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On the algebra of derivations of some nilpotent Leibniz algebras

Abstract. We describe the algebra of derivations of some nilpotent Leibniz algebra, having dimensionality 3.
Key words: Leibniz algebra, dimensionality, derivation

Анотація. Описано алгебру диференціювань деякої нільпотентної алгебри Лейбніца вимірності 3.
Ключові слова: алгебра Лейбніца, вимірність, диференціювання

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1. Introduction

Let L be an algebra over a field F with the binary operations $+$ and $[\cdot,\cdot]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]]$$

for all $a, b, c \in L$. We will also use another form of this identity:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

Leibniz algebras appeared first in the paper of A. Blokh [2], but the term "Leibniz algebra" appears in the book of J.-L. Loday [11], and the article of J.-L. Loday [12]. In [13] J.-L. Loday and T. Pirashvili began the real study of the properties of Leibniz algebras. The theory of Leibniz algebras has developed very intensively in many different directions. Some of the results of this theory were presented in the book [1].

As in Lie algebras, the structure of Leibniz algebras is greatly influenced by their algebras of derivations.

Denote by $\mathbf{End}_{[,]}(L)$ the set of all linear transformations of L, then L is an associative algebra by the operation $+$ and $\circ$. As usual, $\mathbf{End}_{[,]}(L)$ is a Lie algebra by the operations $+$ and $[,]$, where $[f,g] = f \circ g - g \circ f$ for all $f, g \in \mathbf{End}_{[,]}(L)$.

A linear transformation f of a Leibniz algebra L is called a *derivation*, if

$$f([a,b]) = [f(a),b] + [a,f(b)] \text{ for all } a,b \in L.$$

Let $\mathbf{Der}_{[,]}(L)$ be the subset of all derivations of L. It is possible to prove that $\mathbf{Der}_{[,]}(L)$ is a subalgebra of a Lie algebra $\mathbf{End}_{[,]}(L)$. $\mathbf{Der}_{[,]}(L)$ is called the *algebra of derivations* of a Leibniz algebra L.

The following result shows the influence on the structure of the Leibniz algebra of their algebras of derivations: if A is an ideal of a Leibniz algebra, then the factor-algebra of L by the annihilator of A is isomorphic to some subalgebra of $\mathbf{Der}(A)$ [4, Proposition 3.2]. Therefore, elucidating the structure of algebras of derivations of Leibniz algebras is one of the important tasks of this theory. It should be noted that the automorphism groups of Leibniz algebras have been studied very little. It is natural to start studying the algebra of derivations of a Leibniz algebra, the structure of which has been studied quite extensively. A description of the structure of the algebra of derivations of finite dimensional one-generator Leibniz algebras was given in paper [8, 14] and infinite dimensional one-generator Leibniz algebras was given in papers [10]. The question naturally arises about the algebra of derivations of a Leibniz algebra of small dimension. In contrast to Lie algebras, the situation with Leibniz algebras of dimension 3 is very diverse [3, 6, 7]. Leibniz algebras of dimension 3 are mainly described, their most detailed description can be found in the work [5], the description of Leibniz algebras of dimension 4, 5 is carried out in a rather intensive way. The list of papers devoted to these studies is quite large and we will not give it in full here. The description of the algebra of derivations of Leibniz algebras of dimension 3 was started in [9]. This description is quite extensive, describing the algebra of derivations of two types of nilpotent Leibniz algebras of dimension 3.

First type of nilpotent Leibniz algebras, whose dimension is 3, is the nilpotent Leibniz algebras, whose nilpotency class is 3. There exists only one type of such algebras, it is the following algebra:

$$L_1 = \mathbf{Lei}_1(3,F) = Fa_1 \oplus Fa_2 \oplus Fa_3$$

where

$$[a_1,a_1] = a_2, [a_1,a_2] = a_3, [a_1,a_3] = 0,$$
$$[a_2,a_1] = [a_3,a_1] = [a_2,a_2] = [a_2,a_3] = [a_3,a_3] = 0.$$

It is one-generator Leibniz algebra,

$$\mathbf{Leib}(L_1) = \zeta^{\text{left}}(L_1) = [L_1,L_1] = Fa_2 \oplus Fa_2,$$
$$\zeta^{\text{right}}(L_1) = \zeta(L_1) = \gamma_3(L_1) = Fa_3.$$

Let now L be a nilpotent Leibniz algebra, whose nilpotency class is 2. Of course we suppose that L is not a Lie algebra. Then the center $\zeta(L)$ has dimension 2 or 1. Suppose that $\mathbf{dim}_F(\zeta(L)) = 2$. Since L is not a Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. Since a Leibniz algebra of dimension 1 is abelian, $a_3 \in \zeta(L)$. It follows that $[a_1, a_3] = [a_3, a_1] = [a_3, a_3] = 0$. Being an abelian algebra of dimension 2, $\zeta(L)$ has a direct decomposition $\zeta(L) = Fa_2 \oplus Fa_3$ for some element a_3, thus we come to the following type of nilpotent Leibniz algebra:

$$L_2 = \mathbf{Lei}_2(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3$$

where

$$[a_1, a_1] = a_3, [a_1, a_2] = [a_3, a_1] = [a_3, a_3] = 0, [a_2, a_2] = 0,$$
$$[a_2, a_1] = [a_3, a_2] = [a_2, a_3] = [a_1, a_2] = 0.$$

In other words, L_2 is a direct sum of two ideals $A = Fa_1 \oplus Fa_3$ and $B = Fa_2$, moreover, A is a nilpotent cyclic Leibniz algebra of dimension 2, $\mathbf{Leib}(L_2) = [L_2, L_2] = Fa_3, \zeta^{\text{left}}(L_2) = \zeta^{\text{right}}(L_2) = \zeta(L_2) = Fa_2 \oplus Fa_3$.

We note that the structure of the automorphism groups of Leibniz algebras $\mathbf{Lei}_1(3, F)$ and $\mathbf{Lei}_2(3, F)$ has been described in a paper [9].

Suppose now that L is a nilpotent Leibniz algebra, whose nilpotency class is 2 and the center of L has dimension 1. Since L is not a Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. Since a factor-algebra $L/\zeta(L)$ is abelian, $a_3 \in \zeta(L)$. It follows that $[a_1, a_3] = [a_3, a_1] = [a_3, a_3] = 0$. Then $\zeta(L) = Fa_3$. For every element $x \in L$ we have $[a_1, x], [x, a_1] \in \zeta(L) \leq \langle a_1 \rangle = Fa_1 \oplus Fa_3$. It follows that a subalgebra $\langle a_1 \rangle$ is an ideal of L. Since $\mathbf{dim}_F(\langle a_1 \rangle) = 2$, $\langle a_1 \rangle \neq L$. Choose an element b such that $b \notin \langle a_1 \rangle$. We have $[b, a_1] = \gamma a_3$ for some $\gamma \in F$. If $\gamma \neq 0$, then put $b_1 = \gamma^{-1} b - a_1$. Then $[b_1, a_1] = 0$. The choice of b_1 shows that $b_1 \notin \langle a_1 \rangle$. If follows that a subalgebra $\mathbf{Ann}^{\text{left}}_{[,]}(a_1)$ has dimension 2. Suppose first that $\mathbf{Ann}^{\text{left}}_{[,]}(a_1)$ is an abelian subalgebra, then it has a direct decomposition $\mathbf{Ann}^{\text{left}}_{[,]}(a_1) = Fa_2 \oplus Fb_2$ for some element b_2 such that $[b_2, b_2] = 0$. Since $\mathbf{dim}_F(\zeta(L)) = 1, b_2 \notin \zeta(L)$. Then $[a_1, b_2] = \lambda a_3$ where $0 \neq \lambda \in F$. Put now $a_2 = \lambda^{-1} b_2$. Thus we come to the following type of nilpotent Leibniz algebra:

$$L_3 = \mathbf{Lei}_3(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where}$$
$$[a_1, a_1] = [a_1, a_2] = a_3, [a_1, a_3] = [a_3, a_1] = [a_3, a_3] = 0,$$
$$[a_2, a_1] = [a_2, a_2] = 0, [a_3, a_2] = [a_2, a_3] = 0.$$

In other words, L_3 is a direct sum of ideals $A = Fa_1 \oplus Fa_3$ and a subalgebra $B = Fa_2$, moreover, A is a nilpotent cyclic Leibniz algebra of dimension 2, $\mathbf{Leib}(L_3) = [L_3, L_3] = Fa_3$, $\zeta^{\text{left}}(L_3) = Fa_2 \oplus Fa_3$, $\zeta^{\text{right}}(L_3) = \zeta(L_3) = Fa_3$.

The description of this type of nilpotent Leibniz algebras is the subject of this paper.

2. An algebra of derivations of the Leibniz algebra $\mathbf{Lei}_3(3, F)$

Let's start with some general properties of the algebra of derivations of Leibniz algebras. We show here some basic elementary properties of derivations, which have been proved in a paper [10]. We recall some definitions.

Every Leibniz algebra L has one specific ideal. Denote by $\mathbf{Leib}(L)$ the subspace, generated by the elements $[a, a], a \in L$. It is possible to prove that $\mathbf{Leib}(L)$ is an ideal of L. The ideal $\mathbf{Leib}(L)$ is called the *Leibniz kernel* of algebra L. By its definition, a factor-algebra $L/\mathbf{Leib}(L)$ is a Lie algebra. And conversely, if K is an ideal of L such that L/K is a Lie algebra, then K includes a Leibniz kernel.

Let L be a Leibniz algebra. Define the lower central series of L

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \dots \gamma_\alpha(L) \geqslant \gamma_{\alpha+1} \geqslant \dots \gamma_\delta(L) = \gamma_\infty(L)$$

by the following rule: $\gamma_1(L) = L, \gamma_2(L) = [L, L]$, and recursively $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals a and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ. The last term $\gamma_\delta(L) = \gamma_\infty(L)$ is called the *lower hypocenter* of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

As usually, we say that a Leibniz algebra L is called *nilpotent*, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class c* if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$.

The *left* (respectively *right*) *center* $\zeta^{\text{left}}(L)$ (respectively $\zeta^{\text{right}}(L)$) of a Leibniz algebra L is defined by the rule:

$$\zeta^{\text{left}}(L) = \{x \in L \mid [x, y] = 0 \text{ for each element } y \in L\}$$
$$(\text{respectively, } \zeta^{\text{right}}(L) = \{x \in L \mid [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but it is not true for the right center. Moreover, $\mathbf{Leib}(L) \leq \zeta^{\text{left}}(L)$, so that $L/\zeta^{\text{left}}(L)$ is a Lie algebra. The right center is an subalgebra of L, and, in general, the left and right centers are different; they even may have different dimensions (see [4]).

The *center* $\zeta(L)$ of L is defined by the rule:

$$\zeta(L) = \{x \in L \mid [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L.

Starting of the center we can construct the upper central series

$$\langle 0 \rangle = \zeta_0(L) \leq \zeta_1(L) \leq \zeta_2(L) \leq \dots \zeta_\alpha(L) \leq \zeta_{\alpha+1}(L) \leq \dots \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) = \bigcup_{\mu<\lambda} \zeta_\mu(L)$ for the limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\zeta_\infty(L)$ of this series is called the *upper hypercenter* of L. If $L = \zeta_\infty(L)$ then L is called a *hypercentral* Leibniz algebra.

Lemma 1. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\zeta^{\text{left}}(L)) \le \zeta^{\text{left}}(L)$, $f(\zeta^{\text{right}}(L)) \le \zeta^{\text{right}}(L)$ and $f(\zeta(L)) \le \zeta(L)$.*

Corollary 1. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\zeta_\alpha(L)) \le \zeta_\alpha(L)$ for every ordinal α.*

Theorem 1. *Let $\mathbf{D}$ be an algebra of derivations of the Leibniz algebra $\mathbf{Lei}_3(3, F)$. Then the following assertions hold:*

1) $\mathbf{D}$ is a semidirect sum of an ideal $\mathbf{A}$ and a subalgebra of dimension 1, generated by derivation f_1 such that $f_1(a_1) = a_1, f_1(a_2) = a_2, f_1(a_3) = 2a_3$;

2) $\mathbf{A}$ is a semidirect sum of an ideal $\mathbf{N}$ of $\mathbf{D}$ and a subalgebra of dimension 1, generated by derivation f_2 such that $f_2(a_1) = a_2$, $f_2(a_2) = a_2$, $f_2(a_3) = a_3$;

3) an ideal $\mathbf{N}$ is abelian, $\mathbf{N} = Ff_3 \oplus Ff_4$, where

$$f_3(a_1) = a_3, f_3(a_2) = 0, f_3(a_3) = 0, f_4(a_1) = 0, f_4(a_2) = a_3, f_4(a_3) = 0.$$

Moreover, $[f_1, f_4] = f_4$, $[f_1, f_3] = f_3$, $[f_1, f_2] = 0$, $[f_2, f_4] = -f_3$, $[f_2, f_3] = f_3$.

4) An algebra $\mathbf{D}$ is isomorphic to a Lie subalgebra of matrices, having the following form

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_1 + \alpha_2 & 0 \\ \alpha_3 & \beta_3 & 2\alpha_1 + \alpha_2 \end{pmatrix}, \text{ where } \alpha_1, \alpha_2, \alpha_3, \beta_3 \in F.$$

Proof of Theorem 1. Let $L = \mathbf{Lei}_1(3, F)$ and let $f \in \mathbf{Der}(L)$. We have $f(a_1) = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$, then

$$\begin{aligned} f(a_3) = f([a_1, a_1]) = [f(a_1), a_1] + [a_1, f(a_1)] = \\ = [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_1] + [a_1, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] = \\ = \alpha_1[a_1, a_1] + \alpha_2[a_1, a_2] + \alpha_3[a_1, a_3] = (2\alpha_1 + \alpha_2)a_3. \end{aligned}$$

By **Lemma 1** $f(\zeta^{\text{left}}(L)) \le \zeta^{\text{left}}(L)$. It follows that $f(a_2) = \beta_2 a_2 + \beta_3 a_3$. Then

$$\begin{aligned} f(a_3) = f([a_1, a_2]) = [f(a_1), a_2] + [a_1, f(a_2)] = \\ = [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_2] + [a_1, \beta_2 a_2 + \beta_3 a_3] = \\ = \alpha_1[a_1, a_3] + \alpha_2[a_2, a_2] + \alpha_3[a_3, a_2] + \beta_2[a_1, a_2] + \beta_3[a_1, a_3] = \\ = \alpha_1 a_3 + \beta_2 a_3 = (\alpha_1 + \beta_2)a_3. \end{aligned}$$

It follows that $(2\alpha_1 + \alpha_2)a_3 = (\alpha_1 + \beta_2)a_3$ and $2\alpha_1 + \alpha_2 = \alpha_1 + \beta_2$. It follows that $\beta_2 = \alpha_1 + \alpha_2$.

Thus a derivation f has in a basis $\{a_1, a_2, a_3\}$ the following matrice

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_1 + \alpha_2 & 0 \\ \alpha_3 & \beta_3 & 2\alpha_1 + \alpha_2 \end{pmatrix}.$$

Put $\mathbf{N} = \{f \mid f \in \mathbf{Der}(L) \text{ and } f(a_1), f(a_2) \in Fa_3 = \zeta(L)\}$. By **Lemma 1** $f(\zeta(L)) \leq \zeta(L)$ for every derivation f of L. Therefore it is not hard to see, that $\mathbf{N}$ is a subalgebra of $\mathbf{Der}(L)$. Moreover, $\mathbf{N}$ is an ideal of $\mathbf{Der}(L)$. Indeed, let $f \in \mathbf{N}$, $g \in \mathbf{Der}(L)$. Then

$$[g, f](a_1) = (g \circ f - f \circ g)(a_1) = (g \circ f)(a_1) - (f \circ g)(a_1) = g(f(a_1)) - f(g(a_1));$$
$$[g, f](a_2) = (g \circ f - f \circ g)(a_2) = (g \circ f)(a_2) - (f \circ g)(a_2) = g(f(a_2)) - f(g(a_2)).$$

We have $g(a_1) = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3$, $g(a_2) = (2\beta_1 + \beta_2)a_3$, then

$$f(g(a_1)) = f(\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3) = \beta_1 f(a_1) + \beta_2 f(a_2) + \beta_3 f(a_3).$$

Lemma 1 implies that $\beta_3 f(a_3) \in \zeta(L)$. By $f \in \mathbf{N}$, $f(a_1), f(a_2) \in \zeta(L)$, so again, leveraging **Lemma 1** we obtain that $f(g(a_1) \in \zeta(L)$ and $f(g(a_2) \in \zeta(L)$. **Lemma 1** and the fact that $f(a_1), f(a_2) \in \zeta(L)$ imply that $g(f(a_1))g(f(a_2)) \in \zeta(L) \in \zeta^{\text{left}}(L)$. Hence $[g, f](a_1), [g, f](a_2) \in \zeta(L)$. This means that $\mathbf{N}$ is an ideal of D.

Denote by Ξ the canonical monomorphism of $\mathbf{End}_{[,]}(L)$ in $\mathbf{M}_3(F)$ (i.e. $\Xi(f)$ is a matrice of f in the basis $\{a_1, a_2, a_3\}$). Then $\Xi(\mathbf{N})$ is a subset $\mathbf{M}_3(F)$, consisting of the matrices, having a following form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_3 & \beta_3 & 0 \end{pmatrix}.$$

Define a linear transformation f_3 of L by the following rule:

$$\text{if } x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3 \text{ then } f_3(x) = \xi_1 a_3.$$

Let x, y be the arbitrary elements of L, $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, $y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$, where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in F$. Then

$$\begin{gathered} [x, y] = [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] = \\ = \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] + \xi_1\eta_3[a_1, a_3] + \\ \xi_2\eta_1[a_2, a_1] + \xi_2\eta_2[a_2, a_2] + \xi_2\eta_3[a_2, a_3] + \\ +\xi_3\eta_1[a_3, a_1] + \xi_3\eta_2[a_3, a_2] + \xi_3\eta_3[a_3, a_3] = \\ \xi_1\eta_1 a_3 + \xi_1\eta_2 a_3 = \xi_1(\eta_1 + \eta_2)a_3, \\ [f_3(x), y] = [\xi_1 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] = 0, \\ [x, f_3(y)] = [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_3] = 0, \\ f_3([x, y]) = f_3(\xi_1(\eta_1 + \eta_2)a_3) = 0. \end{gathered}$$

In particular, $f_3([x, y]) = [f_3(x), y] + [x, f_3(y)]$, so that f_3 is a derivation of L. By its definition, $f_3 \in \mathbf{N}$.

Define a linear transformation f_4 of L by the following rule:

$$\text{if } x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3 \text{ then } f_4(x) = \xi_2 a_3.$$

Let x, y be arbitrary elements of L, $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, $y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$, where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in F$. Then

$$\begin{aligned} [x, y] &= \xi_1(\eta_1 + \eta_2) a_3, \\ [f_4(x), y] &= [\xi_2 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] = 0, \\ [x, f_4(y)] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_2 a_3] = 0, \\ f_4([x, y]) &= f_4(\xi_1(\eta_1 + \eta_2) a_3) = 0. \end{aligned}$$

In particular, $f_4([x, y]) = [f_4(x), y] + [x, f_4(y)]$, so that f_4 is a derivation of L. By its definition, $f_4 \in \mathbf{N}$. It is not hard to see that the derivations f_3, f_4 are linearly independent. Using a monomorphism Ξ we can see that $\mathbf{N}$ has dimension 2. Hence $\mathbf{N} = F f_3 \oplus F f_4$. Furthermore,

$$\begin{aligned} (f_3 \circ f_4)(x) &= f_3(f_4(x)) = f_3(\xi_2 a_3) = 0, \\ (f_4 \circ f_3)(x) &= f_4(f_3(x)) = f_4(\xi_1 a_3) = 0. \end{aligned}$$

It follows that an ideal $\mathbf{N}$ is abelian.

Let's put $\mathbf{A} = \{f \mid f \in \mathbf{Der}_{[,]}(L) \text{ and } f(a_1) \in F a_2 \oplus F a_2 = \zeta^{\text{left}}(L)\}$. By **Lemma 1** $f(\zeta^{\text{left}}(L)) \leq \zeta^{\text{left}}(L)$ for every derivation f of L. Therefore it is not hard to see, that $\mathbf{A}$ is a subalgebra of $\mathbf{Der}_{[,]}(L)$. Moreover, $\mathbf{A}$ is an ideal of $\mathbf{Der}_{[,]}(L)$. Indeed, let $f \in \mathbf{A}$, $g \in \mathbf{Der}(L)$. Then

$$[g, f](a_1) = (g \circ f - f \circ g)(a_1) = (g \circ f)(a_1) - (f \circ g)(a_1) = g(f(a_1)) - f(g(a_1)).$$

We have $g(a_1) = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3$, then

$$f(g(a_1)) = f(\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3) = \beta_1 f(a_1) + f(\beta_2 a_2 + \beta_3 a_3).$$

Lemma 1 implies that $f(\beta_2 a_2 + \beta_3 a_3) \in \zeta^{\text{left}}(L)$. By $f \in \mathbf{A}$, $f(a_1) \in \zeta^{\text{left}}(L)$, so again, leveraging **Lemma 1** we obtain that $f(g(a_1)) \in \zeta^{\text{left}}(L)$. **Lemma 1** and the fact that $f(a_1) \in \zeta^{\text{left}}(L)$ imply that $g(f(a_1)) \in \zeta^{\text{left}}(L)$. Hence $[g, f](a_1) \in \zeta^{\text{left}}(L)$. This means that $\mathbf{A}$ is an ideal of D. We can see that $\Xi(\mathbf{A})$ is a subset $\mathbf{M}_3(F)$, consisting of the matrices, having a following form

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \alpha_2 & 0 \\ \alpha_3 & \beta_3 & \alpha_2 \end{pmatrix}.$$

Define a linear transformation f_2 of L by the following rule:

$$\text{if } x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3 \text{ then } f_2(x) = (\xi_1 + \xi_2) a_2 + \xi_3 a_3.$$

Let x, y be the arbitrary elements of L, $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, $y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$, where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in F$. As we have proved above

$$[x, y] = \xi_1(\eta_1 + \eta_2)a_3.$$

We have

$$\begin{gathered}[f_2(x), y] = [(\xi_1 + \xi_2)a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] = \\ = (\xi_1 + \xi_2)\eta_1[a_2, a_1] + (\xi_1 + \xi_2)\eta_2[a_2, a_2] + (\xi_1 + \xi_2)\eta_3[a_2, a_3] = 0, \\ [x, f_2(y)] = [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, (\eta_1 + \eta_2)a_2 + \eta_3 a_3] = \\ = \xi_1(\eta_1 + \eta_2)[a_1, a_2] + \xi_2(\eta_1 + \eta_2)[a_2, a_2] = \xi_1(\eta_1 + \eta_2)a_3, \\ f_2([x, y]) = f_2(\xi_1(\eta_1 + \eta_2)a_3) = \xi_1(\eta_1 + \eta_2)a_3.\end{gathered}$$

In particular, $f_2([x, y]) = [f_2(x), y] + [x, f_2(y)]$, so that f_2 is a derivation of L. By its definition, $f_2 \in \mathbf{A}$. It is not hard to see that the derivations f_2, f_4, f_2, f_3 are linearly independent. Using a monomorphism Ξ we can see that $\mathbf{A} = \mathbf{N} \oplus F f_3$. Furthermore,

$$\begin{gathered}(f_2 \circ f_4)(x) = f_2(f_4(x)) = f_2(\xi_2 a_3) = \xi_2 f_2(a_3) = \xi_2 a_3 = f_4(x), \\ (f_4 \circ f_2)(x) = f_4(f_2(x)) = f_4((\xi_1 + \xi_2)a_2 + \xi_3 a_3) = (\xi_1 + \xi_2)a_3 = \\ = \xi_1 a_3 + \xi_2 a_3 = f_3(x) + f_4(x) = (f_3 + f_4)(x); \\ (f_2 \circ f_3)(x) = f_2(f_3(x)) = f_2(\xi_1 a_3) = \xi_1 f_2(a_3) = \xi_1 a_3 = f_3(x), \\ (f_3 \circ f_2)(x) = f_3(f_2(x)) = f_3((\xi_1 + \xi_2)a_2 + \xi_3 a_3) = 0.\end{gathered}$$

Thus

$$\begin{gathered}[f_2, f_4] = f_2 \circ f_4 - f_4 \circ f_2 = f_4 - (f_3 + f_4) = -f_3, \\ [f_2, f_3] = f_2 \circ f_3 - f_3 \circ f_2 = f_3 - 0 = f_3.\end{gathered}$$

Finally, define a linear transformation f_1 of L by the following rule:

$$\text{if } x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3 \text{ then } f_1(x) = \xi_1 a_1 + \xi_2 a_2 + 2\xi_3 a_3.$$

Let x, y be the arbitrary elements of L, $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, $y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$, where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in F$. As we have proved above

$$[x, y] = \xi_1(\eta_1 + \eta_2)a_3.$$

We have

$$\begin{gathered}[f_1(x), y] = [\xi_1 a_1 + \xi_2 a_2 + 2\xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] = \\ = \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] = \xi_1\eta_1 a_3 + \xi_1\eta_2 a_3 = \xi_1(\eta_1 + \eta_2)a_3, \\ [x, f_1(y)] = [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + 2\eta_3 a_3] = \\ = \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] = \xi_1\eta_1 a_3 + \xi_1\eta_2 a_3 = \xi_1(\eta_1 + \eta_2)a_3, \\ f_1([x, y]) = f_1(\xi_1(\eta_1 + \eta_2)a_3) = 2\xi_1(\eta_1 + \eta_2)a_3.\end{gathered}$$

In particular, $f_1([x, y]) = [f_1(x), y] + [x, f_1(y)]$, so that f_1 is a derivation of L. By its definition, $f_2 \in \mathbf{A}$. Using a monomorphism Ξ we can see that the derivations f_1, f_4, f_1, f_3, f_1, f_2 are linearly independent and $\mathbf{D} = \mathbf{A} \oplus Ff_1$. Furthermore,

$$(f_1 \circ f_4)(x) = f_1(f_4(x)) = f_1(\xi_2 a_3) = 2\xi_2 a_3 = 2f_4(x),$$
$$(f_4 \circ f_1)(x) = f_4(f_1(x)) = f_4(\xi_1 a_1 + \xi_2 a_2 + 2\xi_3 a_3) = \xi_2 a_3 = f_4(x),$$
$$(f_1 \circ f_3)(x) = f_1(f_3(x)) = f_1(\xi_1 a_3) = 2\xi_1 a_3 = 2f_3(x),$$
$$(f_3 \circ f_1)(x) = f_3(f_1(x)) = f_3(\xi_1 a_1 + \xi_2 a_2 + 2\xi_3 a_3) = \xi_1 a_3 = f_3(x),$$
$$(f_1 \circ f_2)(x) = f_1(f_2(x)) = f_1((\xi_1 + \xi_2)a_2 + \xi_3 a_3) = (\xi_1 + \xi_2)a_2 + 2\xi_3 a_3,$$
$$(f_2 \circ f_1)(x) = f_2(f_1(x)) = f_2(\xi_1 a_1 + \xi_2 a_2 + 2\xi_3 a_3) = (\xi_1 + \xi_2)a_2 + 2\xi_3 a_3.$$

Thus

$$[f_1, f_4] = f_1 \circ f_4 - f_4 \circ f_1 = 2f_4 - f_4 = f_4,$$
$$[f_1, f_3] = f_1 \circ f_3 - f_3 \circ f_1 = 2f_3 - f_3 = f_3,$$
$$[f_1, f_2] = f_1 \circ f_2 - f_2 \circ f_1 = 0.$$

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