (36). This completes the proof.

Remark 4. The limit (30) actually holds for all real numbers u and v such that $x + u > 0$ and $x + v > 0$. This is deduced from the asymptotic relation (6.1.47) on page 257 of [1] as follows.

$$\begin{aligned}
\lim_{x\to\infty} x^{v-u}\frac{\Gamma_\lambda(x+u)}{\Gamma_\lambda(x+v)} &= \lim_{x\to\infty} x^{v-u}\frac{\lambda^{-(x+u)}\Gamma(x+u)}{\lambda^{-(x+v)}\Gamma(x+v)} \\
&= \lambda^{v-u}\lim_{x\to\infty} x^{v-u}\frac{\Gamma(x+u)}{\Gamma(x+v)} \\
&= \lambda^{v-u}. \qquad (37)
\end{aligned}$$

Theorem 3. *Let $u \geq 0$, $v \geq 0$ such that $x+1 > u+v$. Then the function*

$$\mathcal{P}(x) = \frac{\Gamma_\lambda(x+1)\Gamma_\lambda(x+1-u-v)}{\Gamma_\lambda(x+1-u)\Gamma_\lambda(x+1-v)} \tag{38}$$

is decreasing. Consequently, for $u \geq 0$, $v \geq 0$ and $x+1 > u+v$, the inequality

$$1 \leq \frac{\Gamma_\lambda(x+1)\Gamma_\lambda(x+1-u-v)}{\Gamma_\lambda(x+1-u)\Gamma_\lambda(x+1-v)} \tag{39}$$

holds. And for $u > 0$, $v > 0$ and $x > u+v$, the inequality

$$\frac{\Gamma_\lambda(x+1)\Gamma_\lambda(x+1-u-v)}{\Gamma_\lambda(x+1-u)\Gamma_\lambda(x+1-v)} < \frac{(u+v)^{u+v}}{u^u v^v} \tag{40}$$

holds.

Proof. Taking the logarithmic derivative of $\mathcal{P}(x)$ yields

$$\begin{aligned}
\frac{\mathcal{P}'(x)}{\mathcal{P}(x)} &= \psi_\lambda(x+1) + \psi_\lambda(x+1-u-v) - \psi_\lambda(x+1-u) - \psi_\lambda(x+1-v) \\
&= \int_0^1 \frac{t^{x-u} + t^{x-v} - t^x - t^{x-u-v}}{1-t} dt \\
&= -\int_0^1 \frac{(t^u-1)(t^v-1)}{1-t} t^{x-u-v} dt \\
&\leq 0.
\end{aligned}$$

Hence $\mathcal{P}(x)$ is decreasing. Consequently, applying (37) gives

$$\begin{aligned}
\mathcal{P}(x) &\geq \lim_{x\to\infty} \mathcal{P}(x) \\
&= \lim_{x\to\infty} \frac{\Gamma_\lambda(x+1)\Gamma_\lambda(x+1-u-v)}{\Gamma_\lambda(x+1-u)\Gamma_\lambda(x+1-v)} \\
&= \lim_{x\to\infty} x^{-u} \frac{\Gamma_\lambda(x+1)}{\Gamma_\lambda(x+1-u)} \cdot \lim_{x\to\infty} x^u \frac{\Gamma_\lambda(x+1-u-v)}{\Gamma_\lambda(x+1-v)} \\
&= \lambda^{-u}\lambda^u \\
&= 1
\end{aligned}$$

which yields (39). Next, it is known that [9],

$$\beta(u,v) \geq \frac{u^{u-1}v^{v-1}}{(u+v)^{u+v-1}} \tag{41}$$

for $u > 0$ and $v > 0$. This together with the monotonicity of $\mathcal{P}(x)$ implies that

$$\begin{aligned}\mathcal{P}(x) < \mathcal{P}(u+v) &= \frac{\Gamma_\lambda(u+v+1)\Gamma_\lambda(1)}{\Gamma_\lambda(u+1)\Gamma_\lambda(v+1)} \\ &= \frac{u+v}{uv}\frac{\Gamma_\lambda(u+v)}{\Gamma_\lambda(u)\Gamma_\lambda(v)} \\ &= \frac{u+v}{uv}\frac{1}{\beta_\lambda(u,v)} \\ &\leq \frac{(u+v)^{u+v}}{u^u v^v}\end{aligned}$$

which yields (40). This completes the proof.

Remark 5. In particular, if $u = v = 1$ in (39), then we obtain

$$1 \leq \frac{\Gamma_\lambda(x+1)\Gamma_\lambda(x-1)}{\Gamma_\lambda(x)^2}, \quad x > 1, \tag{42}$$

which can be rearranged as

$$\frac{\lambda^2}{x^2} \leq \frac{\Gamma_\lambda(x-1)}{\Gamma_\lambda(x+1)}, \quad x > 1. \tag{43}$$

Also, if $u = v = \frac{1}{2}$ in (39), then we obtain

$$1 \leq \frac{\Gamma_\lambda(x+1)\Gamma_\lambda(x)}{\Gamma_\lambda(x+\frac{1}{2})^2}, \quad x > 0, \tag{44}$$

and this can also be rearranged as

$$\sqrt{\frac{\lambda}{x}} \leq \frac{\Gamma_\lambda(x)}{\Gamma_\lambda(x+\frac{1}{2})}, \quad x > 0. \tag{45}$$

Remark 6. The logarithmic convexity of the λ-gamma function implies that

$$\Gamma_\lambda(kz_1 + (1-k)z_2) \leq [\Gamma_\lambda(z_1)]^k [\Gamma_\lambda(z_2)]^{1-k} \tag{46}$$

for $z_1 > 0$, $z_2 > 0$ and $k \in [0,1]$. It is interesting to note that if $z_1 = x+1$, $z_2 = x-1$ and $k = \frac{1}{2}$, then we recover (42). Likewise, if $z_1 = x$, $z_2 = x+1$ and $k = \frac{1}{2}$, then we recover (44). More generally, if $u \geq 0$, $z_1 = x+u$, $z_2 = x-u$ and $k = \frac{1}{2}$, then we obtain

$$1 \leq \frac{\Gamma_\lambda(x+u)\Gamma_\lambda(x-u)}{\Gamma_\lambda(x)^2}, \quad u \geq 0,\ x > u, \tag{47}$$

which is equivalent to

$$1 \leq \frac{\Gamma_\lambda(x+1+u)\Gamma_\lambda(x+1-u)}{\Gamma_\lambda(x+1)^2}, \quad u \geq 0,\ x+1 > u. \tag{48}$$

Theorem 4. *Let $u \geq 0$ and $x+1 > u$. Then the function*

$$\mathcal{A}(x) = \frac{\Gamma_\lambda(x+1+u)\Gamma_\lambda(x+1-u)}{\Gamma_\lambda(x+1)^2} \tag{49}$$

is decreasing. Consequently, the inequality

$$\frac{x^2}{x^2-u^2} \leq \frac{\Gamma_\lambda(x+u)\Gamma_\lambda(x-u)}{\Gamma_\lambda(x)^2} \leq \frac{x^2}{x^2-u^2}.\frac{\pi u}{\sin(\pi u)} \tag{50}$$

holds for $x > u$ and $u \in [0,1)$.

Proof. Logarithmic differentiation yields

$$\begin{aligned}
\frac{\mathcal{A}'(x)}{\mathcal{A}(x)} &= \psi_\lambda(x+1+u) + \psi_\lambda(x+1-u) - 2\psi_\lambda(x+1) \\
&= \int_0^1 \frac{2t^x - t^{x+u} - t^{x-u}}{1-t} dt \\
&= -\int_0^1 \frac{(t^u-1)^2}{1-t} t^{x-u} dt \\
&\leq 0.
\end{aligned}$$

Thus, $\mathcal{A}(x)$ is decreasing. Using (48) in conjunction with the monotonicity of $\mathcal{A}(x)$ and identity (11), we obtain

$$\begin{aligned}
1 \leq \frac{\Gamma_\lambda(x+1+u)\Gamma_\lambda(x+1-u)}{\Gamma_\lambda(x+1)^2} &\leq \frac{\Gamma_\lambda(1+u)\Gamma_\lambda(1-u)}{\left(\frac{1}{\lambda}\right)^2} \\
&\leq \frac{\pi u}{\sin(\pi u)}.
\end{aligned}$$

By applying (8), we arrive at the desired result (50) and this completes the proof.

Remark 7. The results of Theorem 3 and Theorem 4 are motivated by the paper [2]. For further information on inequalities of these types, one may refer to that paper.

Theorem 5. *The function*

$$\mathcal{K}(x) = \Gamma_\lambda(x) - \frac{1}{x} \tag{51}$$

is increasing for $x > 0$ and the inequality

$$\frac{1}{x} - (\ln\lambda + \gamma) < \Gamma_\lambda(x) \leq \frac{1}{x} + \frac{1}{\lambda} - 1 \tag{52}$$

holds for $x \in (0,1]$.

Proof. By using (8), we have

$$\mathcal{K}(x) = \frac{\lambda\Gamma_\lambda(x+1) - 1}{x}$$

which implies that

$$\begin{aligned} x^2\mathcal{K}'(x) &= \lambda x\Gamma'_\lambda(x+1) - \lambda\Gamma_\lambda(x+1) + 1 \\ &= \theta(x). \end{aligned}$$

Then

$$\theta'(x) = \lambda x\Gamma''_\lambda(x+1) > 0$$

which shows that $\theta(x)$ is increasing and so

$$\theta(x) > \theta(0) = 0.$$

Hence $\mathcal{K}(x)$ is increasing. Then for $x \in (0,1]$, we have

$$-(\ln\lambda + \gamma) = \lim_{x\to 0}\mathcal{K}(x) < \mathcal{K}(x) \le \mathcal{K}(1) = \frac{1}{\lambda} - 1$$

which gives (52).

Remark 8. The left hand side of (52) actually holds for all $x > 0$ and for $x > 1$, the right hand side reverses with strict inequality.

Lemma 1 ([8])**.** *For $x > 0$, the inequality $\Gamma(x)\Gamma(1/x) \ge 1$ holds.*

Theorem 6. *For $x > 0$, the inequalities*

$$\Gamma_\lambda(x)\Gamma_\lambda(1/x) \ge \lambda^{-(x+\frac{1}{x})}, \tag{53}$$

$$\Gamma_\lambda(x) + \Gamma_\lambda(1/x) \ge 2\lambda^{-\frac{1}{2}(x+\frac{1}{x})} \tag{54}$$

are satisfied. With equality when $x = 1$.

Proof. By applying definition (2) and Lemma 1, we have

$$\begin{aligned} \Gamma_\lambda(x)\Gamma_\lambda(1/x) &= \lambda^{-x}\Gamma(x)\lambda^{-\frac{1}{x}}\Gamma(1/x) \\ &= \lambda^{-(x+\frac{1}{x})}\Gamma(x)\Gamma(1/x) \\ &\ge \lambda^{-(x+\frac{1}{x})} \end{aligned}$$

which gives (53). By applying the arithmetic-geometric mean inequality in conjunction with (53), we obtain

$$\begin{aligned} \Gamma_\lambda(x) + \Gamma_\lambda(1/x) &\ge 2\sqrt{\Gamma_\lambda(x)\Gamma_\lambda(1/x)} \\ &\ge 2\lambda^{-\frac{1}{2}(x+\frac{1}{x})} \end{aligned}$$

which gives (54). This completes the proof.

We conclude the paper by proving the Bohr-Mollerup theorem for the λ-gamma function which characterizes the function uniquely.

Theorem 7. *Let $x > 0$ and $\lambda > 0$. If $Q : (0, \infty) \to (0, \infty)$ satisfies the conditions*

(a) $Q(1) = \frac{1}{\lambda}$,

(b) $Q(x+1) = \frac{x}{\lambda} Q(x)$,

(c) $Q(x)$ *is logarithmically convex,*

then $Q(x) = \Gamma_\lambda(x)$.

Proof. Let $Q(x)$ satisfy conditions (a)–(c). We prove that $Q(x)$ coincides with $\Gamma_\lambda(x)$ on the interval $(0, 1]$. When this happens, then $Q(x)$ will coincide with $\Gamma_\lambda(x)$ throughout its domain as a result of condition (b). Let $x \in (0, 1]$ and $k \in \mathbb{N}$ such that $k \geq 2$. Then $k - 1 < k < k + x \leq k + 1$ and by virtue of the logarithmic convexity of $Q(x)$, we obtain

$$\frac{\ln Q(k-1) - \ln Q(k)}{(k-1) - k} \leq \frac{\ln Q(k+x) - \ln Q(k)}{(k+x) - k} \leq \frac{\ln Q(k+1) - \ln Q(k)}{(k+1) - k}.$$

That is

$$-\ln \frac{Q(k-1)}{Q(k)} \leq \frac{1}{x} \ln \frac{Q(k+x)}{Q(k)} \leq \ln \frac{Q(k+1)}{Q(k)}. \tag{55}$$

But, in view of conditions (a) and (b), we have

$$Q(k+1) = \frac{k!}{\lambda^{k+1}}, \quad Q(k) = \frac{(k-1)!}{\lambda^k} \quad \text{and} \quad Q(k-1) = \frac{(k-2)!}{\lambda^{k-1}}.$$

Also, repeated application of condition (b) yields

$$Q(k+x) = \frac{x(x+1)(x+2)\dots(x+k-1)}{\lambda^k} Q(x).$$

Then (55) becomes

$$x \ln \left(\frac{k-1}{\lambda} \right) \leq \ln \left[\frac{x(x+1)(x+2)\dots(x+k-1)Q(x)}{(k-1)!} \right] \leq x \ln \left(\frac{k}{\lambda} \right)$$

which implies that

$$\frac{(k-1)!\lambda^{-x}(k-1)^x}{x(x+1)(x+2)\dots(x+k-1)} \leq Q(x) \leq \frac{(k-1)!\lambda^{-x}k^x}{x(x+1)(x+2)\dots(x+k-1)}. \tag{56}$$

Substituting k by $k+1$ at the left hand side of (56) and rearranging the right hand side yields

$$\frac{k!\lambda^{-x}k^x}{x(x+1)(x+2)\dots(x+k)} \leq Q(x) \leq \frac{k!\lambda^{-x}k^x}{x(x+1)(x+2)\dots(x+k)} \cdot \frac{x+k}{k} \tag{57}$$

Now, applying Squeeze's theorem on (57) yields

$$Q(x) = \lim_{k\to\infty} \frac{k!\lambda^{-x}k^x}{x(x+1)(x+2)\ldots(x+k)} = \Gamma_\lambda(x)$$

which completes the proof.

References

1. *Abramowitz M., Stegun I.A.*: Handbook of Mathematical Functions with formulas, Graphic and Mathematical Tables. Dover Publications, Inc., New York, 1965.
2. *Al-Jararha M.M., Al-Jararha J.M.*: Inequalities of Gamma Function Appearing in Generalizing Probability Sampling Designs. Aust. J. Math. Anal. Appl. 2020; 17(2) Art 17: pp. 9.
3. *Chaudhry M.A., Zubair S.M.*: Generalized incomplete gamma functions with applications. J. Comput. Appl. Math. 1994; 55: pp. 99–124.
4. *Díaz R., Pariguan E.*: On hypergeometric functions and Pochhammer k-symbol. Divulg. Mat. 2007; 15: pp. 179–192.
5. *Díaz R., Teruel C.*: q, k-generalized gamma and beta functions. J. Nonlinear Math. Phys. 2005; 12(1): pp. 118–134.
6. *Djabang E., Nantomah K., Iddrisu M.M.*: On a v-analogue of the Gamma function and some associated inequalities. J. Math. Comput. Sci. 2020; 11(1): pp. 74–86.
7. *Fu H., Peng Y., Du T.*: Some inequalities for multiplicative tempered fractional integrals involving the λ-incomplete gamma functions. AIMS Mathematics 2021; 6(7): pp. 7456–7478.
8. *Gautschi W.*: A harmonic mean inequality for the gamma function. SIAM J. Math. Anal. 1974; 5: pp. 278–281.
9. *Grenie L., Molteni G.*: Inequalities for the Beta function. Math. Inequal. Appl. 2015; 18(4): pp. 1427–1442.
10. *Krasniqi V., Merovci F.*: Some Completely Monotonic Properties for the (p, q)-Gamma Function. Math. Balkanica (N.S.) 2012; 26: pp. 1–2.
11. *Loc T.G., Tai T.*: The Generalized Gamma Functions. Acta Math. Vietnam. 2012; 37(2): pp. 219–230.
12. *Mohammed P.O., Sarikaya M.Z., Baleanu D.*: On the Generalized Hermite-Hadamard Inequalities via the Tempered Fractional Integrals. Symmetry 2020; 12(4) 595: pp. 1–17.
13. *Nantomah K., Prempeh E., Twum S.B.*: On a (p, k)-analogue of the Gamma function and some associated Inequalities. Moroccan J. of Pure and Appl. Anal. 2016; 2(2): pp. 79–90.
14. *Qi F.*: Bounds for the Ratio of Two Gamma Functions. J. Inequal. Appl., 2010.
15. *Sandor J.*: On certain inequalities for the Gamma function. RGMIA Res. Rep. Coll. 2006; 9(1) Art 11.

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