Res. Math. **31**(2), 2023, p. 62–78
doi:10.15421/242316

ISSN (Print) 2664-4991
ISSN (Online) 2664-5009

UDK 517.5

Divya Waghela*, **Snehal B. Rao***
* Department of Applied Mathematics, Faculty of Technology & Engineering, The Maharaja Sayajirao University of Baroda,
Vadodara, Gujarat, India-390001.
E-mail: divyarwaghela@gmail.com , snehal.b.rao-appmath@msubaroda.ac.in

A Note on Sequence of Functions associated with the Generalized Jacobi polynomial

Abstract. An attempt is made to introduce and use operational techniques to study about a new sequence of functions containing generalized Jacobi polynomial. Some generating relations, finite summation formulae, explicit representation of a sequence of function $S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)$ associated with the generalized Jacobi polynomial $P_{n,\tau}^{(\alpha,\gamma,\beta)}(x)$ have been deduced.

Key words: Jacobi polynomial, generalized Jacobi polynomial, generating relations, finite summation formulae

Анотація. Зроблена спроба представити та використати операційні методи для дослідження нової послідовності функцій, що містить узагальнений поліном Якобі. Були доведені деякі породжуючі співвідношення, формули скінченого підсумовування, явне представлення послідовності функцій $S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)$ пов'язаних з узагальненим поліномом Якобі $P_{n,\tau}^{(\alpha,\gamma,\beta)}(x)$.

Ключові слова: поліном Якобі, узагальнений поліном Якобі, породжуючі співвідношення, формули скінченого підсумовування

MSC2020: Pri 33E20, Sec 44A99, 33C45, 33E30, 33E12, 33E99

1. Introduction and Preliminaries

1.1. Introduction

One of the most important polynomials in the theory of special functions, is the Jacobi polynomial $P_n^{(\alpha,\,\beta)}(x)$. Many special functions of applied mathematics can be expressed in terms of Jacobi polynomial, Legendre polynomial, Gegenbauer polynomial (Ultraspherical polynomial), Chebyshev polynomial of first and second kind, and this has motivated us to define its generalization and study / derive several properties of which. Operational techniques, also known as Operational calculus, have drawn attention of several researchers in the study of sequence of functions and polynomials. By operational technique, the problems in analysis, in particular differential equations are transformed into algebraic problems, usually the problem of solving a polynomial equation. In this paper, a new sequence of functions has been introduced containing Generalized Jacobi polynomial defined as below in (1.13) and obtained its various properties using operational techniques.

Jacobi polynomial is defined [4], as

$$P_n^{(\alpha,\,\beta)}(x) = \left[\frac{(1+\alpha)_n}{n!}\right] {}_2F_1\left(-n,\, \alpha+\beta+n+1;\, 1+\alpha; \frac{1-x}{2}\right), \tag{1.1}$$

where $\mathrm{Re}\,(\alpha) > (-1)\,,\, \mathrm{Re}\,(\beta) > 0, x \in \mathbb{C}; n \in \mathbb{N} \cup \{0\}$.

Equivalent definitions of which are as listed below [4]:

$$P_n^{(\alpha,\,\beta)}(x) = \left[\frac{(1+\alpha)_n}{n!}\right] \left(\frac{1+x}{2}\right)^n {}_2F_1\left(-n,\, -n-\beta;\, 1+\alpha; \frac{x-1}{x+1}\right), \tag{1.2}$$

$$P_n^{(\alpha,\,\beta)}(x) = \frac{(-1)^n\,(1+\beta)_n}{n!} {}_2F_1\left(-n,\, \alpha+\beta+n+1;\, 1+\beta; \frac{x+1}{2}\right), \tag{1.3}$$

$$\begin{aligned} P_n^{(\alpha,\,\beta)}(x) =& \frac{(\alpha+\beta+1)_{2n}}{n!(\alpha+\beta+1)_n} \left(\frac{-1+x}{2}\right)^n \\ &\cdot {}_2F_1\left(-n,\, -\alpha-n;\, -\alpha-2n-\beta; \frac{2}{1-x}\right), \end{aligned} \tag{1.4}$$

$$P_n^{(\alpha,\,\beta)}(x) = \frac{(1+\beta)_n}{n!} \left(\frac{-1+x}{2}\right)^n {}_2F_1\left(-n,\, -\alpha-n;\, 1+\beta; \frac{1+x}{x-1}\right), \tag{1.5}$$

$$\begin{aligned} P_n^{(\alpha,\,\beta)}(x) =& \frac{(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n} \left(\frac{x+1}{2}\right)^n \\ & {}_2F_1\left(-n,\, -\beta-n;\, -2n-\alpha-\beta\,; \frac{2}{1+x}\right). \end{aligned} \tag{1.6}$$

In a nut-shell, Legendre, Chebyshev, Ultraspherical and Gegenbauer polynomials can be viewed as a particular case or can be expressed by use of Jacobi polynomial [4], as listed below:

1. When $\alpha = \beta = 0$, we get the Legendre polynomial

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \left(\frac{1}{2}\right)_{n-k} (2x)^{n-2k}}{k!\,(n-2k)!}. \tag{1.7}$$

2. When $\alpha = \beta = -\frac{1}{2}$, we get the Chebyshev polynomials of the first kind $U_n(x) = (1+n)\,{}_2F_1\left(-n,\ 2+n;\ \frac{3}{2};\ \frac{-x+1}{2}\right)$.

3. When $\alpha = \beta = \frac{1}{2}$, we get the Chebyshev polynomials of the second kind $T_n(x) = {}_2F_1\left(-n, n;\ \frac{1}{2};\ \frac{-x+1}{2}\right)$.

4. When $\alpha = \beta$, we get the Ultraspherical polynomial

$$P_n^{(\alpha,\,\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, 1+2\alpha+n; 1+\alpha; \frac{1-x}{2}\right). \tag{1.8}$$

The Gegenbauer Polynomial
$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{n!}\,{}_2F_1\left(-n,\ 2\lambda+n\,; \lambda+\ \frac{1}{2}\,; \frac{1-x}{2}\right)$,
has the following relation with above Ultraspherical polynomial as given below [4]:

$$P_n^{(\alpha,\,\alpha)}(x) = \frac{(\alpha+1)_n\, C_n^{\alpha\,+\,1/2}(x)}{(2\,\alpha+1)_n}, \tag{1.9}$$

or

$$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n\, P_n^{(\lambda-1/2\ \ ,\ \lambda-1/2\ )}(x)}{(\lambda+1/2\ )_n}. \tag{1.10}$$

The Classical Gauss Hypergeometric function is given by [4]

$${}_2F_1\,(a,b;c;z) \ \ = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k}\left(\frac{z^k}{k!}\right), \tag{1.11}$$

where $\mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0, \mathrm{Re}(c) > 0$ when $|z| < 1$ and $\mathrm{Re}(c-a-b) > 0$ when $|z| = 1$.

Virchenko et al. [19] introduced the generalization of Gauss hypergeometric function in the terms of

$${}_2R_1\,(a,b;c;\tau;z) \ \ = \frac{\Gamma\,(c)}{\Gamma\,(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma\,(b+\tau k)}{\Gamma\,(c+\tau k)}\left(\frac{z^k}{k!}\right), \tag{1.12}$$

where $\mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0, \mathrm{Re}(c) > 0, \tau > 0$ when $|z| < 1$ and $\mathrm{Re}(c-a-b) > 0, \tau > 0$ when $|z| = 1$.
Rao et al. have studied many properties of this Wright-type Generalized Hypergeometric Function in [5, 6, 7, 8, 9, 10].

Using Wright-type generalized Hypergeometric function ${}_2R_1\,(a,b;c;\tau;z)$, the present work aims to generalize the definition of Jacobi polynomial as below.

We define, Generalized Jacobi polynomial as:

$$P_{n,\,\tau}^{(\alpha,\,\gamma,\,\beta)}(x) = \left[\frac{(\alpha+1)_n}{n!}\right] {}_2R_1\left(-n,\ n+\alpha+\beta+1;\ \gamma+1;\ \tau;\ \frac{1-x}{2}\right), \quad (1.13)$$

which reduces to $P_n^{(\alpha,\,\beta)}(x)$ for $\gamma = \alpha$ and $\tau = 1$, where $x, \alpha, \beta, \gamma \in \mathbb{C}$ *and* $\tau > 0$; $n \in \mathbb{N} \bigcup \{0\}$, $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\gamma) > (-1)$ *and* $\mathrm{Re}(\beta) \geq 0$.

1.2. Preliminaries

- Srivastava and Singhal [16] introduced a general class of polynomials in 1971 as

$$G_n^{(\alpha)}(x, r, p, k) = \frac{x^{-\alpha-kn}}{n!} \exp(px^r) \left(x^{k+1}D\right)^n \cdot \left[x^\alpha \exp(-px^r)\right], \quad (1.14)$$

where Laguerre, Hermite and Konhauser polynomials are the special case of (1.14).

- In 1979 Srivastava and Singh [14] introduced a general sequence of functions $\left\{V_n^{(\alpha)}(x; a, k, s)\ /\ n = 0, 1, 2, \ldots\right\}$;

$$V_n^{(\alpha)}(x; a, k, s) = \frac{x^{-\alpha}}{n!} \exp\{p_k(x)\} \left[x^a (s + xD)\right]^n \cdot \left[x^\alpha \exp\{-p_k(x)\}\right], \quad (1.15)$$

Where $p_k(x)$ is a polynomial in x of degree k, a and s are constants.

- In 2007, Shukla and Prajapati [13] introduced a general class of polynomials

$$A_{qn}^{(\alpha,\beta,\gamma,\delta)}(x; a; k, s) = \frac{x^{-\delta-an}}{n!} E_{\alpha,\beta}^{\gamma,q}\{p_k(x)\}\, \theta^n \left[x^\delta E_{\alpha,\beta}^{\gamma,q}\{-p_k(x)\}\right], \quad (1.16)$$

where $p_k(x)$ is a polynomial of degree k in x and $\alpha, \beta, \gamma, \delta, a, k, s$ are real or complex constants, the differential operator defined [3] as $\theta \equiv x^a(s + xD), D \equiv \frac{d}{dx}$ and $E_{\alpha,\beta}^{\gamma,q}(z) = \sum\limits_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$ is a generalized Mittag-Leffler function defined by Shukla and Prajapati [12].

- In 2012 Ajudia and Prajapati [1] introduced a sequence of function as,

$$V_n^{(\alpha,\beta,\delta)}(x;a,k,s) = \frac{x^{-\beta}}{n!} W(\alpha,\delta;p_k(x))\,\theta^n \left(x^{\beta} W(\alpha,\delta;-p_k(x))\right), \tag{1.17}$$

where $W(\alpha,\delta;z)$ is a Wright function defined as

$$W(\alpha,\delta;z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\alpha n+\delta)}. \tag{1.18}$$

- In 2013, Rao & Shukla [9] introduced a sequence of functions containing Generalized Hypergeometric function as:

$$G_{\tau,n}^{(a,b,c,\delta)}(x;\,\alpha,k,s) = \frac{x^{-\delta-\alpha n}}{n!}\,{}_2R_1(a,b;c;\tau;p_k(x))\,({}_\alpha T_x^s)^n \left[x^{\delta}{}_2R_1(a,b;c;\tau;-p_k(x))\right], \tag{1.19}$$

where $p_k(x)$ is a polynomial in x of degree k; $x \in (0,\infty)$; α, k, s are constants; $\tau \in \mathbb{R}_+ = (0,\infty)$; a,b, $c \in \mathbb{C}$; $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b) > 0$, $\mathrm{Re}(c) > 0$; and $({}_\alpha T_x^s)^n = x^{n\,\alpha}(s+x\,D)(s+\alpha+x\,D)(s+2\alpha+x\,D)\ldots(s+(n-1)\,\alpha+x\,D)$, with $D \equiv \frac{d}{dx}$.

- New sequence of functions:
  Let us introduce, newly defined sequence of functions $\left\{S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)\right\}_{n=1}^{\infty}$, containing Generalized Jacobi polynomial $P_{n,\tau}^{(\alpha,\gamma,\beta)}(x)$ as:

$$S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v) = \frac{x^{-\delta-ak}}{k!} P_{n,\tau}^{(\alpha,\gamma,\beta)}(-1-2p_u(x))\,({}_aT_x^v)^k[x^{\delta}P_{n,\tau}^{(\alpha,\gamma,\beta)}(1+2p_u(x))], \tag{1.20}$$

where $p_u(x)$ is a polynomial in x of degree u and $x \in (0,\infty), a,\delta,u,v$ are constants, $\alpha,\beta,\gamma \in \mathbb{C}$ and $\tau \in \mathbb{R}_+ = (0,\infty); k \in \mathbb{N}, u,n \in \mathbb{N}\bigcup\{0\}$,$\mathrm{Re}(\alpha)$, $\mathrm{Re}(\gamma) > (-1)$ *and* $\mathrm{Re}(\beta) \geq 0$ and

$$({}_aT_x^v)^k \equiv x^{ak}(v+xD)(v+a+xD)\ldots(v+(k-1)a+xD) \tag{1.21}$$

with $D \equiv \frac{d}{dx}$.

Special Cases

1. For $\gamma = \alpha, \tau = 1$ in $\left\{S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)\right\}_{n=1}^{\infty}$, the analogue of sequence of functions associated with Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is

$$\left\{S_{n,1,k}^{(\alpha,\beta,\alpha,\delta)}(x;a,u,v)\right\}_{n=1}^{\infty}. \tag{1.22}$$

2. For $\alpha = \beta = \gamma = 0, \tau = 1$ in $\left\{S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)\right\}_{n=1}^{\infty}$, the analogue of sequence of functions associated with Legendre polynomial $P_n(x)$ is

$$\left\{S_{n,1,k}^{(0,0,0,\delta)}(x;a,u,v)\right\}_{n=1}^{\infty}. \tag{1.23}$$

3. For $\alpha = \beta = \gamma = \frac{-1}{2}, \tau = 1$ in $\left\{S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)\right\}_{n=1}^{\infty}$, the analogue of sequence of functions associated with Chebyshev polynomial of the first kind $U_n(x)$ is

$$\left\{S_{n,1,k}^{\left(\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\delta\right)}(x;a,u,v)\right\}_{n=1}^{\infty}. \tag{1.24}$$

4. For $\alpha = \beta = \gamma = \frac{1}{2}, \tau = 1$ in $\left\{S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)\right\}_{n=1}^{\infty}$, the analogue of sequence of functions associated with Chebyshev polynomial of the second kind $T_n(x)$ is

$$\left\{S_{n,1,k}^{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\delta\right)}(x;a,u,v)\right\}_{n=1}^{\infty}. \tag{1.25}$$

Some important results are listed below for our further study

1. The operational formulae based on [2, 3]:

$$e^{t\cdot{}_{\alpha}T_x^s}\left(x^{\beta}\,f(x)\right) = e^{t\,x^{\alpha}(s+xD)}\left(x^{\beta}\,f(x)\right)$$

$$= x^{\beta}\left(1-\alpha x^{\alpha}t\right)^{-\left(\frac{\beta+s}{\alpha}\right)} f\left[x(1-\alpha x^{\alpha}t)^{-1/\alpha}\right], \tag{1.26}$$

$$e^{t\cdot{}_{\alpha}T_x^s}\left(x^{\beta-\alpha n}\,f(x)\right) = e^{t\,x^{\alpha}(s+xD)}\left(x^{\beta-\alpha n}\,f(x)\right)$$

$$= x^{\beta}\left(1+\alpha t\right)^{-1+\left(\frac{\beta+s}{\alpha}\right)} f\left[x(1+\alpha t)^{1/\alpha}\right] \tag{1.27}$$

$$\left({}_{\alpha}T_x^s\right)^n(xuv) = x\sum_{m=0}^{n}\left(\begin{array}{c} n \\ m \end{array}\right)\left({}_{\alpha}T_x^s\right)^{n-m}(v)\,\left({}_{\alpha}T_x^1\right)^m(u), \tag{1.28}$$

$$(1-at)^{-\alpha/a} = (1-at)^{-\beta/a} \sum_{m=0}^{\infty} \left(\frac{\alpha-\beta}{a}\right)_m \frac{(at)^m}{m!}. \tag{1.29}$$

The Stirling number of second kind (Riordan [11]) denoted by $S(n, k)$ is defined as:

$$S(n, k) = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n. \tag{1.30}$$

2. Srivastava [17] proved the following theorem:
Let the sequence $\{\xi_n(x)\}_{n=0}^{\infty}$ be generated by

$$\sum_{k=0}^{\infty} \binom{n+k}{k} \xi_{n+k}(x)\, t^k = f(x, t)\{g(x, t)\}^{-n} \xi_n(h(x, t)), (n \in \mathbb{C} \cup \{0\}), \tag{1.31}$$

where f, g, h are suitable functions of x and t, then the following generating relation in terms of Stirling number $S(n, k)$, holds true, provided each number of following equation (1.28) exists.

$$\sum_{k=0}^{\infty} k^n \xi_k(h(x, -z)) \left(\frac{z}{g(x, -z)}\right)^k = $$

$$= \{f(x, -z)\}^{-1} \cdot \sum_{k=0}^{n} k! S(n, k) \xi_k(x)\, z^k. \tag{1.32}$$

3. In 1975 Srivastava and Lavoie [18] proved the following result:
If sequence $\{\Delta_\mu(x) : \mu \textit{ is a complex number}\}$ is generated by

$$\sum_{n=0}^{\infty} \gamma_{\mu,\, n} \Delta_{\mu+n}(x)\, t^n = \theta(x,t)\{\phi(x,t)\}^{-\mu} \Delta_\mu(\psi(x,t)), \tag{1.33}$$

where $\gamma_{\mu,n}$ are arbitrary constants and θ, ϕ and ψ are arbitrary functions of x and t, and let

$$\Phi_{q,\nu}[x,t] = \sum_{n=0}^{\infty} \delta_{\nu,\, n} \Delta_{\nu+qn}(x)\, t^n, (\delta_{\nu,n} \neq 0), \tag{1.34}$$

where q is a positive integer and ν is an arbitrary complex number, then

$$\sum_{n=0}^{\infty} \Delta_{\mu+n}(x) R_{n,\nu}^{q}(y)\, t^n = \theta(x,t)\{\phi(x,t)\}^{-\nu} \Phi_{q,\nu}\left[\psi(x,t), y\left\{\frac{t}{\phi(x,t)}\right\}^q\right], \tag{1.35}$$

where $R^q_{n,\nu}(y)$ is a polynomial of degree $[n/q\,]$ in y, which is defined as

$$R^q_{n,\nu}(y) = \sum_{k=0}^{[n/q\,]} \gamma_{\nu+qk,\, n-qk}\delta_{\nu,k}\, y^k. \tag{1.36}$$

Srivastava and Manocha [15] mentioned the following series rearrangement technique:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m,n-m). \tag{1.37}$$

4. We are using the operational formula based on [15, 16]

$$({}_aT^v_x)^n(x^k) = a^n\left(\frac{k+v}{a}\right)_n x^{k+na}. \tag{1.38}$$

2. Generating relations for the function $S^{(\alpha,\beta,\gamma,\delta)}_{n,\tau,k}(x;a,u,v)$

A considerably large number of special functions (including all the classical orthogonal polynomials) are known to possess generating relations. Using operational techniques, by employing ${}_aT^v_x$ (as in (1.21)) as a differential operator, we obtain following generating relations of the function $S^{(\alpha,\beta,\gamma,\delta)}_{n,\tau,k}(x;a,u,v)$.

Theorem 1. *Let $p_u(x)$ be a polynomial in x of degree u and $x \in (0,\infty)$, a, u, δ, v are constants and $\alpha,\beta,\gamma,t \in \mathbb{C}$ and $\tau > 0$; $k \in \mathbb{N}, n \in \mathbb{N}\bigcup\{0\}, \operatorname{Re}(\alpha), \operatorname{Re}(\gamma) > (-1)$ and $\operatorname{Re}(\beta) \geq 0$, then*

$$1)\ \sum_{k=0}^{\infty} S^{(\alpha,\beta,\gamma,\delta)}_{n,\tau,k}(x;a,u,v)\,t^k = (1-at)^{-\left((\delta+v)/a\right)} P^{(\alpha,\gamma,\beta)}_{n,\tau}(-1-2p_u(x)) \, P^{(\alpha,\gamma,\beta)}_{n,\,\tau}\left(1+2p_u\left(x(1-at)^{\frac{-1}{a}}\right)\right). \tag{2.1}$$

$$2)\ \sum_{k=0}^{\infty} S^{(\alpha,\beta,\gamma,\delta-ak)}_{n,\tau,k}(x;a,u,v)t^k = (1+at)^{-1+\left((\delta+v)/a\right)} P^{(\alpha,\gamma,\beta)}_{n,\tau}(-1-2p_u(x)) \, P^{(\alpha,\,\gamma,\beta)}_{n,\,\tau}\left(1+2p_u\left(x(1+at)^{\frac{1}{a}}\right)\right). \tag{2.2}$$

$$3)\ \sum_{m=0}^{\infty}\binom{m+k}{m} S^{(\alpha,\beta,\gamma,\delta)}_{n,\tau,m+k}(x;\,a,u,v)\,t^m = (1-at)^{-k-(\delta+v)/a} \frac{P^{(\alpha,\,\gamma,\,\beta)}_{n,\,\tau}(-1-2p_u(x))}{P^{(\alpha,\,\gamma,\,\beta)}_{n,\,\tau}\left(1+2p_u\left(x(1-at)^{-1/a}\right)\right)} \times S^{(\alpha,\beta,\gamma,\delta)}_{n,\tau,k}\left(x(1-at)^{-1/a};a,u,v\right) \tag{2.3}$$

4) $\sum_{k=0}^{\infty} k^m S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)z^k$

$$= (1+az)^{-(\delta+v)/a} \frac{P_{n,\tau}^{(\alpha,\gamma,\beta)}\left(-1-2p_u\left\{x(1+az)^{1/a}\right\}\right)}{P_{n,\tau}^{(\alpha,\gamma,\beta)}(-1-2p_u(x))}$$

$$\sum_{n=0}^{m} k! S\,(k\,,\ m) S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}\left(x(1+az)^{-1/a};a,u,v\right)\left(\frac{z}{1+az}\right)^k \quad (2.4)$$

Proof.
Part: 1) From equation (1.20), we have

$$\sum_{k=0}^{\infty} x^{ak} S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)\,t^k$$
$$= x^{-\delta} P_{n,\tau}^{(\alpha,\gamma,\beta)}(-1-2p_u(x)) e^{t\cdot{}_{\alpha}T_x^v}[x^{\delta}\,P_{n,\tau}^{(\alpha,\gamma,\beta)}(1+2p_u(x))]$$
$$= \left(\frac{(1+\alpha)_n}{n!}\right)^2 x^{-\delta}{}_2R_1\,(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u(x))\,e^{t\cdot{}_{\alpha}T_x^s}$$
$$\left[x^{\delta}{}_2R_1\,(-n,n+\alpha+\beta+1;1+\gamma;\tau;-p_u(x))\right]$$

and use of above identity (1) gives simplification of above equation as:

$$\sum_{k=0}^{\infty} x^{ak} S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)t^k$$
$$= x^{-\delta}\left(\frac{(1+\alpha)_n}{n!}\right)^2 {}_2R_1\,(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u\,(x))\,x^{\delta}$$
$$\cdot\,(1-ax^at)^{-\left((\delta+v)/a\right)}{}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;-p_u\left\{x(1-ax^av)^{-1/a}\right\}\right)$$

Replacing t by $t\cdot x^{-a}$ in this, we get the proof of part 1.

Special cases:

1. Generating relation for $\left\{S_{n,1,k}^{(\alpha,\beta,\alpha,\delta)}(x;a,u,v)\right\}_{n=1}^{\infty}$ associated with Jacobi polynomial:

$$\sum_{k=0}^{\infty} S_{n,1,k}^{(\alpha,\beta,\alpha,\delta)}(x;a,u,v)\,t^k = (1-at)^{-\left((\delta+v)/a\right)} P_n^{(\alpha,\beta)}\,(-1-2p_u(x))$$
$$\cdot P_n^{(\alpha,\beta)}\left(1+2p_u\left\{x(1-at)^{-1/a}\right\}\right). \quad (2.5)$$

2. Generating relation for $\left\{S_{n,1,k}^{(0,0,0,\delta)}(x;a,u,v)\right\}_{n=1}^{\infty}$ associated with Legendre polynomial:

$$\sum_{k=0}^{\infty} S_{n,1,k}^{(0,0,0,\delta)}(x;a,u,v)\,t^k = (1-at)^{-\left(\left(\delta+v\right)/a\right)} P_n\left(-1-2p_u(x)\right) \cdot P_n\left(1+2p_u\left\{x(1-at)^{-1/a}\right\}\right). \quad (2.6)$$

3. Generating relation for $\left\{S_{n,1,k}^{\left(\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\delta\right)}(x;a,u,v)\right\}_{n=1}^{\infty}$ associated with Chebyshev polynomial of the first kind:

$$\sum_{k=0}^{\infty} S_{n,1,k}^{\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\delta\right)}(x;a,u,v)\,t^k = (1-at)^{-\left(\left(\delta+v\right)/a\right)} U_n\left(-1-2p_u(x)\right) U_n\left(1+2p_u\left\{x(1-at)^{-1/a}\right\}\right). \quad (2.7)$$

4. Generating relation for $\left\{S_{n,1,k}^{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\delta\right)}(x;a,u,v)\right\}_{n=1}^{\infty}$ associated with Chebyshev polynomial of the second kind:

$$\sum_{k=0}^{\infty} S_{n,1,k}^{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\delta\right)}(x;a,u,v)\,t^k = (1-at)^{-\left(\left(\delta+v\right)/a\right)} T_n\left(-1-2p_u(x)\right) T_n\left(1+2p_u\left\{x(1-at)^{-1/a}\right\}\right). \quad (2.8)$$

Part: 2) From equation (1.20), we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta-ak)}(x;a,u,v)\,t^k \\
&= x^{-\delta}\left(\frac{(1+\alpha)_n}{n!}\right)^2 {}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+2p_u(x)\right) \\
&\quad e^{t\ {}_aT_x^v}\cdot\left[x^{\delta-ak}\cdot {}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;-p_u(x)\right)\right] \\
&= x^{-\delta}\left(\frac{(1+\alpha)_n}{n!}\right)^2 {}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u(x)\right)x^{\delta} \\
&\quad (1+at)^{-1+\left(\left(\delta+v\right)/a\right)} {}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;-p_u\left\{x(1+at)^{1/a}\right\}\right) \\
&= (1+at)^{-1+\left(\left(\delta+v\right)/a\right)} P_{n,\tau}^{(\alpha,\gamma,\beta)}\left(-1-2p_u(x)\right) \\
&\quad P_{n,\tau}^{(\alpha,\gamma,\beta)}\left(1+2p_u\left(x(1+at)^{1/a}\right)\right).
\end{aligned}$$

This completes the proof of part 2).
Part: 3) Writing equation (1.20), as

$$e^{t\,({}_aT_x^v)}\left(({}_aT_x^v)^k\left[x^\delta\left(\frac{(1+\alpha)_n}{n!}\right){}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;-p_u\left(x\right)\right)\right]\right)$$
$$=k!e^{t\,({}_aT_x^v)}\left[x^{\delta+ak}\frac{S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}\left(x;\,a,u,v\right)}{\left(\frac{(1+\alpha)_n}{n!}\right){}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u\left(x\right)\right)}\right] \quad (2.9)$$

the above equation, using (1) can be re-written as

$$\sum_{m=0}^{\infty}\frac{t^m({}_aT_x^v)^{m+k}}{m!}\left[x^\delta\left(\frac{(1+\alpha)_n}{n!}\right){}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;-p_u\left(x\right)\right)\right]$$
$$=\frac{k!x^{\delta+ak}\left(1-ax^at\right)^{-k-\left(\frac{\delta+v}{a}\right)}\cdot S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}\left(x(1-ax^at)^{-1/a};\,a,u,v\right)}{\left(\frac{(1+\alpha)_n}{n!}\right){}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u\left\{x(1-ax^at)^{-1/a}\right\}\right)} \quad (2.10)$$

Therefore, using (1.20) we have,

$$\sum_{m=0}^{\infty}\frac{(m+k)!}{m!k!}\frac{x^{\delta+a(m+k)}\cdot S_{n,\tau,m+k}^{(\alpha,\beta,\gamma,\delta)}\left(x;\,a,u,v\right)t^m}{\left(\frac{(1+\alpha)_n}{n!}\right){}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u\left(x\right)\right)}$$
$$=\frac{x^{\delta+ak}\left(1-ax^at\right)^{-k-\left(\frac{\delta+v}{a}\right)}\cdot S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}\left(x(1-ax^at)^{-1/a};\,a,u,v\right)}{\left(\frac{(1+\alpha)_n}{n!}\right){}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u\left\{x(1-ax^at)^{-1/a}\right\}\right)} \quad (2.11)$$

Replacing t by $t\cdot x^{-a}$, immediately leads to equation (2.3).
Part: 4) Comparing equations (2.3) and (1), we get
$f\left(x,\,t\right)=\frac{(1-at)^{-(\delta+v)/a}\cdot{}_2R_1(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u(x))}{{}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u\left\{x(1+at)^{-1/a}\right\}\right)}$, $g\left(x,\,t\right)=(1-at)$,
$h\left(x,\,t\right)=x(1-at)^{-1/a}$ and $\xi_k\left(x\right)\mapsto S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)$;

$$\sum_{k=0}^{\infty}k^m\,S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}\left((1+az)^{-1/a};a,u,v\right)\left(\frac{z}{(1+az)}\right)^k$$
$$=\frac{(1+az)^{(\delta+v)/a}\cdot{}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u\left\{x(1-az)^{-1/a}\right\}\right)}{{}_2R_1(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u(x))}$$
$$\cdot\sum_{n=0}^{m}k!S\left(k,m\right)S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)z^k. \quad (2.12)$$

For $z\mapsto z/(1+az)$, $x\mapsto x(1+az)^{-1/a}$, the above equation immediately leads to (2.4) in terms of part 4).

3. Finite summation formulae for the function $S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)$

In this section, we obtain two finite summation formulae for equation (1.13).

Theorem 2. *Let $p_u(x)$ be a polynomial in x of degree u and $x \in (0,\infty)$, a,u,v are constants and $\alpha,\beta,\gamma \in \mathbb{C}$ and $\tau > 0$;$n \in \mathbb{N}\bigcup\{0\}$,$\mathrm{Re}(\alpha)$,$\mathrm{Re}(\gamma) > (-1)$ and $\mathrm{Re}(\beta) \geq 0$, then*

$$1)\; S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;\,a,u,v) = \sum_{m=0}^{k} \frac{1}{m!} a^m \left(\frac{\delta}{a}\right)_m S_{n,\tau,(k-m)}^{(\alpha,\beta,\gamma,0)}(x;\,a,u,v)\,. \tag{3.1}$$

$$2)\; S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v) = \sum_{m=0}^{k} \frac{1}{m!} a^m \left(\frac{\delta-\sigma}{a}\right)_m S_{n,\tau,k-m}^{(\alpha,\beta,\gamma,\sigma)}(x;\,a,u,v). \tag{3.2}$$

Proof.
Part: 1) By employing the operational formula (1.28) to (1.20), we get,

$$\begin{aligned} &S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;\,a,u,v) \\ &= \frac{x^{-\delta-ak}}{k!}\left(\frac{(1+\alpha)_n}{n!}\right)^2 {}_2R_1(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u(x))\,x \\ &\cdot \sum_{m=0}^{k}\binom{k}{m}\left({}_aT_x^v\right)^{k-m} {}_2R_1(-n,n+\alpha+\beta+1;1+\gamma;\tau;-p_u(x))\left({}_aT_x^1\right)^m x^{\delta-1} \\ &= \left(\frac{(1+\alpha)_n}{n!}\right)^2 {}_2R_1(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u(x)) \sum_{m=0}^{k} \tfrac{x^{1-\delta}}{m!\,(k-m)!} \\ &\cdot \prod_{i=0}^{k-m-1}(v+ia+xD)\,{}_2R_1(-n,n+\alpha+\beta+1;1+\gamma;\tau;-p_u(x))\,a^m\left(\frac{\delta}{a}\right)_m x^{\delta-1} \end{aligned} \tag{3.3}$$

Using (1.20), we have

$$\begin{aligned} &\frac{1}{(k-m)!}\prod_{i=0}^{k-m-1}(v+ia+xD)\,{}_2R_1(-n,n+\alpha+\beta+1;1+\gamma;\tau;-p_u(x)) \\ &= \frac{S_{n,\tau,(k-m)}^{(\alpha,\beta,\gamma,0)}(x;a,u,v)}{{}_2R_1(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u(x))}. \end{aligned} \tag{3.4}$$

Part: 2) Using (1), in (1.20) gives

$$\begin{aligned} &\sum_{k=0}^{\infty} x^{ak} S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)t^k \\ &= (1-ax^a t)^{-\left(\frac{\delta+v}{a}\right)}\left(\frac{(1+\alpha)_n}{n!}\right)^2 {}_2R_1(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u(x)) \\ &\cdot {}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;-p_u\left(x(1-ax^a t)^{-1/a}\right)\right) \end{aligned} \tag{3.5}$$

which on application of (1.29) yields

$$\begin{aligned}
&\sum_{k=0}^{\infty} x^{ak} S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)t^k \\
&= \left[(1-ax^a t)^{-\left(\frac{\sigma+v}{a}\right)} \sum_{m=0}^{\infty} \left(\frac{\delta-\sigma}{a}\right)_m \frac{(ax^a t)^m}{m!}\right] \left(\frac{(1+\alpha)_n}{n!}\right)^2 \\
&\cdot {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; 1+p_u(x)\right) \\
&\cdot\ {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; -p_u\left(x(1-ax^a t)^{-1/a}\right)\right) \qquad (3.6)
\end{aligned}$$

Use of (1) again gives simplification of above expression in the form of

$$\begin{aligned}
&\sum_{k=0}^{\infty} x^{ak} S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)t^k \\
&= \left[\sum_{m=0}^{n} \left(\frac{\delta-\sigma}{a}\right)_m \frac{(ax^a t)^m}{m!}\right] \left(\frac{(1+\alpha)_n}{n!}\right)^2 \\
&\cdot {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; 1+p_u(x)\right) x^{-\sigma} e^{t({}_aT_x^v)} \\
&\cdot x^{\sigma}\, {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; -p_u(x)\right) \qquad (3.7)
\end{aligned}$$

Further simplification and using, (1.37), we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} x^{ak} S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;\, a,u,v)\, t^k \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{k} \left(\frac{\delta-\sigma}{a}\right)_m \frac{(ax^a)^m t^k}{m!\,(k-m)!} x^{-\sigma} \left(\frac{(1+\alpha)_n}{n!}\right)^2 \\
&\cdot\ {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; 1+p_u(x)\right) \\
&\cdot ({}_aT_x^v)^{k-m} \left(x^{\sigma}\, {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; -p_u(x)\right)\right) \qquad (3.8)
\end{aligned}$$

Equating the coefficients of t^k on left and right sides of above expression, we have

$$\begin{aligned}
&S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v) \\
&= \sum_{m=0}^{k} \frac{1}{m!} a^m \left(\frac{\delta-\sigma}{a}\right)_m \frac{x^{-\sigma-a(k-m)}}{(k-m)!} \left(\frac{(1+\alpha)_n}{n!}\right)^2 \\
&\cdot {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; 1+p_u(x)\right) \\
&\cdot ({}_aT_x^v)^{k-m} \left(x^{\sigma}\, {}_2R_1\left(-n, n+\alpha+\beta+1; 1+\gamma; \tau; -p_u(x)\right)\right) \\
&= \sum_{m=0}^{k} \frac{1}{m!} a^m \left(\frac{\delta-\sigma}{a}\right)_m S_{n,\tau,k-m}^{(\alpha,\beta,\gamma,\sigma)}(x;\, a,u,v).
\end{aligned}$$

4. Bilateral-generating relation for the function $S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)$

Comparing (1.33) with (2.3) and replacing μ of (1.33) by k and n of (1.33) by m, observe that we have $\gamma_{k,m} = \begin{pmatrix} k+m \\ m \end{pmatrix}$, $\phi(x,t) = 1$,

$$\theta(x,t) \quad = \quad (1-at)^{-k-((\delta+v)/a)} \frac{{}_2R_1(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u(x))}{{}_2R_1\left(-n,n+\alpha+\beta+1;1+\gamma;\tau;1+p_u\left\{x(1-at)^{-1/a}\right\}\right)},$$

$\Delta_k(x) = S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)$ and $\psi(x,t) = x(1-at)^{-1/a}$, then from (1.35) we get the following bilateral-generating relation for $S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)$, which is given by

$$\sum_{m=0}^{\infty} S_{n,\tau,(m+k)}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v) R_{m,\nu}^{q}(y)\, t^m$$
$$= \theta(x,t)\{\phi(x,t)\}^{-\nu} \Phi_{q,\nu}\left[\psi(x,t), y\left\{\frac{t}{\phi(x,t)}\right\}^{q}\right] \tag{4.1}$$

where $\Phi_{q,\nu}[x,t]$ and $R_{m,\nu}^{q}(y)$ are as defined in (1.34) and (1.36) respectively, with q is a positive integer and ν is an arbitrary complex number.

5. Explicit representation of $S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)$

In this section we introduce the Explicit Representation of $S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)$ as defined in (1.20), in terms of the following result.

Theorem 3.

$$S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)$$
$$= \frac{a^k}{(k!)}\left[\frac{(\alpha+1)_n}{n!}\right]^2 \cdot \sum_{m=0}^{n}\sum_{l=0}^{n}\left(\frac{(-n)_m(\Gamma(\gamma+1))^2\Gamma(n+\alpha+\beta+1+\tau m)}{\Gamma(\gamma+1+\tau m)(\Gamma(n+\alpha+\beta+1))^2 m!}\right.$$
$$\left. \cdot \frac{(-1)^l(-n)_l\Gamma(n+\alpha+\beta+1+\tau l)}{\Gamma(\gamma+1+\tau l)\, l!} x^{ul}(1+x^u)^m\left(\frac{\delta+v+ul}{a}\right)_k\right), \tag{5.1}$$

where $p_u(x)$ is a polynomial in x of degree u and $x \in (0,\infty), a, \delta$ are constants and $\alpha,\beta,\gamma \in \mathbb{C}$ and $\tau > 0$; $n \in \mathbb{N}\bigcup\{0\}$, $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\gamma) > (-1)$ and $\mathrm{Re}(\beta) \geq 0$; and $({}_aT_x^v)^k \equiv x^{ak}(v+xD)(v+a+xD)...(v+(k-1)a+xD)$ with $D \equiv \frac{d}{dx}$.

The following is the *proof* of above theorem.

For simplicity purpose, putting $p_u(x) = x^u$ in (1.20), we get

$$
\begin{aligned}
&S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)\\
&=\frac{x^{-\delta-ak}}{k!}P_{n,\tau}^{(\alpha,\gamma,\beta)}(-1-2p_u(x))({}_aT_x^v)^k\left[x^\delta P_{n,\tau}^{(\alpha,\gamma,\beta)}(1+2p_u(x))\right]\\
&=\frac{x^{-\delta-ak}}{k!}\left[\frac{(\alpha+1)_n}{n!}\right]^2{}_2R_1\left(-n,n+\alpha+\beta+1;\gamma+1;\tau;1+p_u(x)\right)\\
&\quad({}_aT_x^v)^k\left[x^\delta{}_2R_1\left(-n,n+\alpha+\beta+1;\gamma+1;\tau;-p_u(x)\right)\right]\\
&=\frac{x^{-\delta-ak}}{k!}\left[\frac{(\alpha+1)_n}{n!}\right]^2{}_2R_1\left(-n,\ n+\alpha+\beta+1;\ \gamma+1;\ \tau;\ 1+\ x^u\right)\\
&\quad({}_aT_x^v)^k\ \left[x^\delta{}_2R_1\left(-n,\ n+\alpha+\beta+1;\ \gamma+1;\ \tau;\ -x^u\right)\right]\\
&=\frac{x^{-\delta-ak}}{k!}\left[\frac{(\alpha+1)_n}{n!}\right]^2{}_2R_1\left(-n,\ n+\alpha+\beta+1;\ \gamma+1;\ \tau;\ 1+x^u\right)\\
&\quad\sum_{l=0}^{n}\frac{(-1)^l(-n)_l\,\Gamma(\gamma+1)\,\Gamma(n+\alpha+\beta+1+\tau l)}{\Gamma(\gamma+1+\tau l)\,\Gamma(n+\alpha+\beta+1)\,l!}({}_aT_x^v)^k\left(x^{\delta+ul}\right)
\end{aligned}
$$

Using (1.38) gives us,

$$
\begin{aligned}
&S_{n,\tau,k}^{(\alpha,\beta,\gamma,\delta)}(x;a,u,v)\\
&=\frac{x^{-\delta-ak}}{k!}\left[\frac{(\alpha+1)_n}{n!}\right]^2{}_2R_1\left(-n,\ n+\alpha+\beta+1;\ \gamma+1;\ \tau;\ 1+x^u\right)\\
&\quad\sum_{l=0}^{n}\frac{(-1)^l(-n)_l\,\Gamma(\gamma+1)\,\Gamma(n+\alpha+\beta+1+\tau l)}{\Gamma(\gamma+1+\tau l)\,\Gamma(n+\alpha+\beta+1)\,l!}a^k\left(\frac{\delta+v+ul}{a}\right)_k x^{\delta+ul+ka}\\
&=\frac{a^k}{k!}\left[\frac{(\alpha+1)_n}{n!}\right]^2{}_2R_1\left(-n,\ n+\alpha+\beta+1;\ \gamma+1;\ \tau;\ 1+x^u\right)\\
&\quad\sum_{l=0}^{n}\frac{(-1)^l(-n)_l\,\Gamma(\gamma+1)\,\Gamma(n+\alpha+\beta+1+\tau l)}{\Gamma(\gamma+1+\tau l)\,\Gamma(n+\alpha+\beta+1)\,l!}x^{ul}\left(\frac{\delta+v+ul}{a}\right)_k
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{a^k}{k!}\left[\frac{(\alpha+1)_n}{n!}\right]^2\sum_{m=0}^{n}\frac{(-n)_m\,(\Gamma(\gamma+1))^2\,\Gamma(n+\alpha+\beta+1+\tau m)}{\Gamma(\gamma+1+\tau m)\,(\Gamma(n+\alpha+\beta+1))^2\,m!}(1+x^u)^m\\
&\quad\sum_{l=0}^{n}\frac{(-1)^l(-n)_l\ \Gamma(n+\alpha+\beta+1+\tau l)}{\Gamma(\gamma+1+\tau l)\ l!}x^{ul}\left(\frac{\delta+v+ul}{a}\right)_{kn}
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{a^k}{k!}\left[\frac{(\alpha+1)_n}{n!}\right]^2\sum_{m=0}^{n}\sum_{l=0}^{n}\frac{(-n)_m\,(\Gamma(\gamma+1))^2\,\Gamma(n+\alpha+\beta+1+\tau m)}{\Gamma(\gamma+1+\tau m)\,(\Gamma(n+\alpha+\beta+1))^2\,m!}\\
&\quad\frac{(-1)^l(-n)_l\ \Gamma(n+\alpha+\beta+1+\tau l)}{\Gamma(\gamma+1+\tau l)\ l!}x^{ul}(1+x^u)^m\left(\frac{\delta+v+ul}{a}\right)_k.
\end{aligned}
$$

This proves the identity (5.1).

Some particular cases

1. Explicit representation for sequence of functions $\left\{S_{n,1,k}^{(\alpha,\beta,\alpha,\delta)}(x;a,u,v)\right\}_{n=1}^{\infty}$, associated with the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$: $S_{n,1,k}^{(\alpha,\beta,\alpha,\delta)}(x;a,u,v) =$

$$= \frac{a^k}{k!}\left[\frac{(\alpha+1)_n}{n!}\right]^2 \cdot \sum_{m=0}^{n}\sum_{l=0}^{n} \frac{(-n)_m\,(\Gamma(\alpha+1))^2\,\Gamma(n+\alpha+\beta+1+m)}{\Gamma(\alpha+1+m)\,(\Gamma(n+\alpha+\beta+1))^2\,m!}$$
$$\frac{(-1)^l(-n)_l\,\Gamma(n+\alpha+\beta+1+l)}{\Gamma(\alpha+1+l)\;l!} x^{ul}(1+x^u)^m\left(\frac{\delta+v+ul}{a}\right)_k. \tag{5.2}$$

2. Explicit representation for sequence of functions $\left\{S_{n,1,k}^{(0,0,0,\delta)}(x;a,u,v)\right\}_{n=1}^{\infty}$, associated with the Legendre polynomial $P_n(x)$ is given as:

$$S_{n,1,k}^{(0,0,0,\delta)}(x;a,u,v) = \frac{a^k}{k!}\sum_{m=0}^{n}\sum_{l=0}^{n} \frac{(-n)_m\,\Gamma(n+1+m)}{\Gamma(1+m)\,(\Gamma(n+1))^2\,m!}$$
$$\frac{(-1)^l(-n)_l\,\Gamma(n+1+l)}{\Gamma(1+l)\;l!} x^{ul}(1+x^u)^m\left(\frac{\delta+v+ul}{a}\right)_k \tag{5.3}$$

3. Explicit representation for sequence of functions $\left\{S_{n,1,k}^{\left(\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\delta\right)}(x;a,u,v)\right\}_{n=1}^{\infty}$, associated with the Chebyshev polynomial of the 1st kind $U_n(x)$ is given as:

$$S_{n,1,k}^{\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\delta\right)}(x;a,u,v) = \frac{a^k}{k!}\sum_{m=0}^{n}\sum_{l=0}^{n} \frac{(-n)_m\left(\Gamma\left(\frac{1}{2}\right)\right)^2\Gamma(n+m)}{\Gamma\left(\frac{1}{2}+m\right)\,(\Gamma(n))^2\,m!}x^{um}$$
$$\frac{(-1)^l(-n)_l\,\Gamma(n+l)}{\Gamma\left(\frac{1}{2}+l\right)\;l!} x^{ul}(1+x^u)^m\left(\frac{\delta+v+ul}{a}\right)_k \tag{5.4}$$

4. Explicit representation for sequence of functions $\left\{S_{n,1,k}^{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\delta\right)}(x;a,u,v)\right\}_{n=1}^{\infty}$, associated with the Chebyshev polynomial of the 2nd kind $T_n(x)$ is given as:

$$S_{n,1,k}^{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\delta\right)}(x;a,u,v) = \frac{a^k}{k!}\left[\frac{\left(\frac{3}{2}\right)_n}{n!}\right]^2 \cdot \sum_{m=0}^{n}\sum_{l=0}^{n} \frac{(-n)_m\left(\Gamma\left(\frac{3}{2}\right)^2\right)\Gamma(n+2+m)}{\Gamma\left(\frac{3}{2}+m\right)(\Gamma(n+2))^2\,m!}$$
$$\frac{(-1)^l(-n)_l\,\Gamma(n+2+l)}{\Gamma\left(\frac{3}{2}+l\right)\;l!} x^{ul}(1+x^u)^m\left(\frac{\delta+v+ul}{a}\right)_k \tag{5.5}$$

References

1. *Ajudia N.K., Prajapati J.C.*: On a Sequence of functions $V_n^{(\alpha,\beta,\delta)}(x;a,k,s)$. Proyecciones Journal of Mathematics, 2016; 35(4): pp. 417–436.

2. *Mittal H.B.*: Bilinear and bilateral generating functions. American Journal of Mathematics, 1977; 99: pp. 23–55.
3. *Patil K.R., Thakare N.K.*: Operational formulas for a function defined by a generalized Rodrigues formula-II. Scientific Journal of Shivaji University, 1975; 15: pp. 1–10.
4. *Rainville E.D.*: Special functions. New York: The MacMillan Company, 1960.
5. *Rao S.B.*: Explicit Representation and some Integral transforms of sequence of functions associated with the Wright type Generalized Hypergeometric Function. Global Journal of Pure and Applied Mathematics, 2017; 13(9): pp. 6703–6712.
6. *Patil K.R., Thakare N.K.*: Operational formulas for a function defined by a generalized Rodrigues formula-II. Scientific Journal of Shivaji University, 1975; 15: pp. 1–10.
7. *Rao S.B., Prajapati J.C., Patel A.D., Shukla A.K.*: Some Properties of Wright-type Generalized Hypergeometric Function via fractional calculus. Advances in Difference Equations, 2014; 2014: 119.
8. *Rao S.B., Prajapati J.C., Shukla A.K.*: Wright type Hypergeometric functions and its properties. Advances in Pure Mathematics, 2013; 3(3): pp. 335–342.
9. *Rao S.B., Salehbhai I.A., Shukla A.K.*: On sequence of functions Containing Generalized Hypergeometric Function. Mathematical Sciences Research Journal, 2013; 17(4): pp. 98–110.
10. *Rao S.B., Shukla A.K.*: Note on Generalized Hypergeometric Function. Integral transforms and special functions, 2013; 24(11): pp. 896–904.
11. *Riordan J.*: Combinatorial identities. Wiley: (New York/ London/ Sydney), 1968.
12. *Shukla A.K., Prajapati J.C.*: A general class of polynomial associated with generalized Mittag-Leffler function. Integral transforms and Special functions, 2008; 19(1): pp. 23–34.
13. *Shukla A.K., Prajapati J.C.*: Generalization of Mittag-Leffler function and their properties. Journal of Mathematical Analysis and applications, 2007; 336: pp. 797–811.
14. *Srivastava A.N., Singh S.N.*: Some generating relations connected with a function defined by a generalized Rodrigues formula. Indian Journal of Pure and Applied Mathematics, 1979; 10(10): pp. 1312–1317.
15. *Srivastava H.M., Manocha H.L.*: A treatise on generating functions. Ellis Horwood Limited, Chichester, 1984.
16. *Srivastava H.M., Singhal J.P.*: A class of polynomials defined by generalized Rodrigues' formula. Annali di Mathematica Pura ed Applicata, 1971; 90(4): pp. 75–85.
17. *Srivastava H.M.*: Some families of generating functions associated with the Stirling numbers of second kind. Journal of Mathematical Analysis and Applications, 2000; 251: pp. 752–769.
18. *Srivastava H.M., Lavoie J.L.*: A certain method of obtaining bilateral generating functions. Nederlandse Akademic Wetenschvan appen Proceedings A; 78(4): and Indagationes Mathematical, 1975; 37(4), pp. 304–320.
19. *Virchenko N., Kalla S.L., Al-Zamel A.*: Some Results On a Generalized Hypergeometric Function. Integral Transforms and Special Functions, 2001; 12(1): pp. 89–100.

Received: 09.07.2023. *Accepted:* 15.09.2023