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O. O. Bezushchak
Taras Shevchenko National University of Kyiv,
Kyiv 01033. *E-mail: bezushchak@knu.ua*

Derivations of rings of infinite matrices

Abstract. We describe derivations of several important associative and Lie rings of infinite matrices over general rings of coefficients.
Key words: ring of infinite matrices, derivation

Анотація. У роботі описуються диференціювання деяких важливих класів асоціативних та лієвих кілець нескінченних матриць над загальними кільцями коефіцієнтів.
Ключові слова: кільце нескінченних матриць, диференціювання

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1. Introduction

Let R be an associative ring with unit 1_R, and let I be an infinite set.

Consider the ring $M(I,R)$ of $(I\times I)$-matrices over the ring R having finitely many nonzero entries in each column. The ring $M(I,R)$ is isomorphic to the ring of endomorphisms of a free R^{op}-module of rank $\operatorname{card}(I)$. Here, R^{op} is a ring that is anti-isomorphic to R; and $\operatorname{card}(I)$ is the cardinality of the set I.

Consider also the subring $M_\infty(I,R) < M(I,R)$ of all $(I\times I)$-matrices over R having finitely many nonzero entries, and the subring $M_{rcf}(I,R) < M(I,R)$ of all $(I\times I)$-matrices over R having finitely many nonzero entries in each row and in each column.

Recall that an additive mapping $d : R \to R$ is called a *derivation* if

$$d(ab) = d(a)b + ad(b) \quad \text{for arbitrary elements} \quad a,\ b\in R.$$

Let V be a bi-module over a ring R. An additive mapping $d : R \to V$ is called a *derivation* or a 1-*cocycle* if $d(ab) = d(a)b + ad(b)$ for arbitrary elements $a, b \in R$. For an element $v \in V$ the mapping $d_v : R \to V$, $d_v(a) = av - va$ is a derivation.

The purpose of this paper is to determine derivations of the rings $M_\infty(I,R)$, $M_{rcf}(I,R)$, $M(I,R)$. Recall that all derivations of a ring form a Lie ring; see [6].

Every derivation of the ring R gives rise to a derivation of the ring $M(I,R)$ that leaves $M_{rcf}(I,R)$ and $M_\infty(I,R)$ invariant. Hence, the Lie ring $\mathrm{Der}(R)$ lies in each of the Lie rings $\mathrm{Der}(M(I,R))$, $\mathrm{Der}(M_{rcf}(I,R))$, $\mathrm{Der}(M_\infty(I,R))$.

For an element $a \in M(I,R)$, let

$$\mathrm{ad}(a) : x \mapsto [a,x] = ax - xa$$

denote the *inner derivation*. Since $M_\infty(I,R)$ is a two-sided ideal in $M_{rcf}(I,R)$, it follows that $M_\infty(I,R)$ is invariant under any inner derivation $\mathrm{ad}(a)$, $a \in M_{rcf}(I,R)$.

Theorem 1.

(a) *An arbitrary derivation d of the ring $M_\infty(I,R)$ is of the type*

$$d = \mathrm{ad}(a) + u, \quad \textit{where} \quad a \in M_{rcf}(I,R), \quad u \in \mathrm{Der}(R);$$

(b) *an arbitrary derivation d of the ring $M(I,R)$ (resp. $M_{rcf}(I,R)$) is of the type*

$$d = \mathrm{ad}(a) + u,\ \textit{where } a \in M(I,R)\ (\textit{resp. } a \in M_{rcf}(I,R)),\ u \in \mathrm{Der}(R).$$

In [4], we proved Theorem 1 in the case when R is a field and the derivation d is linear. W. Hołubowski and S. Żurek proved Theorem 1 under the assumptions that (i) the set I is countable, (ii) the ring R is commutative, and (iii) the derivation d is R-linear; see [5].

An arbitrary associative ring R gives rise to the Lie ring

$$R^{(-)} = (R,\ [a,b] = ab - ba).$$

Using the proof of Herstein's Conjectures by K.I. Beidar, M. Brešar, M.A. Chebotar and W.S. Martindale (see [1, 2, 3]) and Theorem 1, we obtain descriptions of derivations of Lie rings

$$\mathfrak{gl}_{rcf}(I,R) = M_{rcf}(I,R)^{(-)}, \quad \mathfrak{gl}(I,R) = M(I,R)^{(-)},$$

$$\mathfrak{sl}_\infty(I,R) = [\mathfrak{gl}_\infty(I,R), \mathfrak{gl}_\infty(I,R)]$$

under the assumption that $\frac{1}{2} \in R$.

Theorem 2.

(a) *An arbitrary derivation d of the Lie ring $\mathfrak{sl}_\infty(I,R)$ is of the type*

$$d = \mathrm{ad}(a) + u, \quad \textit{where} \quad a \in \mathfrak{gl}_{rcf}(I,R), \quad u \in \mathrm{Der}(R);$$

(b) *an arbitrary derivation d of the Lie ring $\mathfrak{gl}(I,R)$ (resp. $\mathfrak{gl}_{rcf}(I,R)$) is of the type*

$$d = \mathrm{ad}(a) + u,\ \textit{where} \quad a \in \mathfrak{gl}(I,R)\ (\textit{resp. } a \in \mathfrak{gl}_{rcf}(I,R)),\ u \in \mathrm{Der}(R).$$

For an element $a \in R$ and indices $i, j \in I$ let $e_{ij}(a)$ denote the $(I \times I)$-matrices having the element a at the intersection of the i-th row and the j-th column, all other entries are equal to zero.

The following lemma is straightforward.

Lemma 1. *Let $A = (a_{ij})_{i,j \in I} \in M(I, R)$, $a_{ij} \in R$. Then*

$$A = e_{kk}(1_R)A + Ae_{kk}(1_R)$$

if and only if $a_{ij} = 0$ whenever $i \neq k$, $j \neq k$ or $i = j = k$.

Let $\mathbb{Z}$ be the ring of integers. Then $\mathbb{Z} \cdot 1_R$ is a subring of the ring R. The ring $M(I, R)$ is a bimodule over the ring $M_\infty(I, \mathbb{Z} \cdot 1_R)$.

Lemma 2. *For an arbitrary derivation $d : M_\infty(I, \mathbb{Z} \cdot 1_R) \to M(I, R)$ there exists a matrix $v \in M_\infty(I, R)$ such that*

$$\big(d - d_v\big)\big(e_{kk}(1_R)\big) = 0$$

for an arbitrary index $k \in I$.

Proof. Let

$$d\big(e_{kk}(1_R)\big) = (a_{ij}^{(k)})_{i,j \in I}, \quad k \in I, \quad a_{ij}^{(k)} \in R.$$

We have

$$d\big(e_{kk}(1_R)\big) = d\big(e_{kk}(1_R)\big)e_{kk}(1_R) + e_{kk}(1_R)d\big(e_{kk}(1_R)\big).$$

By Lemma 1, $a_{ij}^{(k)} = 0$ whenever $i \neq k$, $j \neq k$ or $i = j = k$. Choose distinct indices $p, q \in I$. Then $e_{pp}(1_R)e_{qq}(1_R) = 0$ implies

$$(a_{ij}^{(p)})_{i,j \in I}\ e_{qq}(1_R) + e_{pp}(1_R)\ (a_{kl}^{(q)})_{k,l \in I} = 0.$$

The (p, q)-entry of the matrix on the left hand side is

$$a_{pq}^{(p)} + a_{pq}^{(q)} = 0.$$

Let $X = (x_{ij})_{i,j \in I}$, $x_{ij} \in R$. The (p, q)-entry of the matrix $[X, e_{kk}(1_R)]$ is equal to 0 if $p \neq k$, $q \neq k$; is equal to $-x_{pq}$ if $k = p$; and to x_{pq} if $k = q$. The diagonal entries of the matrix $[X, e_{kk}(1_R)]$ are equal to zero.

Define $a_{ij} = a_{ij}^{(j)}$ for $i \neq j$; $a_{ii} = 0$; and $v = (a_{ij})_{i,j \in I}$. By the above,

$$d\big(e_{kk}(1_R)\big) = (a_{ij}^{(k)})_{i,j \in I} = \big[v, e_{kk}(1_R)\big]$$

for an arbitrary index $k \in I$.

The j-th column of the matrix v is equal to the j-th column of the matrix $d\big(e_{kk}(1_R)\big) \in M(I, R)$. Hence, only finitely many entries in the j-th column are different from zero, hence $v \in M(I, R)$. This completes the proof of the lemma.

Lemma 3. *Let $v \in M(I,R)$. Then*

(*a*) $\big[v, M_\infty(I,R)\big] \subseteq M_{rcf}(I,R)$ *if and only if* $v \in M_{rcf}(I,R)$;

(*b*) *if* $v \in M_{rcf}(I,R)$, *then in fact* $\big[v, M_\infty(I,R)\big] \subseteq M_\infty(I,R)$.

Proof. The assertion (b) immediately follows from the fact that $M_\infty(I,R)$ is a two-sided ideal in the ring $M_{rcf}(I,R)$.

Let

$$v = (a_{ij})_{i,j\in I} \in M(I,R), \quad \big[v, e_{kk}(1_R)\big] \in M_{rcf}(I,R) \quad \text{for any} \quad k \in I.$$

All entries in the k-th row of the matrix v, except for the diagonal one, are negatives of the corresponding entries in the k-th row of the matrix $[v, e_{kk}(1_R)]$. Since $\big[v, e_{kk}(1_R)\big] \in M_{rcf}(I,R)$, it follows that only finitely many entries of the k-th row of the matrix v are different from zero. Hence, $v \in M_{rcf}(I,R)$. This completes the proof of the lemma.

Now, we are ready to prove Theorem 1.

Proof. Let d be a derivation of the ring $M_\infty(I,R)$. By Lemma 2, there exists a matrix $v \in M(I,R)$ such that $d' = d - d_v$ maps all matrix units $e_{kk}(1_R)$ to zero. This implies that

$$d'\big(e_{ij}(R)\big) \subseteq e_{ij}(R)$$

for all indices $i,j \in I$.

Define the additive mapping $d_{ij} : R \to R$ via

$$d'\big(e_{ij}(a)\big) = e_{ij}\big(d_{ij}(a)\big).$$

Clearly, $d_{ij}(1_R) = 0$. Let $a,b \in R$; $i,j,k \in I$. We have $e_{ij}(ab) = e_{ik}(a)e_{kj}(b)$. Hence,

$$d_{ij}(ab) = d_{ik}(a)\cdot b + a \cdot d_{kj}(b).$$

Let $b = 1_R$. Then $d_{ij}(a) = d_{ik}(a)$. This implies that d_{ij} does not depend on i, j.

Define $d = d_{ij}$; $i,j \in I$. The mapping d is a derivation of the ring R. Hence, $d' \in \mathrm{Der}(R)$. All derivations from $\mathrm{Der}(R)$ map $M_\infty(I,R)$ into itself. Therefore,

$$d_v\big(M_\infty(I,R)\big) \subseteq M_\infty(I,R).$$

By Lemma 3(a), it implies that $v \in M_{rcf}(I,R)$ and completes the proof of the first part of the theorem.

Let d be a derivation of the ring $M(I,R)$. Arguing as above, we find a matrix $v \in M(I,R)$ such that $u = d - d_v \in \mathrm{Der}(R)$. If d is a derivation of the ring $M_{rcf}(I,R)$, then the mapping d_v maps $M_\infty(I,R)$ to $M_{rcf}(I,R)$. By Lemma 3(a), $v \in M_{rcf}(I,R)$. This completes the proof of the theorem.

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