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Virtual endomorphisms of the group pg

Abstract. A virtual endomorphism of a group G is a homomorphism of the form $\phi : H \to G$, where $H < G$ is a subgroup of finite index. A virtual endomorphism $\phi : H \to G$ is called simple if there are no nontrivial normal ϕ-invariant subgroups, that is, the ϕ-core is trivial. We describe all virtual endomorphisms of the plane group pg, also known as the fundamental group of the Klein bottle. We determine which of these virtual endomorphisms are simple, and apply these results to the self-similar actions of the group. We prove that the group pg admits a transitive self-similar (as well as finite-state) action of degree d if and only if $d \geq 2$ is not an odd prime, and admits a self-replicating action of degree d if and only if $d \geq 6$ is not a prime or a power of 2.
Key words: virtual endomorphism, plane group, self-similar action

Анотація. Віртуальним ендоморфізмом групи G називається гомоморфізм вигляду $\phi : H \to G$, де $H < G$ — підгрупа скінченного індексу. Віртуальний ендоморфізм $\phi : H \to G$ називається простим, якщо не існує нетривіальних нормальних ϕ-інваріантних підгруп, тобто ϕ-серцевина є тривіальною. Ми описуємо всі віртуальні ендоморфізми плоскої групи pg, також відомої як фундаментальна група пляшки Кляйна. Ми визначаємо, які віртуальні ендоморфізми є простими, і застосовуємо ці результати до самоподібних дій групи. Ми доводимо, що група pg допускає транзитивну самоподібну (також скінченностанову) дію степеня d тоді і лише тоді, коли $d \geq 2$ не є непарним простим числом, та допускає рекурентну дію степеня d тоді і лише тоді, коли $d \geq 6$ не є простим числом або степенем двійки.
Ключові слова: віртуальний ендоморфізм, плоска група, самоподібна дія

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1. Introduction

A virtual endomorphism of a group G is a homomorphism $\phi : H \to G$, where $H < G$ is a subgroup of finite index. Virtual endomorphisms arise naturally in relation to self-coverings of topological spaces, lattices in Lie groups, groups acting on trees, complex dynamics (see [5, 9, 10]).

Virtual endomorphisms are strongly connected to self-similar group actions. A group G admits a faithful transitive self-similar action if and only if it possesses a simple virtual endomorphism ϕ, where simple means that there are no nontrivial normal ϕ-invariant subgroups. The corresponding self-similar action is produced via iterations of ϕ. This connection was used to analyze self-similar actions for a wide range of groups: free abelian groups [11], finitely generated nilpotent groups [2, 3], solvable groups [1], wreath products of abelian groups [6, 7], the affine group $GL_n(\mathbb{Z}) \ltimes \mathbb{Z}^2$ [4], irreducible lattices in semisimple algebraic groups [9], p-adic analytic pro-p groups [12].

In this paper, we study virtual endomorphisms of the plane group K with number pg in IUC notation. The group K is the fundamental group of the Klein bottle. We describe virtual endomorphisms of K (see Section 4), and determine which of them are simple (see Theorem 4). These results are applied to the self-similar actions of the group. We determine which degrees are possible for self-similar, self-replicating, and finite-state actions of K (see Theorems 5 and 6). In contrast to the abelian groups, we show that K admits faithful self-similar actions for non-injective virtual endomorphisms (see Example 4) and finite-state actions that are not contracting (see Example 7).

2. Crystallographic groups

We review basic information about crystallographic groups (see [8, 13] for more details).

The Euclidean group $E(n)$ is the group of isometries of $\mathbb{R}^n$. The translation group $T(n) \cong \mathbb{R}^n$ of $\mathbb{R}^n$ is a normal subgroup of $E(n)$. The group $E(n)$ decomposes into the semidirect product:

$$E(n) = O_n(\mathbb{R}) \ltimes \mathbb{R}^n,$$

where $O_n(\mathbb{R})$ is the orthogonal group. The group $E(n)$ is a subgroup of the affine group $A_n(\mathbb{R})$ of $\mathbb{R}^n$, which is the semidirect product

$$A_n(\mathbb{R}) = GL_n(\mathbb{R}) \ltimes \mathbb{R}^n.$$

The elements of $A_n(\mathbb{R})$ are written as pairs $g = \left(A \mid a\right)$ for $A \in GL_n(\mathbb{R})$ and $a \in \mathbb{R}^n$; here A is called the linear part of g and a its translation part. The product of elements written in this form can be performed by the rule

$$\left(A \mid a\right) \cdot \left(B \mid b\right) = \left(AB \mid Ab + a\right).$$

We identify $a \in \mathbb{R}^n$ and the translation $\left(E \mid a\right)$.

Definition 1. A *crystallographic group* of dimension n is a discrete cocompact subgroup of $E(n)$. A *plane group* is a crystallographic group of dimension 2.

Definition 2. Let G be a crystallographic group. The *translation subgroup* of G is $T(G) = G \cap T(n) < \mathbb{R}^n$. The *point group* of G is $P(G) = G/T(G) < O_n(\mathbb{R})$. The point group $P(G)$ is a finite group consisting of linear parts of elements of G.

The fundamental properties of crystallographic groups were determined by Bieberbach (1912):

Theorem 1 (Bieberbach)**.** *1) The translation subgroup $T(G)$ of an n-dimensional crystallographic group G is isomorphic to $\mathbb{Z}^n$ and is a maximal abelian and normal subgroup of finite index.*

2) Every isomorphism between n-dimensional crystallographic groups is a conjugation by an element of the affine group $A_n(\mathbb{R})$.

3) For every $n \in \mathbb{N}$, there are only finitely many crystallographic groups of dimension n up to isomorphism.

We will use the following properties of subgroups in crystallographic groups (see Theorems 4 and 17 in [8]).

Theorem 2. *Let G be a crystallographic group and $H \leq G$ a subgroup.*

1) If H has finite index, then H is crystallographic and $T(H) = H \cap T(G)$.

2) If H is normal, then H is crystallographic and $T(H) = H \cap T(G)$.

3. The group pg and its subgroups of finite index

There are 17 plane groups up to isomorphism. We will be interested in one of them — the group pg in IUC notation. We denote this group by K.

The group K is generated by two elements

$$a = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right), \; b = \left(\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & -1 & 0 \end{array}\right) \in A_2(\mathbb{Q})$$

and has finite presentation $K = \langle a, b | aba = b \rangle$. The group K consists of the following elements:

$$K = \left\{ a^m b^{2n}, a^m b^{2n+1} : n, m \in \mathbb{Z} \right\} =$$
$$= \left\{ \left(\begin{array}{cc|c} 1 & 0 & n \\ 0 & 1 & m \end{array}\right), \left(\begin{array}{cc|c} 1 & 0 & n+1/2 \\ 0 & -1 & m \end{array}\right) : n, m \in \mathbb{Z} \right\}.$$

The translation subgroup of K is $T(K) = \langle a, b^2 \rangle = \mathbb{Z}^2$ and the point group is

$$P(K) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Note that the normalizer of $P(K)$ in the group $GL_2(\mathbb{R})$ consists of diagonal matrices. The group K is torsion-free, the quotient $\mathbb{R}^2/K$ is homeomorphic to the Klein bottle, so K is the fundamental group of the Klein bottle.

Let us describe finite index subgroups of K.

Theorem 3. *Let $H \leq K$ be a subgroup of finite index. Then H is isomorphic to either $\mathbb{Z}^2$ or K. More precisely:*

1) *Every subgroup $H \leq K$ of finite index with $H \cong \mathbb{Z}^2$ is contained in $T(K) = \mathbb{Z}^2$ and $H = A\mathbb{Z}^2$ for an integer matrix $A \in GL_2(\mathbb{Q})$, here $[K : H] = 2|\det A|$.*

2) *Every subgroup $H \leq K$ of finite index with $H \cong K$ is of the form $H = gKg^{-1}$ for*

$$g = \left(\begin{array}{cc|c} 2n_1+1 & 0 & 0 \\ 0 & n_2 & \frac{1}{2}n_3 \end{array}\right) \in A_2(\mathbb{Q}),$$
$$[K : H] = |(2n_1+1)n_2|, \text{ where } n_i \in \mathbb{Z}.$$

Proof. By Theorem 2 item 1), H is a plane group. The point group $P(H)$ is either trivial or is equal to $P(K)$. The only plane groups with such property are the groups $p1$, pm and pg in IUC notation. The group $p1$ is isomorphic to $\mathbb{Z}^2$, and the group pm is not torsion-free and cannot be a subgroup of K.

If $H \cong \mathbb{Z}^2$ then $P(H) = E$ and $H = T(H) \leq T(K) = \mathbb{Z}^2$. Every subgroup of $\mathbb{Z}^2$ of rank two is of the form $A\mathbb{Z}^2$ for an integer matrix $A \in GL_2(\mathbb{Q})$ and has finite index $2|\det(A)|$ in K.

If $H \cong K$, then H and K are conjugate in the affine group $A_2(\mathbb{Q})$ by the Bieberbach theorem. Let us determine the elements $g \in A_2(\mathbb{Q})$ such that $gKg^{-1} \leq K$. Write $g = (A|t)$, then A belongs to the normalizer of $P(K)$, which consists of diagonal matrices. Put $A = diag(d_1, d_2)$ and $t = (a_1, a_2)$, and conjugate elements of K:

$$g\left(\begin{array}{cc|c} 1 & 0 & n \\ 0 & 1 & m \end{array}\right)g^{-1} = \left(\begin{array}{cc|c} 1 & 0 & d_1n \\ 0 & 1 & d_2m \end{array}\right) \in K,$$
$$g\left(\begin{array}{cc|c} 1 & 0 & n+1/2 \\ 0 & -1 & m \end{array}\right)g^{-1} = \left(\begin{array}{cc|c} 1 & 0 & d_1(n+1/2) \\ 0 & -1 & d_2m+2a_2 \end{array}\right) \in K.$$

for all $n, m \in \mathbb{Z}$. It follows that $d_1, d_2 \in \mathbb{Z}$, $2a_2 \in \mathbb{Z}$, d_1 should be odd, and it is enough to consider t with $a_1 = 0$.

In order to describe simple virtual endomorphisms of K, we will use the structure of its normal subgroups.

Proposition 1. A subgroup $N \leq \mathbb{Z}^2$ is normal in K if and only if $N = \langle (n, m), (n, -m) \rangle$ or $N = \langle (n, 0), (0, m) \rangle$ for some $n, m \in \mathbb{Z}$. The subgroup N has a finite index when $n, m \neq 0$.

Proof. A subgroup $N \leq \mathbb{Z}^2$ is normal in K if $b^{-1}Nb \leq N$. We compute:

$$b^{-1}\left(\begin{array}{cc|c} 1 & 0 & n \\ 0 & 1 & m \end{array}\right)b = \left(\begin{array}{cc|c} 1 & 0 & n \\ 0 & 1 & -m \end{array}\right).$$

Therefore, $N \leq \mathbb{Z}^2$ is normal in K when $(n,m) \in N$ whenever $(n,-m) \in N$. Hence, the subgroups $\langle (n,m),(n,-m)\rangle$ and $N = \langle (n,0),(0,m)\rangle$ are normal in K for all $n,m \in \mathbb{Z}$.

Conversely, let $N \leq \mathbb{Z}^2$ be normal in K. If N is cyclic, then $N = \langle (n,0)\rangle$ or $N = \langle (0,m)\rangle$ (otherwise the property above does not hold). Assume N contains elements $(n,0)$ and $(0,m)$ for some $n,m \geq 1$; let $n,m \geq 1$ be the smallest numbers with this property. If $N \neq \langle (n,0),(0,m)\rangle$, then N contains an element (k,l) for $0 < k < n$, $0 < l < m$. Notice that such an element is unique. Then $(2k,0),(0,2l) \in N$ and $(2k-n,0),(0,2l-m) \in N$. By the minimality of n,m, we get $n = 2k$ and $m = 2l$. By adding/subtracting $(n,0)$, $(0,m)$ from any given element of N, we can obtain either (k,l) or $(0,0)$. Hence $N = \langle (k,l),(k,-l)\rangle$.

4. Virtual endomorphisms of the group pg

Let us describe virtual endomorphisms $\phi : H \to K$ of the group K. We consider separately the cases when $H \cong \mathbb{Z}^2, K$ and ϕ is injective/non-injective. In each case, we define the matrix $B_\phi \in M_2(\mathbb{Q})$ that will be used to determine the simplicity of ϕ.

(1) Let $H \cong \mathbb{Z}^2$. Then $H \leq \mathbb{Z}^2$ and $H = A\mathbb{Z}^2$ for an integer matrix $A \in GL_2(\mathbb{Q})$.

(1a) If ϕ is injective, then $Im(\phi) \leq T(K) = \mathbb{Z}^2$ and $\phi : H \to \mathbb{Z}^2$ is of the form

$$\phi(x) = Bx \quad \text{for } B \in GL_2(\mathbb{Q}),$$

where B is admissible whenever BA has integer coefficients. We put $B_\phi = B$.

(1b) If ϕ is not injective, then $Im(\phi)$ is cyclic, because K is torsion-free. Put $H = \langle a_1, b_1\rangle$ and $Im(\phi) = \langle g\rangle$ for $g \in K$. Then $\phi(a_1) = g^n$, $\phi(b_1) = g^m$ and $\phi : H \to K$ is of the form:

$$\phi(a_1^k b_1^l) = g^{kn+lm}, \quad k,l \in \mathbb{Z}, \tag{4.1}$$

where all $g \in K$ and $n,m \in \mathbb{Z}$ are admissible. Let $g^2 = (a_2,b_2) \in \mathbb{Z}^2$ and put

$$B_\phi = \tfrac{1}{2}\begin{pmatrix} na_2 & ma_2 \\ nb_2 & mb_2 \end{pmatrix} A^{-1}.$$

Then $\phi(x) = B_\phi x$ for every $x \in 2A\mathbb{Z}^2$. Note that if $g \notin T(K)$, then $b_2 = 0$.

(2) Let $H \cong K$. Then $H = g_1 K g_1^{-1}$ for some

$$g_1 = \left(\begin{array}{cc|c} 2n_1+1 & 0 & 0 \\ 0 & n_2 & \frac{1}{2}n_3 \end{array}\right) \in A_2(\mathbb{Q}),$$

where $n_i \in \mathbb{Z}$, $n_2 \neq 0$.

(2a) If ϕ is injective, then ϕ is an isomorphism between two crystallographic groups H and $Im(\phi)$, and by the Bieberbach theorem ϕ is of the form

$$\phi(x) = gxg^{-1} \quad \text{for } g \in A_2(\mathbb{Q}),$$

where g is such that $gg_1Kg_1^{-1}g^{-1} \le K$. The admissible g is of the form

$$g = \left(\begin{matrix} d_1 & 0 \\ 0 & d_2 \end{matrix} \middle| \begin{matrix} 0 \\ a_2 \end{matrix}\right) \in A_2(\mathbb{Q}),$$

where $d_1, d_2 \in \mathbb{Q}^*$, $a_2 \in \mathbb{Q}$ satisfy the conditions:

$$\begin{gathered} d_1(2n_1+1),\ d_2n_2,\ 2a_2 + d_2n_3 \in \mathbb{Z} \\ \text{and}\ \ d_1(2n_1+1)\ \text{ is odd.} \end{gathered} \tag{4.2}$$

Put $B_\phi = diag(d_1, d_2)$, the linear part of g. Then $\phi(x) = B_\phi(x)$ for $x \in H \cap \mathbb{Z}^2$.

(2b) If ϕ is not injective, then, as in the case (1b), put $H = \langle a_1, b_1 \rangle$, and $\phi : H \to K$ is of the form (4.1). By checking the defining relation, we get

$$g^n g^m g^n = g^m \ \Rightarrow\ g^{2n} = e \ \Rightarrow\ n = 0,$$

because K is torsion-free. Therefore, ϕ is of the form

$$\phi(a_1^k b_1^l) = g^{lm} \text{ for } k, l \in \mathbb{Z},$$

where every $g \in K$ and $m \in \mathbb{Z}$ are admissible. In this case ϕ is never simple (see Remark 2 below), but we still define

$$B_\phi = \tfrac{1}{n_2} \begin{pmatrix} 0 & ma_2 \\ 0 & mb_2 \end{pmatrix},$$

where $g^2 = (a_2, b_2) \in \mathbb{Z}^2$. Then $\phi(x) = B_\phi(x)$ for $x \in H \cap \mathbb{Z}^2$.

Definition 3. The *ϕ-core* of a virtual endomorphism $\phi : H \to K$ is the maximal normal ϕ-invariant subgroup N of K.

A virtual endomorphism ϕ is called *simple* if the ϕ-core is trivial, that is, there are no nontrivial ϕ-invariant subgroups that are normal in K.

Lemma 1. *Let $B \in M_2(\mathbb{Q})$ and $N \le \mathbb{Z}^2$ be the maximal B-invariant subgroup, i.e., $BN \le N$. Then:*

1) N has finite index in $\mathbb{Z}^2$ if and only if $\chi_B(x)$ has integer coefficients;

2) N is infinite cyclic if and only if exactly one eigenvalue of B is an integer;

3) N is trivial if and only if $\chi_B(x) \notin \mathbb{Z}[x]$ and $\chi_B(x)$ has no integer roots.

Proof. 1) If $\chi_B(x) = x^2 + ax + b \in \mathbb{Z}[x]$, then $B^2 = -bE - aB$, and $H = \langle v, Bv \rangle$ is B-invariant for every $v \in \mathbb{Z}^2$. Hence N has a finite index. Conversely, let $N = \langle v, u \rangle$ for linearly independent $v, u \in \mathbb{Z}^2$. Since $BN \le N$, the matrix of B in the basis (v, u) has integer coefficients, and hence $\chi_B(x) \in \mathbb{Z}[x]$.

2) The nontrivial subgroup $\langle v \rangle$ is B-invariant if and only if v is an eigenvector of B with integer eigenvalue. The other eigenvalue is non-integer, since otherwise $\chi_B(x) \in \mathbb{Z}[x]$ and N is of finite index by item 1).

The item 3) follows immediately from the items 1) and 2) (also, see Theorem 2.9.2 in [11]).

Let us determine which matrices preserve a normal subgroup of K.

Lemma 2. *Let $B \in M_2(\mathbb{Q})$ and $n, m \in \mathbb{Z} \setminus \{0\}$. Then:*

1) $H = \langle (n,0),(0,m) \rangle$ is B-invariant if and only if B is of the form

$$\begin{pmatrix} \alpha & \frac{n}{m}\beta \\ \frac{m}{n}\gamma & \delta \end{pmatrix} \tag{4.3}$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$.

2) $H = \langle (n,m),(n,-m) \rangle$ is B-invariant if and only if B is of the form

$$\frac{1}{2}\begin{pmatrix} \alpha+\beta+\gamma+\delta & \frac{n}{m}(\alpha+\beta-\gamma-\delta) \\ \frac{m}{n}(\alpha-\beta+\gamma-\delta) & \alpha-\beta-\gamma+\delta \end{pmatrix} \tag{4.4}$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$.

3) $H = \langle (n,0) \rangle$ or $H = \langle (0,m) \rangle$ is B-invariant if and only if B is of the form

$$\begin{pmatrix} k & b_1 \\ 0 & b_2 \end{pmatrix} \text{ or } \begin{pmatrix} b_1 & 0 \\ b_2 & k \end{pmatrix} \tag{4.5}$$

for $k \in \mathbb{Z}$, $b_i \in \mathbb{Q}$.

Proof. A subgroup $H = A\mathbb{Z}^2$ for an integer matrix $A \in GL_2(\mathbb{Q})$ is B-invariant if and only if $A^{-1}BA = C$ has integer coefficients. Then we obtain 1),2) by direct computation of $B = ACA^{-1}$ for an integer matrix C and

$$A = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix} \text{ and } \begin{pmatrix} n & n \\ m & -m \end{pmatrix}.$$

The item 3) implies that $(1,0)$ or $(0,1)$ is an eigenvector of B with an integer eigenvalue, and B has the required form.

The next theorem determines simple virtual endomorphisms of K.

Theorem 4. *Let $\phi : H \to K$ be a virtual endomorphism and $B_\phi \in M_2(\mathbb{Q})$ the associated matrix. Then ϕ is simple if and only if B_ϕ is not of the forms (4.3), (4.4), (4.5).*

Proof. In all the cases $(1a),(1b),(2a),(2b)$, the matrix B_ϕ has the following property: there exists $H_1 \leq H \cap \mathbb{Z}^2$ of finite index such that

$$\phi|_{H_1} : H_1 \to K,\ \phi(x) = B_\phi x \text{ for } x \in H_1.$$

If B_ϕ has one of the forms (4.3), (4.4), (4.5), then there exists a nontrivial B_ϕ-invariant subgroup $N \leq \mathbb{Z}^2$ that is normal in K. Since H_1 is of finite index, there exists $d \in \mathbb{N}$ such that $dN \leq H_1$. Then dN is a normal ϕ-invariant subgroup, and ϕ is not simple.

Conversely, assume the ϕ-core N is nontrivial. If ϕ is injective, then $N_1 = N \cap \mathbb{Z}^2$ is a nontrivial ϕ-invariant subgroup, and it is normal in K as an intersection of normal subgroups. Choose $d \in \mathbb{N}$ such that $dN_1 \leq H_1$. Then dN_1 is B_ϕ-invariant and normal in K. Hence B_ϕ has one of the forms (4.3), (4.4), (4.5).

If ϕ is not injective, then ϕ is of the form (1b) or (2b). In the case (1b), $N \leq H \leq \mathbb{Z}^2$ and we can repeat the same arguments as above. In the case (2b), the matrix B_ϕ always has the form (4.5), and ϕ is never simple (here $\phi(a_1) = 0$ for $a_1 = (0, n_2)$ and $\langle (0, n_2) \rangle$ is a normal ϕ-invariant subgroup).

Corollary 1. *If B_ϕ satisfies the condition of Lemma 1 item 3), then ϕ is simple.*

Remark 1. In the case (2a), the matrix $B_\phi = diag(d_1, d_2)$ satisfies the condition of Theorem 4 if and only if $d_1, d_2 \notin \mathbb{Z}$.

Remark 2. In the case (2b), the matrix B_ϕ always has the form (4.5), and ϕ is never simple.

Let us construct a few examples.

Example 1. The case (1a). Let $H = 2\mathbb{Z} \times \mathbb{Z}$ and $\phi_i : H \to \mathbb{Z}^2$, $\phi_i(x) = B_i x$, $i = 1, 2$, where

$$B_1 = \begin{pmatrix} 1/2 & 3 \\ 0 & -2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & -2 \\ 1/2 & -3 \end{pmatrix}.$$

The endomorphism ϕ_1 is simple, but ϕ_2 is not, here $\langle (2,1), (2,-1) \rangle$ is a normal ϕ_2-invariant subgroup.

Example 2. The case (1b). Let $H = \langle ab^2, ab^{-2} \rangle = \langle (1,1), (-1,1) \rangle$ and $\phi : H \to K$, $\phi((1,1)) = g^3$ and $\phi((-1,1)) = g^{-2}$ for

$$g = \left(\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & -1 & 0 \end{array}\right), \quad \text{here } B_\phi = \begin{pmatrix} 5/2 & 1/2 \\ 0 & 0 \end{pmatrix}.$$

Then ϕ is simple.

Example 3. The case (2a). Let $H = gKg^{-1}$ and $\phi : H \to K$, $\phi(x) = g^{-1}xg$ for

$$g = \left(\begin{array}{cc|c} 3 & 0 & 0 \\ 0 & 1 & 1/2 \end{array}\right), \quad \text{here } B_\phi = \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then ϕ is not simple, here $\langle (0,1) \rangle$ is a normal ϕ-invariant subgroup.

5. Self-similar actions of the group pg

Let X be an alphabet. Let X^* be the free monoid generated by X, that is, the space of finite words over X with the operation of concatenation.

Definition 4. A faithful action of a group G on the space X^* is called *self-similar* if for every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w) \text{ for every } w \in X^*.$$

The size of the alphabet $d = |X|$ is called the *degree* of a self-similar action.

The element h is uniquely determined by g and x, and it is called the *section of g at x*, denoted $g|_x := h$. The section of an element can be defined at every $v \in X^*$ recursively by the rule $g|_{xw} = (g|_x)|_w$ for $x \in X, w \in X^*$.

For every $x \in X$, the map

$$\phi_x : St_G(x) \to G,\ \phi_x(g) = g|_x$$

is a virtual endomorphism of G, here $St_G(x)$ is the stabilizer of x and $[G : St_G(x)] \le |X|$.

Definition 5. A self-similar action (G, X^*) is called *transitive* if G acts transitively on X.

Definition 6. A transitive self-similar action (G, X^*) is called *self-replicating* if the associated virtual endomorphism ϕ_x is surjective for some (equiv., every) $x \in X$.

For transitive self-similar actions, the associated virtual endomorphisms ϕ_x are simple. And vice versa, if $\phi : H \to G$ is a simple virtual endomorphism of a group G, then G admits a transitive self-similar action of degree $d = [G : H]$ (see Prop. 2.7.5 in [11]). The construction is the following. Choose a set D of coset representatives for H in G (called a *digit set*), and identify the alphabet X with D. The action (G, X^*) is defined recursively: for $g \in G$ and $x \in X$,

$$g(xw) = yh(w) \text{ for } w \in X^*,$$

where $y \in X$ is the unique element such that $y^{-1}gx \in H$ and $h = \phi(y^{-1}gx)$.

We determine possible degrees for transitive self-similar actions of K.

Theorem 5. *1) The group K admits a transitive self-similar action of degree d if and only if $d \ge 2$ is not an odd prime.*

2) The group K admits a self-replicating action of degree d if and only if $d \ge 6$ is not a prime or a power of 2.

Proof. We need to determine which indices $d = [K : H]$ are possible for simple virtual endomorphisms $\phi : H \to K$.

For $d = 2$, a simple endomorphism is constructed in Example 4 below.

For every $d = 2n$, $n \ge 2$, put $H = n\mathbb{Z} \times \mathbb{Z}$ and consider $\phi : H \to K$,

$$\phi(x) = \begin{pmatrix} 0 & 1 \\ 1/n & 0 \end{pmatrix} x.$$

Then ϕ corresponds to the case (1a), and the matrix B_ϕ satisfies item 3) of Lemma 1. Hence, ϕ is simple by Corollary 1.

An odd d is possible only in the case (2a). Here $d = |(2n_1 + 1)n_2|$ and $B_\phi = diag(d_1, d_2)$, where the coefficients satisfy the restrictions (4):

$$\begin{gathered} d_1(2n_1+1),\ d_2n_2,\ 2a_2+d_2n_3 \in \mathbb{Z}, \\ d_1(2n_1+1) \ \text{ is odd, } \ \text{and} \\ d_1, d_2 \notin \mathbb{Z},\ n_i \in \mathbb{Z}, n_2 \neq 0,\ a_2 \in \mathbb{Q}. \end{gathered}$$

We get $n_2 \neq \pm 1$, $2n_1 + 1 \neq \pm 1$, and d cannot be prime or a power of 2. Conversely, for $n_2 \neq \pm 1$, $2n_1 + 1 \neq \pm 1$, the numbers

$$d_1 = \frac{1}{2n_1+1},\ d_2 = \frac{1}{n_2},\ a_2 = -\frac{1}{2}d_2n_3$$

satisfy the restrictions, and ϕ is simple.

2) Let ϕ be surjective. Then H is isomorphic to K, and by Remark 2, ϕ could be simple only in the case when it is injective. Then ϕ is an affine conjugacy:

$$\phi : gKg^{-1} \to K,\ \phi(x) = g^{-1}xg,$$

where $g \in A_2(\mathbb{Q})$ is of the form

$$g = \left(\begin{array}{cc|c} d_1 & 0 & 0 \\ 0 & d_2 & a_2 \end{array}\right)$$

for $d_1, d_2 \in \mathbb{Z} \setminus \{0\}$, d_1 is odd, and $2a_2 \in \mathbb{Z}$ (see the proof of Theorem 3). The ϕ is simple when $1/d_1, 1/d_2 \notin \mathbb{Z}$ (see the case (2a)). Since the degree is $d = [K : gKg^{-1}] = |d_1d_2|$, the result follows.

Example 4. Let $H = \mathbb{Z}^2$ and $\phi : H \to K$ be given by $\phi(a) = b$, $\phi(b^2) = b$. Then ϕ is not injective, here $Ker(\phi) = \langle ab^{-2}\rangle$. The ϕ restricted to $2\mathbb{Z} \times 2\mathbb{Z}$ is of the form

$$\phi(x) = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 0 \end{pmatrix} x \ \text{ for } x \in 2\mathbb{Z} \times 2\mathbb{Z},$$

The matrix satisfies the condition in Theorem 4, and hence ϕ is simple. Choose $D = \{e, b\}$. The self-similar action of K associated to (ϕ, D) is given by the following recursion over the binary alphabet $X = \{0, 1\}$:

$$\begin{aligned} a(0w) &= 0b(w), & b(0w) &= 1b(w), \\ a(1w) &= 1b^{-1}(w), & b(1w) &= 0w. \end{aligned}$$

Example 5. Let $H = 2\mathbb{Z} \times \mathbb{Z}$ and $\phi : H \to \mathbb{Z}^2$ be given by

$$\phi(x) = \begin{pmatrix} 3/2 & 1 \\ 0 & 2 \end{pmatrix} x.$$

Then ϕ is not simple as a virtual endomorphism of $\mathbb{Z}^2$, here $N = \langle (2,1) \rangle$ is a ϕ-invariant subgroup. However, ϕ is simple as a virtual endomorphism of K. Choose $D = \{e, b, b^2, b^3\}$. The associated self-similar action of K is defined by the following recursion over $X = \{1, 2, 3, 4\}$:

$$\begin{aligned} a(1w) &= 1(a^2b^2)(w), & b(1w) &= 2w, \\ a(2w) &= 2(a^{-2}b^{-2})(w), & b(2w) &= 3w, \\ a(3w) &= 3(a^2b^2)(w), & b(3w) &= 4w, \\ a(4w) &= 4(a^{-2}b^{-2})(w), & b(4w) &= 1b^6(w). \end{aligned}$$

Example 6. Let us construct a self-replicating action of degree $d = 6$. We define

$$\begin{aligned} &\phi : gKg^{-1} \to K, \\ &\phi(x) = g^{-1}xg, \end{aligned} \quad \text{for} \quad g = \left(\begin{array}{cc|c} 3 & 0 & 0 \\ 0 & 2 & 1/2 \end{array}\right).$$

Then ϕ is simple, here

$$B_\phi = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Choose $D = \{e, a, b, b^2, ab, ab^2\}$. The respective self-replicating action over the alphabet $X = \{1, 2, 3, 4, 5, 6\}$:

$$\begin{aligned} a(1w) &= 2w, & b(1w) &= 3w, \\ a(2w) &= 1a(w), & b(2w) &= 5a(w), \\ a(3w) &= 5w, & b(3w) &= 4w, \\ a(4w) &= 6w, & b(4w) &= 2(a^{-1}b)(w), \\ a(5w) &= 3a^{-1}(w), & b(5w) &= 6a^{-1}(w), \\ a(6w) &= 4a(w), & b(6w) &= 1(a^{-1}b)(w). \end{aligned}$$

Definition 7. A self-similar action (G, X^*) is called *finite-state* if for every $g \in G$ the set of its sections $S(g) = \{g|_v : v \in X^*\}$ is finite.

Definition 8. A self-similar action (G, X^*) is called *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all $v \in X^*$ of length $\geq n$.

A contracting action is finite-state, but not vice versa. A finitely generated group G admits a finite-state action if and only if G can be generated by a finite Mealy automaton. The states of the automaton are the sections of generators.

The next theorem characterizes contracting and finite-state actions of the group K.

Theorem 6. *Let (K, X^*) be a transitive self-similar action and B_ϕ the matrix of the associated virtual endomorphism ϕ.*

1) The (K, X^*) is contracting if and only if the eigenvalues of B_ϕ are less than 1 in modulus.

2) If (K, X^*) is self-replicating, then (K, X^*) is finite-state if and only if it is contracting.

3) The group K admits a transitive finite-state (contracting) action of degree d if and only if $d \geq 2$ is not an odd prime.

Proof. 1) The spectral radius of B is less than 1 if and only if the contacting coefficient of ϕ is less than 1 in the sense of Def. 2.11.9 from [11], and we can apply Prop. 2.11.11 from [11].

2) In this case, $B_\phi = diag(d_1, d_2)$ and $d_1, d_2 \notin \mathbb{Z}$, see Remark 1. If $|d_i| < 1$, then the action is contracting and finite-state. Conversely, if $|d_1| > 1$ or $|d_2| > 1$, then either a or b^2 has infinitely many sections.

3) It is sufficient to notice that all the actions constructed in the proof of Theorem 5 item 1) are finite-state and contracting.

The actions in Examples 4 and 6 are finite-state and contracting, but it is not finite-state in Example 5. The next examples demonstrate that the action of K can be finite-state and not contracting, in contrast to free abelian groups, and the property of being finite-state depends not only on the virtual endomorphism ϕ, in contract to the contracting property, but also on the choice of a digit set D.

Example 7. Let $H = A\mathbb{Z}^2$ and $\phi : H \to \mathbb{Z}^2$, $\phi(x) = Bx$, where

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}.$$

Then ϕ is a simple virtual endomorphism of K, the degree is 4. For the digit set $D = \{e, a, b, ba\}$, we have the self-similar action over $X = \{1, 2, 3, 4\}$:

$$\begin{aligned} a(1w) &= 2w, & b(1w) &= 3w, \\ a(2w) &= 1a(w), & b(2w) &= 4w, \\ a(3w) &= 4a^{-1}(w), & b(3w) &= 2b^2(w), \\ a(4w) &= 3w, & b(4w) &= 1(b^2a)(w). \end{aligned}$$

This action is finite-state, here

$$S(a) = \{e, a, a^{-1}\} \quad \text{and} \quad S(b) = \{e, b, b^2, b^2a\}.$$

However, the action is not contacting, because the element $g = b^2a = (1, 1)$ satisfies

$$g(1) = 1, g|_1 = g \quad \Rightarrow \quad g^n|_1 = g^n \quad \text{for } n \in \mathbb{N}.$$

For the digit set $D = \{e, b, b^2, b^3\}$, we get:

$$
\begin{aligned}
a(1w) &= 3b^{-2}(w), & b(1w) &= 2w,\\
a(2w) &= 4(a^{-1}b^{-2})(w), & b(2w) &= 3w,\\
a(3w) &= 1(ab^2)(w), & b(3w) &= 4w,\\
a(4w) &= 2b^2(w), & b(4w) &= 1(ab^4)(w).
\end{aligned}
$$

This action is not finite-state: the sections of b^2 along the word $v = 33\ldots3$ are

$$b^2 \longrightarrow ab^4 \longrightarrow a^2b^6 \longrightarrow a^3b^8 \longrightarrow a^4b^{10} \longrightarrow \ldots.$$

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