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A study on unification of generalized hypergeometric function and Mittag-Leffler function with certain integral transforms of generalized basic hypergeometric function [1]

Abstract. This research article explores some new properties of generalized hypergeometric function and its q-analogue. The connections between ${}_2R_1{}^{\upsilon}(\mathfrak{z})$, the Wright function, and generalized Mittag-Leffler functions are explored. The authors introduce the q-analogue of generalized hypergeometric function denoted by ${}_2R_1{}^{\upsilon,q}(\mathfrak{z})$ and discuss its properties and connections with q-Wright function and q-versions of generalized Mittag-Leffler functions. We get the q-integral transforms such as q-Mellin, q-Euler (beta), q-Laplace, q-sumudu, and q-natural transforms of Wright-type generalized q-hypergeometric function. This article contributes to the understanding of hypergeometric functions in q-calculus.

Key words: basic hypergeometric functions in one variable ${}_r\phi_s$, q-calculus and related topics, Mittag-Leffler functions and generalizations, integral transforms of special functions

Анотація. В даній науковій статті досліджуються деякі нові властивості узагальненої гіпергеометричної функції та її q-аналога. Досліджуються зв'язки між ${}_2R_1{}^{\upsilon}(\mathfrak{z})$, функцією Райта та узагальненими функціями Міттаґ-Леффлера. Автори вводять q-аналог узагальненої гіпергеометричної функції, позначеної як ${}_2R_1{}^{\upsilon,q}(\mathfrak{z})$, та обговорюють її властивості та зв'язки з q-функцією Райта та q-версіями узагальнених функцій Міттаґ-Леффлера. Ми отримуємо q-інтегральні перетворення, такі як q-Меллінове, q-Ейлерове (бета), q-Лапласове, q-сумуду, та q-натуральні перетворення узагальненої q-гіпергеометричної функції типу Райта. Ця стаття сприяє розумінню гіпергеометричних функцій у q-численні.

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Ключові слова: основні гіпергеометричні функції в одній змінній ${}_r\phi_s$, q-числення та пов'язані теми, функції Міттаґ-Леффлера та їх узагальнення, інтегральні перетворення спеціальних функцій

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1. Introduction

C. F. Gauss [10] derived the hypergeometric function as

$$ {}_2F_1(\mu,\eta;\lambda;\mathfrak{z}) = \sum_{\ell=0}^{\infty} \frac{(\mu)_\ell(\eta)_\ell}{(\lambda)_\ell}\frac{\mathfrak{z}^\ell}{\ell!} \tag{1.1}$$

Heine [12, 13] derived the q-analogue of ${}_2F_1$ (also known as basic hypergeometric function) as

$$ {}_2\phi_1(\mu,\eta;\lambda;q,\mathfrak{z}) = \sum_{\ell=0}^{\infty} \frac{(\mu;q)_\ell(\eta;q)_\ell}{(\lambda;q)_\ell(q;q)_\ell}\mathfrak{z}^\ell, \tag{1.2}$$

where $\Re(\mu),\Re(\eta),\Re(\lambda) > 0, 0 < |q| < 1$ and $|\mathfrak{z}| < 1$. Ernst [6] reintroduced (1.2) by taking q^μ instead of μ and gave the new notation $(q^\mu;q)_\ell \equiv \langle\mu;q\rangle_\ell$ to make the expression more natural and gave the comprehensive treatment of q-calculus.

The generalization of (1.1) is derived as [27]

$$ {}_2R_1{}^{\upsilon}(\mathfrak{z}) \equiv {}_2R_1(\mu,\eta;\lambda;\upsilon;\mathfrak{z}) = \frac{\Gamma(\lambda)}{\Gamma(\eta)}\sum_{\ell=0}^{\infty} \frac{(\mu)_\ell\Gamma(\mu+\upsilon\ell)}{\Gamma(\lambda+\upsilon\ell)}\frac{\mathfrak{z}^\ell}{\ell!}, \tag{1.3}$$

where $\Re(\mu),\Re(\eta),\Re(\lambda) > 0, \upsilon > 0$ and $|\mathfrak{z}| < 1$. Rao et al. [19–24] studied extensively this generalization, which also has been used effectively in the study of generalization of Jacobi polynomial [28]. In this article, we derived some important connections of ${}_2R_1{}^{\upsilon}(\mathfrak{z})$ with Wright function [31], Mittag-Leffler (M-L) function [16] and its generalizations [17, 26, 30].

In this sequel of study, we established the q-extension of ${}_2R_1{}^{\upsilon}(\mathfrak{z})$ which is denoted and defined as below in (1.4).

$$ {}_2R_1{}^{\upsilon,q}(\mathfrak{z}) \equiv {}_2R_1(\mu,\eta;\lambda;\upsilon;q,\mathfrak{z}) = \frac{\Gamma_q(\lambda)}{\Gamma_q(\eta)}\sum_{\ell=0}^{\infty} \frac{\langle\mu;q\rangle_\ell}{\langle 1;q\rangle_\ell}\frac{\Gamma_q(\eta+\upsilon\ell)}{\Gamma_q(\lambda+\upsilon\ell)}\mathfrak{z}^\ell, \tag{1.4}$$

where $\Re(\mu),\Re(\eta),\Re(\lambda) > 0, \upsilon > 0, 0 < |q| < 1$ and $|\mathfrak{z}| < 1$. Also, Some properties of ${}_2R_1{}^{\upsilon,q}(\mathfrak{z})$ and its relations between q-wright function [4], q-M-L function and its generalizations [4, 8] have been included in this article.

Jackson [14] provided a thorough explanation of q-Jackson integral and methodically created a q-calculus. Due to its shown applicability in numerous, at first glance disparate domains of science and engineering, q-calculus has consequently grown in significance and popularity. In theory of quantum calculus, some significant integral transforms have various q-analogues. Among those, we are introducing the integral transforms of ${}_2R_1{}^{\upsilon,q}(\mathfrak{z})$ which are q-Euler (beta) transform [9], q-extension of Laplace transform [11], the quantum sumudu integral transform [1, 2] and quantum natural transform [3].

2. Preliminaries and Notations

Here, we mentioned some definition and important results that have been used in our study.

Definition 1. For $\mathfrak{z} \in \mathbb{C}$ and $0 < |q| < 1$, the q-shifted factorial is given by [9]

$$(\mathfrak{z}; q)_{\mathfrak{r}} = \begin{cases} 1, \mathfrak{r} = 0 \\ \prod\limits_{\ell=0}^{\mathfrak{r}-1} \left(1 - \mathfrak{z} q^{\ell}\right), \mathfrak{r} \in \mathbb{N} \end{cases} \tag{2.1}$$

Ernst [6] reintroduced the q-shifted factorial for $\mathfrak{z} \in \mathbb{C}$ as below;

$$\langle \mathfrak{z}; q \rangle_{\mathfrak{r}} = \begin{cases} 1, \mathfrak{r} = 0 \\ \prod\limits_{\ell=0}^{\mathfrak{r}-1} \left(1 - q^{\mathfrak{z}+\ell}\right), \mathfrak{r} \in \mathbb{N} \end{cases} \tag{2.2}$$

Here, observe that, $\langle \mathfrak{z}; q \rangle_{\ell} \equiv (q^{\mathfrak{z}}; q)_{\ell}$. For $[\mathfrak{z}]_q \cdot [\mathfrak{z}+1]_q \cdot \ldots \cdot [\mathfrak{z}+\ell-1]_q$, here we adopt the notation $[\mathfrak{z}]_q^{\ell}$ (namely KS-q-pochhammer symbol), thus

$$[\mathfrak{z}]_q^{\ell} = [\mathfrak{z}]_q \cdot [\mathfrak{z}+1]_q \cdot \ldots \cdot [\mathfrak{z}+\ell-1]_q, \text{where } [\mathfrak{z}]_q = \frac{1-q^{\mathfrak{z}}}{1-q}. \tag{2.3}$$

Definition 2. For $|\mathfrak{z}| < 1$ and $0 < |q| < 1$, the q-extension of binomial series is given by [6, 9]

$${}_1\phi_0\left(\mu; -; q, \mathfrak{z}\right) \equiv \sum_{\ell=0}^{\infty} \frac{\langle \mu; q \rangle_{\ell}}{\langle 1; q \rangle_{\ell}} \mathfrak{z}^{\ell} = \frac{(q^{\mu}\mathfrak{z}; q)_{\infty}}{(q; q)_{\infty}} = \frac{1}{(\mathfrak{z}; q)_{\mu}}. \tag{2.4}$$

Definition 3. The q-integration of $f(\mathfrak{z})$ is defined as [9]

$$\int_{\mathfrak{a}}^{\mathfrak{b}} f(\mathfrak{z})\, d_q\mathfrak{z} = \int_0^{\mathfrak{b}} f(\mathfrak{z})\, d_q\mathfrak{z} - \int_0^{\mathfrak{a}} f(\mathfrak{z})\, d_q\mathfrak{z}, \tag{2.5}$$

where

$$\int_0^{\mathfrak{n}} f(\mathfrak{z})\, d_q\mathfrak{z} = \mathfrak{n}(1-q) \sum_{\ell=0}^{\infty} f\left(\mathfrak{n} q^{\ell}\right) q^{\ell}. \tag{2.6}$$

Definition 4. For $\mathfrak{z} \in \mathbb{C}$ and $0 < |q| < 1$, the q-formula of gamma function is given as [9]

$$\Gamma_q(\mathfrak{z}) = \frac{\langle 1; q\rangle_\infty}{\langle \mathfrak{z}; q\rangle_\infty}(1-q)^{1-\mathfrak{z}} \equiv \frac{(q;q)_\infty}{(q^{\mathfrak{z}};q)_\infty}(1-q)^{1-\mathfrak{z}}, \tag{2.7}$$

$\Gamma_q(\mathfrak{z})$ has poles at $\mathfrak{z} = 0, -1, -2, ...$ The integral expression of q-gamma is given as [7]

$$\Gamma_q(\mathfrak{z}) = q^{-\frac{\mathfrak{z}(\mathfrak{z}+1)}{2}} \int_0^\infty \mathfrak{x}^{\mathfrak{z}-1} e_q(-\mathfrak{x})\, d_q\mathfrak{x}, \tag{2.8}$$

where $e_q(\mathfrak{x})$, the q-exponential function, is defined as [7]

$$e_q(\mathfrak{x}) = \sum_{\mathfrak{j}=0}^{\infty} \frac{\mathfrak{x}^{\mathfrak{j}} \cdot (1-q)^{\mathfrak{j}}}{(q;q)_{\mathfrak{j}}}. \tag{2.9}$$

The relation between q-gamma and q-shifted factorial is given by [5]

$$\langle \mathfrak{z}; q\rangle_{\mathfrak{n}} \equiv (q^{\mathfrak{z}};q)_{\mathfrak{n}} = \frac{\Gamma_q(\mathfrak{z}+\mathfrak{n})}{\Gamma_q(\mathfrak{z})}(1-q)^{\mathfrak{n}}. \tag{2.10}$$

Definition 5. The q-expression of beta function is given as [6, 9]

$$B_q(\mathfrak{z}_1, \mathfrak{z}_2) = (1-q)\sum_{\ell=0}^{\infty} \frac{\langle \ell+1; q\rangle_\infty}{\langle \ell+\mathfrak{z}_2; q\rangle_\infty} q^{\ell\mathfrak{z}_1} \equiv (1-q)\sum_{\ell=0}^{\infty} \frac{(q^{\ell+1};q)_\infty}{(q^{\ell+\mathfrak{z}_2};q)_\infty} q^{\ell\mathfrak{z}_1}, \tag{2.11}$$

where $\Re(\mathfrak{z}_1), \Re(\mathfrak{z}_2) > 0$ and its integral expression is given as [9]

$$B_q(\mathfrak{z}_1, \mathfrak{z}_2) = \int_0^1 \mathfrak{x}^{\mathfrak{z}_1-1} \frac{(\mathfrak{x}q;q)_\infty}{(\mathfrak{x}q^{\mathfrak{z}_2};q)_\infty} d_q\mathfrak{x}. \tag{2.12}$$

The q-beta function is described in terms of q-gamma function as [9]

$$B_q(\mathfrak{z}_1, \mathfrak{z}_2) = \frac{\Gamma_q(\mathfrak{z}_1)\Gamma_q(\mathfrak{z}_2)}{\Gamma_q(\mathfrak{z}_1+\mathfrak{z}_2)}. \tag{2.13}$$

Definition 6. For $\mathfrak{h} \in \mathbb{C}; \Re(\mathfrak{h}) > 0, 0 < |q| < 1$ and $\left|\mathfrak{z}(1-q)^{\mathfrak{h}}\right| < 1$, the q-analogue of Mittag-Leffler function is given as [4]

$$e_{\mathfrak{h}}(\mathfrak{z};q) = \sum_{\ell=0}^{\infty} \frac{\mathfrak{z}^\ell}{\Gamma_q(\ell\mathfrak{h}+1)}, \quad (\mathfrak{z} \in \mathbb{C}). \tag{2.14}$$

For $\mathfrak{h}, \mathfrak{g} \in \mathbb{C}; \Re(\mathfrak{h}) > 0, 0 < |q| < 1$ and $\left|\mathfrak{z}(1-q)^{\mathfrak{h}}\right| < 1$, Mansour [15] introduced the q-M-L function as

$$e_{\mathfrak{h},\mathfrak{g}}(\mathfrak{z};q) = \sum_{\ell=0}^{\infty} \frac{\mathfrak{z}^\ell}{\Gamma_q(\ell\mathfrak{h}+\mathfrak{g})}, \quad (\mathfrak{z} \in \mathbb{C}). \tag{2.15}$$

For $\mathfrak{z}, \mathfrak{h}, \mathfrak{g}, \mathfrak{d} \in \mathbb{C}; \Re(\mathfrak{h}) > 0, \Re(\mathfrak{d}) > 0, 0 < |q| < 1$ and $|\mathfrak{z}| < (1-q)^{-\mathfrak{h}}$, then the q-extension of M-L function of three parameter is defined as [25]

$$e^{\mathfrak{d}}_{\mathfrak{h},\mathfrak{g}}(\mathfrak{z}; q) = \sum_{\ell=0}^{\infty} \frac{(q^{\mathfrak{d}}; q)_\ell \, \mathfrak{z}^\ell}{\Gamma_q(\mathfrak{h}\ell + \mathfrak{g})(q; q)_\ell}, \quad (\mathfrak{z} \in \mathbb{C}). \tag{2.16}$$

Garg et al. [8] introduced the q-version of four parameter Mittag-Leffler function as

$$e^{\mathfrak{d},\zeta}_{\mathfrak{h},\mathfrak{g}}(\mathfrak{z}; q) = \sum_{\ell=0}^{\infty} \frac{(q^{\mathfrak{d}}; q)_{\ell\zeta} \, \mathfrak{z}^\ell}{\Gamma_q(\mathfrak{h}\ell + \mathfrak{g}) \, [\ell]!}, \quad (\mathfrak{z} \in \mathbb{C}), \tag{2.17}$$

where $\mathfrak{z}, \mathfrak{h}, \mathfrak{g}, \mathfrak{d} \in \mathbb{C}; \Re(\mathfrak{h}), \Re(\mathfrak{g}), \Re(\mathfrak{d}) > 0, \zeta \in \mathbb{N} \cup \{0\}, 0 < |q| < 1$ and $|\mathfrak{z}| < (1-q)^{-\mathfrak{h}}$. Since $\lim_{q\to 1} e^{\mathfrak{d},\zeta}_{\mathfrak{h},\mathfrak{g}}(\mathfrak{z}(1-q)^{-\mathfrak{d}}; q) = \mathcal{E}^{\mathfrak{d},\zeta}_{\mathfrak{h},\mathfrak{g}}(\mathfrak{z})$ defined by Shukla and Prajapati given as in [26].

Definition 7. For $\Re(\mathfrak{h}), \Re(\mathfrak{g}) > 0, 0 < |q| < 1$ and $|\mathfrak{z}| < (1-q)^{-\mathfrak{h}}$, the q-generalization of Wright function is given as [4]

$$w_{\mathfrak{h},\mathfrak{g}}(\mathfrak{z}; q) = \sum_{\ell=0}^{\infty} \frac{\mathfrak{z}^\ell}{\Gamma_q(\mathfrak{h}\ell + \mathfrak{g})\Gamma_q(\ell+1)}, \quad (\mathfrak{z} \in \mathbb{C}). \tag{2.18}$$

Definition 8. The q-extension of Fox-Wright function is defined by [4]

$$\begin{aligned} &{}_r\psi_s\left[\begin{array}{c} (\mathfrak{a}_1, \mathsf{A}_1), \ldots, (\mathfrak{a}_r, \mathsf{A}_r) \\ (\mathfrak{b}_1, \mathsf{B}_1), \ldots, (\mathfrak{b}_s, \mathsf{B}_s) \end{array} \mid q, \mathfrak{z}\right] \\ &= \sum_{\ell=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma_q(\mathfrak{a}_i + \mathsf{A}_i k)}{\prod_{j=1}^{s} \Gamma_q(\mathfrak{b}_j + \mathsf{B}_j k)} \frac{\left[q^{\binom{k}{2}}\right]^p \mathfrak{z}^k}{\Gamma_q(k+1)}, \end{aligned} \tag{2.19}$$

where $\Re(\mathfrak{a}_i), \Re(\mathfrak{b}_j) > 0, i = 1, 2, \ldots, r, j = 1, 2, \ldots, s$, and $p = \sum_{j=1}^{s} \mathsf{B}_j - \sum_{i=1}^{r} \mathsf{A}_i + 1 \geq 0$.

If $\sum_{j=1}^{s} \mathfrak{b}_j > \sum_{i=1}^{r} \mathfrak{a}_i - 1$ then the domain of convergence is $\sum_{j=1}^{s} \mathsf{B}j = \sum_{i=1}^{r} \mathsf{A}_i - 1$ and the series in (2.19) converges only for $|\mathfrak{z}| < 1$.

Definition 9. The q-Euler (beta) transform of $f(\mathfrak{z})$ [9]

$$\mathcal{B}_q\{f(\mathfrak{z}); \mathfrak{a}, b\} = \int_0^1 \mathfrak{z}^{\mathfrak{a}-1} \frac{(\mathfrak{z}q; q)_\infty}{(\mathfrak{z}q^{\mathfrak{b}}; q)_\infty} f(\mathfrak{z}) \, d_q\mathfrak{z}, \quad (\Re(\mathfrak{a}), \Re(\mathfrak{b}) > 0). \tag{2.20}$$

Definition 10. Hahn [11] derived the Laplace transform in q-theory as

$$ {}_q\mathfrak{L}\left(f(\mathfrak{t})\right)(\mathfrak{s}) = \frac{1}{(1-q)} \int_0^\infty e_q(-\mathfrak{s}\mathfrak{t}) f(\mathfrak{t}) d_q\mathfrak{t}, \quad (\Re(\mathfrak{s}) > 0). \tag{2.21} $$

Definition 11. The sumudu transform of $f(t)$ over the set $\mathfrak{A}_2$ in q-calculus is defined as [1, 2]

$$ {}_q\mathfrak{S}(f(\mathfrak{t}))(\mathfrak{u}) = \frac{1}{\mathfrak{u}(1-q)} \int_0^\infty f(\mathfrak{t}) e_q\left(-\frac{\mathfrak{t}}{\mathfrak{u}}\right) d_q\mathfrak{t}, \quad (\mathfrak{u} \in (\iota_1, \iota_2)), \tag{2.22} $$

where

$$ \mathfrak{A}_2 = \left\{ f(\mathfrak{t}) \,|\exists M, \iota_1, \iota_2 > 0, | f(\mathfrak{t}) |< M \cdot e_q\left(\frac{|\mathfrak{t}|}{\iota_j}\right), \mathfrak{t} \in (-1)^j \times [0, \infty) \right\}. \tag{2.23} $$

Definition 12. For $\Re(\mathfrak{s}) > 0, \mathfrak{u} \in (\iota_1, \iota_2)$, the natural transform of $f(\mathfrak{t})$ over the set defined in (2.23) (i.e., $\mathfrak{A}_2$) in q-calculus is defined as [3]

$$ {}_q\mathfrak{N}(f(\mathfrak{t}))(\mathfrak{u}; \mathfrak{s}) = \frac{1}{(1-q)} \int_0^\infty f(\mathfrak{t}) e_q\left(-\frac{\mathfrak{s}}{\mathfrak{u}}\mathfrak{t}\right) d_q\mathfrak{t}. \tag{2.24} $$

3. Wright-type generalized basic hypergeometric function

The Wright-type generalized q-hypergeometric function is defined as in (1.4).

Special Cases

1. (1.4) reduces to ${}_2R_1^{\upsilon}(\mathfrak{z})$ as $q \to 1$. Thus, ${}_2R_1^{\upsilon,q}(\mathfrak{z})$ is q-analogue of ${}_2R_1^{\upsilon}(\mathfrak{z})$.
2. By taking $\upsilon = 1$, the q-analogue of ${}_2R_1^{\upsilon}(\mathfrak{z})$ reduces to (1.2).
3. If $\lambda = \eta$ then ${}_2R_1\left(\mu, \eta; \lambda; \upsilon; q, \mathfrak{z}\right)$ reduces to the q-binomial series.
4. If $\mu = 1$ and $\lambda = \eta$ then (1.4) reduces to the geometric series.

3.1. Convergence of ${}_2\mathbf{R}_1^{\upsilon,\mathbf{q}}(\mathfrak{z})$

In this section, we discussed the convergence of (1.4) using ratio test. To show the convergence for $|\mathfrak{z}| < 1$, let us begin by considering

$$ \begin{aligned} {}_2R_1{}^{\upsilon,q}(\mathfrak{z}) \equiv {}_2R_1\left(\mu, \eta; \lambda; \upsilon; q, \mathfrak{z}\right) &= \frac{\Gamma_q(\lambda)}{\Gamma_q(\eta)} \sum_{\ell=0}^{\infty} \frac{\langle \mu; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q(\eta + \upsilon\ell)}{\Gamma_q(\lambda + \upsilon\ell)} \mathfrak{z}^\ell, \\ &= 1 + \sum_{\ell=1}^{\infty} \frac{\langle \mu; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q(\eta + \upsilon\ell)}{\Gamma_q(\lambda + \upsilon\ell)} \frac{\Gamma_q(\lambda)}{\Gamma_q(\eta)} \mathfrak{z}^\ell. \end{aligned} $$

Let

$$\sum_{\ell=1}^{\infty} u_\ell = \sum_{\ell=1}^{\infty} \frac{\langle \mu; q\rangle_\ell}{\langle 1; q\rangle_\ell} \frac{\Gamma_q(\eta + \upsilon\ell)}{\Gamma_q(\lambda + \upsilon\ell)} \frac{\Gamma_q(\lambda)}{\Gamma_q(\eta)} \mathfrak{z}^\ell.$$

Now,

$$\begin{aligned}
\lim_{\ell\to\infty} \left| \frac{u_{\ell+1}}{u_\ell} \right| &= \lim_{\ell\to\infty} \left| \frac{\frac{\langle \mu; q\rangle_{\ell+1}}{\langle 1; q\rangle_{\ell+1}} \frac{\Gamma_q(\eta+\upsilon(\ell+1))}{\Gamma_q(\lambda+\upsilon(\ell+1))} \frac{\Gamma_q(\lambda)}{\Gamma_q(\eta)} \mathfrak{z}^{\ell+1}}{\frac{\langle \mu; q\rangle_{\ell}}{\langle 1; q\rangle_{\ell}} \frac{\Gamma_q(\eta+\upsilon\ell)}{\Gamma_q(\lambda+\upsilon\ell)} \frac{\Gamma_q(\lambda)}{\Gamma_q(\eta)} \mathfrak{z}^{\ell}} \right| \\
&= \lim_{\ell\to\infty} \left| \frac{\left(1 - q^{\mu+\ell}\right) \cdot \langle \eta + \upsilon\ell; q\rangle_\infty \cdot \langle \lambda + \upsilon(\ell+1); q\rangle_\infty}{\left(1 - q^{\ell+1}\right) \cdot \langle \eta + \upsilon(\ell+1); q\rangle_\infty \cdot \langle \lambda + \upsilon\ell; q\rangle_\infty} \right| |\mathfrak{z}| \\
&= 1 \cdot |\mathfrak{z}|.
\end{aligned}$$

So, if $|\mathfrak{z}| < 1$ then $\sum_{\ell=1}^{\infty} u_\ell$ converges.

Now, we'll show that $\lim_{\ell\to\infty} \left| \frac{1-q^{\mu+\ell}}{1-q^{\ell+1}} \frac{\langle \eta+\upsilon\ell; q\rangle_\infty}{\langle \eta+\upsilon(\ell+1); q\rangle_\infty} \frac{\langle \lambda+\upsilon(\ell+1); q\rangle_\infty}{\langle \lambda+\upsilon\ell; q\rangle_\infty} \right| = 1.$
Let us begin with

$$\begin{aligned}
&\lim_{\ell\to\infty} \left| \frac{\left(1 - q^{\mu+\ell}\right) \cdot \langle \eta + \upsilon\ell; q\rangle_\infty \cdot \langle \lambda + \upsilon(\ell+1); q\rangle_\infty}{\left(1 - q^{\ell+1}\right) \cdot \langle \eta + \upsilon(\ell+1); q\rangle_\infty \cdot \langle \lambda + \upsilon\ell; q\rangle_\infty} \right| \\
&= 1 \cdot \lim_{\ell\to\infty} \left| \frac{\prod_{\mathfrak{m}=0}^{\infty} \left(1 - q^{\eta+\upsilon\ell+\mathfrak{m}}\right) \prod_{\mathfrak{m}=0}^{\infty} \left(1 - q^{\lambda+\upsilon(\ell+1)+\mathfrak{m}}\right)}{\prod_{\mathfrak{m}=0}^{\infty} \left(1 - q^{\eta+\upsilon(\ell+1)+\mathfrak{m}}\right) \prod_{\mathfrak{m}=0}^{\infty} \left(1 - q^{\lambda+\upsilon\ell+\mathfrak{m}}\right)} \right| \\
&= \lim_{\ell\to\infty} \left| \frac{\lim_{\mathfrak{n}\to\infty} \prod_{\mathfrak{m}=0}^{\mathfrak{n}} \left(1 - q^{\eta+\upsilon\ell+\mathfrak{m}}\right)}{\lim_{\mathfrak{n}\to\infty} \prod_{\mathfrak{m}=0}^{\mathfrak{n}} \left(1 - q^{\eta+\upsilon(\ell+1)+\mathfrak{m}}\right)} \frac{\lim_{\mathfrak{n}\to\infty} \prod_{\mathfrak{m}=0}^{\mathfrak{n}} \left(1 - q^{\lambda+\upsilon(\ell+1)+\mathfrak{m}}\right)}{\lim_{\mathfrak{n}\to\infty} \prod_{\mathfrak{m}=0}^{\mathfrak{n}} \left(1 - q^{\lambda+\upsilon\ell+\mathfrak{m}}\right)} \right| \\
&= \lim_{\ell\to\infty} \left| \lim_{\mathfrak{n}\to\infty} \exp\left(\sum_{\mathfrak{m}=0}^{\mathfrak{n}} \log \frac{\left(1 - q^{\eta+\upsilon\ell+\mathfrak{m}}\right)\left(1 - q^{\lambda+\upsilon(\ell+1)+\mathfrak{m}}\right)}{\left(1 - q^{\eta+\upsilon(\ell+1)+\mathfrak{m}}\right)\left(1 - q^{\lambda+\upsilon\ell+\mathfrak{m}}\right)} \right) \right| = 1.
\end{aligned}$$

4. Some new results on ${}_2\mathbf{R}_1^{\upsilon}(\mathfrak{z})$ and ${}_2\mathbf{R}_1^{\upsilon,\mathbf{q}}(\mathfrak{z})$

4.1. Relations of ${}_2\mathbf{R}_1^{\upsilon}(\mathfrak{z})$ and ${}_2\mathbf{R}_1^{\upsilon,\mathbf{q}}(\mathfrak{z})$

In this subsection, we have defined relations of generalized hypergeometric function and its q-analogue with Mittag-Leffler, Wright function and their q-analogues. These established relations may prove potentially valuable in various applications, particularly when one considers limiting cases.

Theorem 1. *For* $a = c = 1, \Re(b) > 0, \upsilon \in \mathbb{N}$ *and* $\mathfrak{z} \in \mathbb{C}$, *if* $|\mathfrak{z}| < b^{\upsilon}$ *then*

$$\lim_{b\to\infty} {}_2R_1\left(1, b; 1; \upsilon; \frac{\mathfrak{z}}{b^{\upsilon}}\right) = \mathcal{E}_{\upsilon}(\mathfrak{z}), \tag{4.1}$$

where $\mathcal{E}_{\upsilon}(\mathfrak{z})$ *is in the form of Mittag-Leffler function as defined in [16].*

Proof. To prove this, let us consider

$$\begin{aligned}
\lim_{b\to\infty} {}_2R_1{}^{\upsilon}\left(\frac{\mathfrak{z}}{b^{\upsilon}}\right)\Bigg|_{a=c=1} &\equiv \lim_{b\to\infty} {}_2R_1\left(1, b; 1; \upsilon; \frac{\mathfrak{z}}{b^{\upsilon}}\right) \\
&= \lim_{b\to\infty} \frac{\Gamma(1)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(1)_k}{k!} \frac{\Gamma(b+\upsilon k)}{\Gamma(1+\upsilon k)} \left(\frac{\mathfrak{z}}{b^{\upsilon}}\right)^k \\
&= \lim_{b\to\infty} \sum_{k=0}^{\infty} \frac{(b)_{\upsilon k}}{b^{\upsilon k}} \frac{\mathfrak{z}^k}{\Gamma(1+\upsilon k)} \\
&= \sum_{k=0}^{\infty} \frac{\mathfrak{z}^k}{\Gamma(\upsilon k + 1)} = \mathcal{E}_{\upsilon}(\mathfrak{z}).
\end{aligned}$$

Theorem 2 to theorem 5 can be proved in the same manner.

Theorem 2. *For* $a = 1, \Re(b), \Re(c) > 0, \upsilon \in \mathbb{N}$ *and* $\mathfrak{z} \in \mathbb{C}$, *if* $|\mathfrak{z}| < b^{\upsilon}$ *then*

$$\frac{1}{\Gamma(c)} \lim_{b\to\infty} {}_2R_1\left(1, b; c; \upsilon; \frac{\mathfrak{z}}{b^{\upsilon}}\right) = \mathcal{E}_{\upsilon,c}(\mathfrak{z}), \tag{4.2}$$

where $\mathcal{E}_{\upsilon,c}(\mathfrak{z})$ *is in the form of generalized Mittag-Leffler function given by Wiman [30].*

Theorem 3. *For* $\Re(a), \Re(b), \Re(c) > 0, \upsilon \in \mathbb{N}$ *and* $\mathfrak{z} \in \mathbb{C}$, *if* $|\mathfrak{z}| < b^{\upsilon}$ *then*

$$\frac{1}{\Gamma(c)} \lim_{b\to\infty} {}_2R_1\left(a, b; c; \upsilon; \frac{\mathfrak{z}}{b^{\upsilon}}\right) = \mathcal{E}^{a}_{\upsilon,c}(\mathfrak{z}), \tag{4.3}$$

where $\mathcal{E}^{a}_{\upsilon,c}(\mathfrak{z})$ *is in the form of generalized Mittag-Leffler function of three parameters given by Prabhakar [17].*

Theorem 4. *For* $\Re(a), \Re(b), \Re(c) > 0, \upsilon \in \mathbb{N}$ *and* $\mathfrak{z} \in \mathbb{C}$, *if* $|\mathfrak{z}| < a$ *then*

$$\frac{1}{\Gamma(c)} \lim_{a\to\infty} {}_2R_1\left(a, b; c; \upsilon; \frac{\mathfrak{z}}{a}\right) = \mathcal{E}^{b,\upsilon}_{\upsilon,c}(\mathfrak{z}), \tag{4.4}$$

where $\mathcal{E}^{b,\upsilon}_{\upsilon,c}(\mathfrak{z})$ *is in the form of generalized Mittag-Leffler function of four parameters given by Shukla and Prajapati [26].*

Theorem 5. *For* $\Re(a), \Re(b), \Re(c) > 0, \upsilon \in \mathbb{N}$ *and* $\mathfrak{z} \in \mathbb{C}$, *if* $|\mathfrak{z}| < a$ *then*

$$\frac{1}{\Gamma(c)} \lim_{a\to\infty} \lim_{b\to\infty} {}_2R_1\left(a, b; c; \upsilon; \frac{\mathfrak{z}}{ab^{\upsilon}}\right) = \phi(\upsilon, c, \mathfrak{z}), \tag{4.5}$$

where $\phi(\upsilon, c, \mathfrak{z})$ *is in the form of Wright function given by E. M. Wright [31].*

Theorem 6. *For $a = c = 1, \Re(b) > 0, \upsilon > 0, 0 < |q| < 1$ and $\mathfrak{z} \in \mathbb{C}$, if $|\mathfrak{z}(1-q)^{\upsilon}| < 1$ then*

$$\lim_{b\to\infty} {}_2R_1\left(a, b; c; \upsilon; q, (1-q)^{\upsilon}\mathfrak{z}\right)\Big|_{a=c=1} = e_{\upsilon}\left(\mathfrak{z}; q\right). \tag{4.6}$$

Proof. Let us consider

$$\begin{aligned}\lim_{b\to\infty} {}_2R_1{}^{\upsilon,q}\left((1-q)^{\upsilon}\mathfrak{z}\right)\Big|_{a=c=1} &\equiv \lim_{b\to\infty} {}_2R_1\left(1, b; 1; \upsilon; q, (1-q)^{\upsilon}\mathfrak{z}\right)\\ &= \lim_{b\to\infty} \frac{\Gamma_q(1)}{\Gamma_q(b)} \sum_{k=0}^{\infty} \frac{\langle 1; q\rangle_k}{\langle 1; q\rangle_k} \frac{\Gamma_q(b+\upsilon k)}{\Gamma_q(1+\upsilon k)} (1-q)^{\upsilon k} \mathfrak{z}^k\\ &= \sum_{k=0}^{\infty} \frac{\mathfrak{z}^k}{\Gamma_q(\upsilon k+1)} = e_{\upsilon}\left(\mathfrak{z}; q\right) \quad \text{(by using (2.14))}.\end{aligned}$$

In the same manner, we can prove theorem 7 to theorem 10.

Theorem 7. *For $a = 1, \Re(b), \Re(c) > 0, \upsilon > 0, 0 < |q| < 1$ and $\mathfrak{z} \in \mathbb{C}$, if $|\mathfrak{z}(1-q)^{\upsilon}| < 1$ then*

$$\frac{1}{\Gamma_q(c)} \lim_{b\to\infty} {}_2R_1\left(a, b; c; \upsilon; q, (1-q)^{\upsilon}\mathfrak{z}\right)\Big|_{a=1} = e_{\upsilon,c}\left(\mathfrak{z}; q\right). \tag{4.7}$$

Theorem 8. *For $\Re(a), \Re(b), \Re(c) > 0, \upsilon > 0, 0 < |q| < 1$ and $\mathfrak{z} \in \mathbb{C}$, if $|\mathfrak{z}(1-q)^{\upsilon}| < 1$ then*

$$\frac{1}{\Gamma_q(c)} \lim_{b\to\infty} {}_2R_1\left(a, b; c; \upsilon; q, (1-q)^{\upsilon}\mathfrak{z}\right) = e^{a}_{\upsilon,c}\left(\mathfrak{z}; q\right). \tag{4.8}$$

Theorem 9. *For $\Re(a), \Re(b), \Re(c) > 0, \upsilon > 0, 0 < |q| < 1$ and $\mathfrak{z} \in \mathbb{C}$, if $|\mathfrak{z}(1-q)^{\upsilon}| < 1$ then*

$$\frac{1}{\Gamma_q(c)} \lim_{a\to\infty} {}_2R_1\left(a, b; c; \upsilon; q, (1-q)^{\upsilon}\mathfrak{z}\right) = e^{b,\upsilon}_{\upsilon,c}\left(\mathfrak{z}; q\right). \tag{4.9}$$

Theorem 10. *For $\Re(a), \Re(b), \Re(c) > 0, \upsilon > 0, 0 < |q| < 1$ and $\mathfrak{z} \in \mathbb{C}$, if $|\mathfrak{z}(1-q)^{\upsilon}| < 1$ then*

$$\frac{1}{\Gamma_q(c)} \lim_{b\to\infty} {}_2R_1\left(a, b; c; \upsilon; q, (1-q)^{\upsilon}\mathfrak{z}\right) = w_{\upsilon,c}(\mathfrak{z}; q). \tag{4.10}$$

4.2. Some important on ${}_2\mathbf{R}_1^{\upsilon,\mathbf{q}}$

In 1910, Watson [29] gave the q-analogue of Barnes' contour integral of ${}_2F_1$ using Euler reflection formula for $0 < |q| < 1$ as

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(a,b;q)_\infty}{(q,c;q)_\infty}\left(\frac{-1}{2\pi i}\right)\int_{-i\infty}^{i\infty}\frac{\left(q^{1+s},cq^s;q\right)_\infty}{(aq^s,bq^s;q)_\infty}\frac{\pi(-z)^s}{\sin \pi s}ds. \qquad (4.11)$$

By Euler reflection formula $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$, (4.11) can be expressed as [6]

$$ {}_2\phi_1(a,b;c;q,z) = $$
$$ = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma_q(a+s)\Gamma_q(b+s)\Gamma(-s)\Gamma(1+s)}{\Gamma_q(c+s)\Gamma_q(1+s)}(-z)^s ds. \qquad (4.12)$$

Theorem 11. *(Barnes' contour integral representation of ${}_2R_1^{\upsilon,q}(z)$) For $\Re(a),\Re(b),\Re(c),\upsilon > 0, |z| < 1$ and $\Re(s) > 0$, Barnes' contour integral representation of ${}_2R_1^{\upsilon,q}(z)$ (1.4) is given by*

$$ {}_2R_1(a,b;c;\upsilon;q,z) = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}\cdot\frac{1}{2\pi i}\times $$
$$ \times\int_{-i\infty}^{+i\infty}\frac{\Gamma_q(a-s)\Gamma_q(b-\upsilon s)\Gamma(s)\Gamma(1-s)}{\Gamma_q(c-\upsilon s)\Gamma_q(1-s)}(-z)^{-s}ds. \qquad (4.13)$$

where $\Re(a),\Re(b),\Re(c) > 0, \upsilon > 0$ and $|z| < 1$.

Proof. Let us consider the q-analogue of Wright-type generalized hypergeometric function as

$$ {}_2R_1(a,b;c;\upsilon;q,z) = \frac{\Gamma_q(c)}{\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_k\Gamma_q(b+\upsilon k)}{\langle 1;q\rangle_k\Gamma_q(c+\upsilon k)}z^k $$
$$ = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma_q(a+k)\Gamma_q(b+\upsilon k)}{\Gamma_q(1+k)\Gamma_q(c+\upsilon k)}(-z)^k. \qquad (4.14)$$

Now, consider the right hand side of (11);

$$\frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}\cdot\frac{1}{2\pi i}\cdot\int_{-i\infty}^{+i\infty}\frac{\Gamma_q(a-s)\Gamma_q(b-\upsilon s)\Gamma(s)\Gamma(1-s)}{\Gamma_q(c-\upsilon s)\Gamma_q(1-s)}(-z)^{-s}ds$$
$$=\frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}\sum_{k=0}^{\infty}\underset{s=-k}{res}\left[\frac{\Gamma_q(a-s)\Gamma_q(b-\upsilon s)\Gamma(s)\Gamma(1-s)}{\Gamma_q(c-\upsilon s)\Gamma_q(1-s)}(-z)^{-s}\right]$$
$$=\frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}\sum_{k=0}^{\infty}\lim_{s\to-k}\left[\frac{\pi(s+k)}{\sin(\pi s)}\frac{\Gamma_q(a-s)\Gamma_q(b-\upsilon s)}{\Gamma_q(c-\upsilon s)\Gamma_q(1-s)}(-z)^{-s}\right]$$
$$=\frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma_q(a+k)\Gamma_q(b+\upsilon k)}{\Gamma_q(1+k)\Gamma_q(c+\upsilon k)}(-z)^k, \tag{4.15}$$

(4.14) and (4.15) completes the proof of (11), which is the Barnes' contour integral representation of ${}_2R_1(a,b;c;\upsilon;q,z)$, where $0<|q|<1$, $|z|<1$, $|arg(-z)|\le\pi-\delta$, $\delta>0$. The contour integration runs from $-i\infty$ to $+i\infty$, and separate the poles of integrand at $s=-k, k\in\mathbb{N}\cup\{0\}$ to the left and all the poles $s=n+a, n\in\mathbb{N}\cup\{0\}$ as well as $s=\frac{n+b}{\upsilon}, n\in\mathbb{N}\cup\{0\}$ to the right.

Note that, there are three special cases of (11). If $\upsilon=1$, then (4.12) is first special case. Letting limit $q\to1$, we get the Mellin-Barnes integral representation of ${}_2R_1^{\upsilon}(z)$ as defined in [22], and if we combine both special cases, we get the barnes' contour integral representation of ${}_2F_1(z)$ [18].

Theorem 12. *For* $\Re(a),\Re(b),\Re(c),\Re(\delta)>0,\upsilon>0,0<|q|<1,\mathfrak{z}\in\mathbb{C}$, *if* $|\mathfrak{z}u^{\upsilon}|<1$ *then*

$$\int_0^1 u^{c-1}\frac{(uq;q)_\infty}{(uq^\delta;q)_\infty}{}_2R_1^{\upsilon,q}(\mathfrak{z}u^\upsilon)\,d_qu=B_q(c,\delta)\cdot{}_2R_1(a,b;c+\delta;\upsilon;q,\mathfrak{z}). \tag{4.16}$$

Proof. We have

$$\int_0^1 u^{c-1}\frac{(uq;q)_\infty}{(uq^\delta;q)_\infty}{}_2R_1^{\upsilon,q}(\mathfrak{z}u^\upsilon)\,d_qu$$
$$=\frac{\Gamma_q(c)}{\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_k}{\langle 1;q\rangle_k}\frac{\Gamma_q(b+\upsilon k)}{\Gamma_q(c+\upsilon k)}\mathfrak{z}^kB_q(c+\upsilon k,\delta)$$
$$=B_q(c,\delta)\frac{\Gamma_q(c+\delta)}{\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_k}{\langle 1;q\rangle_k}\frac{\Gamma_q(b+\upsilon k)}{\Gamma_q(c+\delta+\upsilon k)}\mathfrak{z}^k.$$

Theorem 13. *For* $\Re(a),\Re(b),\Re(c)>0,\upsilon>0,0<|q|<1,\mathfrak{z}\in\mathbb{C}$, *if* $|\omega s^{\upsilon}|<1$ *then*

$$[c]_q\int_0^{\mathfrak{z}}s^{c-1}{}_2R_1^{\upsilon,q}(a,b;c;\upsilon;q,\omega s^\upsilon)d_qs=\mathfrak{z}^c{}_2R_1^{\upsilon,q}(a,b;c+1;\upsilon;q,\omega\mathfrak{z}^\upsilon). \tag{4.17}$$

Proof. Consider,

$$\int_0^{\mathfrak{z}} s^{c-1}{}_2R_1^{\upsilon,q}(\omega s^{\upsilon})\,d_qs = \frac{\Gamma_q(c)}{\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_k}{\langle 1;q\rangle_k}\frac{\Gamma_q(b+\upsilon k)}{\Gamma_q(c+\upsilon k)}\omega^k\int_0^{\mathfrak{z}} s^{c+\upsilon k-1}d_qs$$
$$= \frac{\mathfrak{z}^c}{[c]_q}\frac{\Gamma_q(c+1)}{\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_k}{\langle 1;q\rangle_k}\frac{\Gamma_q(b+\upsilon k)}{\Gamma_q(c+1+\upsilon k)}(\omega\mathfrak{z}^{\upsilon})^k.$$

Theorem 14. *For* $\Re(a), \Re(b), \Re(c) > 0, \upsilon > 0$ *and* $0 < |q| < 1$, *if* $|q^{\mathfrak{z}}| < 1$ *and* $|q^b| < 1$ *then*

$${}_2R_1(a,b;c;\upsilon;q,q^{\mathfrak{z}}) = \frac{\langle a+\mathfrak{z};q\rangle_\infty\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty\langle\mathfrak{z};q\rangle_\infty}{}_2R_1\left(c-b,\mathfrak{z};a+\mathfrak{z};\upsilon;q,q^b\right). \quad (4.18)$$

Proof. Starting the proof by taking $q^{\mathfrak{z}}$ instead of $\mathfrak{z}$ in (1.4), we get

$${}_2R_1(a,b;c;\upsilon;q,q^{\mathfrak{z}}) = \frac{\Gamma_q(c)}{\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_k}{\langle 1;q\rangle_k}\frac{\Gamma_q(b+\upsilon k)}{\Gamma_q(c+\upsilon k)}q^{\mathfrak{z}k}$$
$$= \frac{\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_k}{\langle 1;q\rangle_k}\frac{\langle c+\upsilon k;q\rangle_\infty}{\langle b+\upsilon k;q\rangle_\infty}q^{\mathfrak{z}k}$$
$$= \frac{\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty}\sum_{j=0}^{\infty}\frac{\langle c-b;q\rangle_j}{\langle 1;q\rangle_j}q^{jb}\frac{\langle\mathfrak{z}+a+j\upsilon;q\rangle_\infty}{\langle\mathfrak{z}+j\upsilon;q\rangle_\infty}$$
$$= \frac{\langle a+\mathfrak{z};q\rangle_\infty\langle b;q\rangle_\infty}{\langle c;q\rangle_\infty\langle\mathfrak{z};q\rangle_\infty}$$
$$\cdot\frac{\Gamma_q(\mathfrak{z}+a)}{\Gamma_q(\mathfrak{z})}\sum_{j=0}^{\infty}\frac{\langle c-b;q\rangle_j}{\langle 1;q\rangle_j}\frac{\Gamma_q(\mathfrak{z}+\upsilon j)}{\Gamma_q(\mathfrak{z}+a+\upsilon j)}\left(q^b\right)^j,$$

which leads us to (4.18).

5. Some q-integral transforms on generalized hypergeometric function

Theorem 15. *(q-Euler (beta) transform of* ${}_2R_1^{\upsilon,q}(xs^{\sigma})$*) For* $\Re(\alpha), \Re(\beta), \Re(a), \Re(b), \Re(c) > 0; \tau > 0$ *and* $|xs^{\sigma}| < 1$, *the q-Euler integral transform of* ${}_2R_1{}^{\tau,q}(xs^{\sigma}) \equiv {}_2R_1(a,b;c;\tau;xs^{\sigma})$ *can be obtained as*

$$\mathcal{B}_q\left\{{}_2R_1{}^{\upsilon,q}(xs^{\sigma}):\alpha,\beta\right\}$$

$$= \frac{\Gamma_q(c)\,\Gamma_q(\beta)}{\Gamma_q(a)\,\Gamma_q(b)}{}_3\psi_2\left[\begin{array}{c}(a,1),(b,\upsilon),(\alpha,\sigma)\\(c,\upsilon),(\alpha+\beta,\sigma)\end{array}\;\middle|\;q,x\right]. \quad (5.1)$$

Proof. We start this proof by taking ${}_2R_1\left(a,b;c;\upsilon;xs^{\sigma}\right)$ instead of $f(\mathfrak{z})$ in (2.20), so we get

$$\begin{aligned}
&\mathcal{B}_q\{{}_2R_1{}^{\upsilon,q}\left(xs^{\sigma}\right):\alpha,\beta\} \\
&\quad=\int\limits_0^1 s^{\alpha-1}\frac{(sq;q)_\infty}{(sq^{\beta};q)_\infty}\left(\frac{\Gamma_q\left(c\right)}{\Gamma_q\left(b\right)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_k\Gamma_q\left(b+\upsilon k\right)}{\langle 1;q\rangle_k\Gamma_q\left(c+\upsilon k\right)}(xs^{\sigma})^k\right)d_qs \\
&\quad=\frac{\Gamma_q\left(c\right)}{\Gamma_q\left(b\right)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_k\Gamma_q\left(b+\upsilon k\right)}{\langle 1;q\rangle_k\Gamma_q\left(c+\upsilon k\right)}x^k\int\limits_0^1 s^{\alpha+\sigma k-1}\frac{(sq;q)_\infty}{(sq^{\beta};q)_\infty}d_qs \\
&\quad=\frac{\Gamma_q\left(c\right)\Gamma_q\left(\beta\right)}{\Gamma_q\left(a\right)\Gamma_q\left(b\right)}\sum_{k=0}^{\infty}\frac{\Gamma_q\left(a+k\right)\cdot\Gamma_q\left(b+\upsilon k\right)\cdot\Gamma_q\left(\alpha+\sigma k\right)\cdot x^k}{\Gamma_q\left(c+\upsilon k\right)\cdot\Gamma_q\left(\alpha+\beta+\sigma k\right)\cdot\Gamma_q\left(1+k\right)}. \qquad (5.2)
\end{aligned}$$

Using (2.19), we get

$$\mathcal{B}_q\left\{{}_2R_1{}^{\upsilon,q}\left(xs^{\sigma}\right):\alpha,\beta\right\}=\frac{\Gamma_q\left(c\right)\Gamma_q\left(\beta\right)}{\Gamma_q\left(a\right)\Gamma_q\left(b\right)}{}_3\psi_2\left[\begin{array}{c}(a,1),(b,\upsilon),(\alpha,\sigma)\\ (c,\upsilon),(\alpha+\beta,\sigma)\end{array}\mid q,x\right],$$

which is our q-Euler (beta) transform of ${}_2R_1{}^{\upsilon,q}(xs^{\sigma})$.

Theorem 16. *(q-Laplace transform of ${}_2R_1{}^{\upsilon,q}(t)$) For $\Re(s)>0$, the q-Laplace transform of ${}_2R_1{}^{\tau,q}(t)\equiv{}_2R_1(a,b;c;\tau;q,t)$ is given as*

$$\begin{aligned}
&{}_q\mathfrak{L}\left({}_2R_1{}^{\tau,q}(t)\right)(s) \\
&\quad=\frac{(s(1-q))^{-1}\Gamma_q\left(c\right)}{\Gamma_q\left(a\right)\Gamma_q\left(b\right)}\sum_{k=0}^{\infty}\frac{\Gamma_q\left(a+k\right)\Gamma_q\left(b+\tau k\right)}{\Gamma_q\left(c+\tau k\right)}\frac{q^{\frac{(k+1)(k+2)}{2}}}{s^k}, \qquad (5.3)
\end{aligned}$$

where $\Re(a),\Re(b),\Re(c)>0;\tau>0$ and $|t|<1$.

Proof. By taking $f(t)={}_2R_1{}^{\upsilon,q}(t)$ in (2.21), we get

$$\begin{aligned}
{}_q\mathfrak{L}\left({}_2R_1{}^{\upsilon,q}(t)\right)(s)&=\frac{1}{(1-q)}\int\limits_0^{\infty}e_q(-st)\frac{\Gamma_q\left(c\right)}{\Gamma_q\left(b\right)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_k\Gamma_q\left(b+\upsilon k\right)}{\langle 1;q\rangle_k\Gamma_q\left(c+\upsilon k\right)}t^kd_qt \\
&=\frac{1}{(1-q)}\frac{\Gamma_q\left(c\right)}{\Gamma_q\left(b\right)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_k\Gamma_q\left(b+\upsilon k\right)}{\langle 1;q\rangle_k\Gamma_q\left(c+\upsilon k\right)}\int\limits_0^{\infty}t^ke_q(-st)d_qt.
\end{aligned}$$

Let $st=x$, we have $t=\frac{x}{s}$. Thus, when $x=0\Rightarrow t=0,x\to\infty\Rightarrow t\to\infty$ and $d_qt=\frac{d_qx}{s}$. Finally the above expression is written as

$$\begin{aligned}
&{}_q\mathfrak{L}\left({}_2R_1{}^{\upsilon,q}(t)\right)(s) \\
&\quad=\frac{1}{s(1-q)}\frac{\Gamma_q\left(c\right)}{\Gamma_q\left(b\right)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_k\Gamma_q\left(b+\upsilon k\right)}{\langle 1;q\rangle_k\Gamma_q\left(c+\upsilon k\right)}\left(\frac{1}{s}\right)^k\int\limits_0^{\infty}x^ke_q(-x)d_qx.
\end{aligned}$$

By using (2.8), we get

$$ {}_{q}\mathfrak{L}\left({}_{2}R_{1}{}^{\upsilon,q}(t)\right)(s)=\frac{1}{s(1-q)}\frac{\Gamma_{q}\left(c\right)}{\Gamma_{q}\left(b\right)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_{k}\Gamma_{q}\left(b+\upsilon k\right)}{\langle 1;q\rangle_{k}\Gamma_{q}\left(c+\upsilon k\right)}\frac{q^{\frac{(k+1)(k+2)}{2}}}{(s)^{k}}\Gamma_{q}\left(k+1\right). $$

Which leads us to (5.3).

Theorem 17. *(q-Sumudu transform of ${}_{2}R_{1}{}^{\upsilon,q}\left(t\right)$) The q-sumudu transform of ${}_{2}R_{1}{}^{\tau,q}\left(t\right)\equiv{}_{2}R_{1}(a,b;c;\tau;q,t)$ on the set $\mathfrak{A}_{2}$ is defined as*

$$ {}_{q}\mathfrak{S}\left({}_{2}R_{1}{}^{\tau,q}\left(t\right)\right)(u) $$

$$ =\frac{1}{(1-q)}\frac{\Gamma_{q}\left(c\right)}{\Gamma_{q}\left(a\right)\Gamma_{q}\left(b\right)}\sum_{k=0}^{\infty}\frac{\Gamma_{q}\left(a+k\right)\Gamma_{q}\left(b+\tau k\right)}{\Gamma_{q}\left(c+\tau k\right)}q^{\frac{(k+1)(k+2)}{2}}u^{k}, \tag{5.4} $$

where $\Re(a),\Re(b),\Re(c),\tau>0,|t|<1,u\in(-c_{1},c_{2})$ *and*

$$ \mathfrak{A}_{2}=\left\{f(t)\,|\exists M,c_{1},c_{2}>0,|\,f(t)\,|<M\cdot e_{q}\left(\frac{|t|}{c_{j}}\right),t\in(-1)^{j}\times[0,\infty)\right\}. $$

Proof. Let $f\left(t\right)={}_{2}R_{1}{}^{\upsilon,q}\left(t\right)$ in (2.22), we get

$$ {}_{q}\mathfrak{S}\left({}_{2}R_{1}{}^{\upsilon,q}\left(t\right)\right)(u) $$

$$ =\frac{1}{(1-q)u}\int_{0}^{\infty}\left(\frac{\Gamma_{q}\left(c\right)}{\Gamma_{q}\left(b\right)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_{k}\Gamma_{q}\left(b+\upsilon k\right)}{\langle 1;q\rangle_{k}\Gamma_{q}\left(c+\upsilon k\right)}t^{k}\right)e_{q}\left(-\frac{t}{u}\right)d_{q}t $$

$$ =\frac{1}{(1-q)u}\frac{\Gamma_{q}\left(c\right)}{\Gamma_{q}\left(b\right)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_{k}\Gamma_{q}\left(b+\upsilon k\right)}{\langle 1;q\rangle_{k}\Gamma_{q}\left(c+\upsilon k\right)}\int_{0}^{\infty}t^{k}e_{q}\left(-\frac{t}{u}\right)d_{q}t. $$

Let $\frac{t}{u}=x\Rightarrow d_{q}t=u\cdot d_{q}x$. Thus, $t=0\Rightarrow x=0$ and $t\to\infty\Rightarrow x\to\infty$. Hence, we can rewrite the above equation as

$$ {}_{q}\mathfrak{S}\left({}_{2}R_{1}{}^{\upsilon,q}\left(t\right)\right)(u) $$

$$ =\frac{1}{(1-q)u}\frac{\Gamma_{q}\left(c\right)}{\Gamma_{q}\left(b\right)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_{k}\Gamma_{q}\left(b+\upsilon k\right)}{\langle 1;q\rangle_{k}\Gamma_{q}\left(c+\upsilon k\right)}u^{k+1}\int_{0}^{\infty}x^{k}e_{q}\left(-x\right)d_{q}x. $$

By applying (2.8), we get

$$ {}_{q}\mathfrak{S}\left({}_{2}R_{1}{}^{\upsilon,q}\left(t\right)\right)(u) $$

$$ =\frac{1}{(1-q)}\frac{\Gamma_{q}\left(c\right)}{\Gamma_{q}\left(b\right)}\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_{k}\Gamma_{q}\left(b+\upsilon k\right)}{\langle 1;q\rangle_{k}\Gamma_{q}\left(c+\upsilon k\right)}u^{k}q^{\frac{(k+1)(k+2)}{2}}\Gamma_{q}\left(k+1\right). $$

Theorem 18. *(q-Natural transform of ${}_2R_1{}^{\upsilon,q}(t)$) The q-natural transform of ${}_2R_1{}^{\tau,q}(t) \equiv {}_2R_1(a,b;c;\tau;q,t)$ on the set $\mathfrak{A}_2$ (2.23) is defined as*

$$
\begin{aligned}
&{}_q\mathfrak{N}\left({}_2R_1{}^{\tau,q}(t)\right)(u;s)\\
&= \frac{1}{s(1-q)} \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)} \sum_{k=0}^{\infty} \frac{\Gamma_q(a+k)\Gamma_q(b+\tau k)}{\Gamma_q(c+\tau k)} q^{\frac{(k+1)(k+2)}{2}} \left(\frac{u}{s}\right)^k, \qquad (5.5)
\end{aligned}
$$

where $\Re(a), \Re(b), \Re(c), \Re(s), \tau > 0, |t| < 1, u \in (-c_1, c_2)$ *and*

$$
\mathfrak{A}_2 = \left\{ f(t) \,|\exists M, c_1, c_2 > 0, |\, f(t)\,| < M \cdot e_q\left(\frac{|t|}{c_j}\right), t \in (-1)^j \times [0,\infty) \right\}.
$$

Proof. Taking $f(t) = {}_2R_1{}^{\upsilon,q}(t)$ in (2.24), we get

$$
\begin{aligned}
&{}_q\mathfrak{N}\left({}_2R_1{}^{\upsilon,q}(t)\right)(u;s)\\
&= \frac{1}{(1-q)} \int_0^{\infty} \left(\frac{\Gamma_q(c)}{\Gamma_q(b)} \sum_{k=0}^{\infty} \frac{\langle a;q\rangle_k \Gamma_q(b+\upsilon k)}{\langle 1;q\rangle_k \Gamma_q(c+\upsilon k)} t^k \right) e_q\left(-\frac{s}{u}t\right) d_q t\\
&= \frac{1}{(1-q)} \frac{\Gamma_q(c)}{\Gamma_q(b)} \sum_{k=0}^{\infty} \frac{\langle a;q\rangle_k \Gamma_q(b+\upsilon k)}{\langle 1;q\rangle_k \Gamma_q(c+\upsilon k)} \int_0^{\infty} t^k e_q\left(-\frac{s}{u}t\right) d_q t.
\end{aligned}
$$

Let $\frac{s}{u}t = x \Rightarrow d_q t = \frac{u}{s} \cdot d_q x$. Thus, $t = 0 \Rightarrow x = 0$ and $t \to \infty \Rightarrow x \to \infty$. Hence, the above expression can be rewritten as

$$
\begin{aligned}
&{}_q\mathfrak{N}\left({}_2R_1{}^{\upsilon,q}(t)\right)(u;s)\\
&= \frac{1}{s(1-q)} \frac{\Gamma_q(c)}{\Gamma_q(b)} \sum_{k=0}^{\infty} \frac{\langle a;q\rangle_k \Gamma_q(b+\upsilon k)}{\langle 1;q\rangle_k \Gamma_q(c+\upsilon k)} \left(\frac{u}{s}\right)^k \int_0^{\infty} x^k e_q(-x) d_q x.
\end{aligned}
$$

By using (2.8), we get

$$
\begin{aligned}
&{}_q\mathfrak{N}\left({}_2R_1{}^{\upsilon,q}(t)\right)(u;s)\\
&= \frac{1}{s(1-q)} \frac{\Gamma_q(c)}{\Gamma_q(b)} \sum_{k=0}^{\infty} \frac{\langle a;q\rangle_k \Gamma_q(b+\upsilon k)}{\langle 1;q\rangle_k \Gamma_q(c+\upsilon k)} \left(\frac{u}{s}\right)^k q^{\frac{(k+1)(k+2)}{2}} \Gamma_q(k+1),
\end{aligned}
$$

which leads us to (5.5). Here, we can conclude that

$$
{}_q\mathfrak{N}\left({}_2R_1{}^{\upsilon,q}(t)\right)(1;s) = {}_q\mathfrak{L}\left({}_2R_1{}^{\upsilon,q}(t)\right)(s), \qquad (5.6)
$$

$$
{}_q\mathfrak{N}\left({}_2R_1{}^{\upsilon,q}(t)\right)(u;1) = {}_q\mathfrak{S}\left({}_2R_1{}^{\upsilon,q}(t)\right)(u). \qquad (5.7)
$$

References

1. *Albayrak D., Purohit S.D., Uçar F.*: On q-analogues of sumudu transform. Analele ştiin. Univ. Ovidius Const., Ser. Matem., 2013; 21(1): pp. 239–259. doi:10.2478/auom-2013-0016

2. *Albayrak D., Purohit S.D., Uçar F.*: On q-sumudu transforms of certain q-polynomials. Filomat, 2013; 27(2): pp. 411–427. doi:10.2298/FIL1302411A
3. *Al-Omari S.K.*: On q-analogues of the natural transform of certain q-bessel functions and some application. Filomat, 2017; 31(9), pp. 2587–2598. doi:10.2298/FIL1709587A
4. *Annaby M.H., Mansour Z.S.*: q-Fractional Calculus and Equations vol. 2056. Springer, Berlin, Heidelberg, 2012.
5. *Bhardwaj H., Sharma P.*: An application of g-hypergeometric series. GANITA, 2021; 71(1): pp. 161–169.
6. *Ernst T.*: A comprehensive treatment of q-calculus. Springer Science & Business Media, 2012.
7. *Exton H.*: q-Hypergeometric functions and applications. John Wiley & Sons, Inc., 605 Third ave., New York, NY 10158, USA, 1983.
8. *Garg M., Chanchlani L., Kalla S.*: A q-analogue of generalized Mittag-Leffler function. Algebras Groups Geom., 2011; 28(2): pp. 205–223.
9. *Gasper G., Rahman M.*: Basic Hypergeometric Series. Cambridge University Press, Cambridge, 1990.
10. *Gauss C.F.*: Disquistiones Generales circa Seriem Infinitam 1+ α β/1. γx+ α (α+ 1) β (β+ 1)/1.2. γ (γ+ 1) xx+ α (α+ 1)(α+ 2) β (β+ 1)(β+ 2)/1.2. 3. γ (γ+ 1)(γ+ 2) x^ 3+ etc. Thesis, Gottingen, 1866.
11. *Hahn W.*: Beiträge zur Theorie der Heineschen Reihen. Die 24 Integrale der hypergeometrischen q-Differenzengleichung. Das q-Analogon der Laplace-Transformation. Math. Nachrichten, 1949; 2(6): pp. 340–379.
12. *Heine E.*: Über die Reihe 1+(qα- 1)(qβ- 1)(q- 1)(qγ- 1) z+(qα- 1)(qα+ 1- 1)(qβ- 1)(qβ+ 1- 1)(q- 1)(q2- 1)(qγ- 1)(qγ+ 1- 1) z2+... J. Reine Angew. Math., 1846; 32: pp. 210–212.
13. *Heine E.*: Untersuchungen über die Reihe. J. Reine Angew. Math, 1847; 34: pp. 285–328. doi:10.1515/crll.1847.34.285
14. *Jackson F.H.*: On a q-definite integrals. Quart. J. Pure Appl. Math., 1910; 41: pp. 193–203.
15. *Mansour Z.S.*: Linear sequential q-difference equations of fractional order. Fract. Calc. Appl. Anal, 2009; 12(2): pp. 159–178.
16. *Mittag-Leffler G.M.*: Sur la nouvelle fonction $E_\alpha(x)$. CR Acad. Sci. Paris, 1903; 137(2): pp. 554–558.
17. *Prabhakar T.R.*: A singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama math. J., 1971; 19(1): pp. 7–15.
18. *Rainville E.D.*: Special functions, The Macmillan Company, New York, 1960.
19. *Rao S.B.*: Explicit representation and some integral transforms of sequence of functions associated with the Wright type generalized hypergeometric function. Global J. Pure Appl. Math., 2017; 13(9): pp. 6703–6712.
20. *Rao S.B., Patel A.D., Prajapati J.C., Shukla A.K.*: Some properties of generalized hypergeometric function. Comm. Korean Math. Soc., 2014; 28(2): pp. 303–317. doi:10.4134/CKMS.2013.28.2.303
21. *Rao S.B., Prajapati J.C., Patel A.D., Shukla A.K.*: Some properties of wright-type generalized hypergeometric function via fractional calculus. Adv. Differ. Equ., 2014; 119: pp. 1–11. doi:10.1186/1687-1847-2014-119
22. *Rao S.B., Prajapati J.C., Shukla A.K.*: Wright type hypergeometric function and its properties. Adv. Pure Math., 2013; 3(3): pp. 335–342. doi:10.4236/apm.2013.33048

23. *Rao S.B., Salehbhai I.A., Shukla A.K.*: On sequence of functions containing generalized hypergeometric function. Math. Sci. Res. J., 2013; 17(4): pp. 98–110.
24. *Rao S.B., Shukla A.K.*: Note on generalized hypergeometric function. Int. Trans. Spec. Func., 2013; 24(11): pp. 896–904. doi:10.1080/10652469.2013.773327
25. *Sharma S.K., Jain R.*: On some properties of generalized q-Mittag Leffler function. Math. Aeterna, 2014; 4(6): pp. 613–619.
26. *Shukla A.K., Prajapati J.C.*: On a generalization of Mittag-Leffler function and its properties. J. Math. Analysis Appl., 2007; 336(2): pp. 797–811. doi:10.1016/j.jmaa.2007.03.018
27. *Virchenko N., Kalla S.L., Al-Zamel A.M.*: Some results on a generalized hypergeometric function. Int. Trans. Spec. Func., 2001; 12(1): pp. 89–100. doi:10.1080/10652460108819336
28. *Waghela D., Rao S.B.*: A Note on Sequence of Functions associated with the Generalized Jacobi polynomial. Res. Math., 2023; 31(2): pp. 62–78. doi:10.15421/242316
29. *Watson G.N.*: The continuation of functions defined by generalized hypergeometric series. Trans. Camb. Phil. Soc, 1910; 21: pp. 281–299.
30. *Wiman A.*: Über den Fundamentalsatz in der Teorie der Funktionen $E_a(x)$. Acta Math., 1905; 29: pp. 191–201. doi:10.1007/BF02403202
31. *Wright E.M.*: On the coefficients of power series having exponential singularities. J. Lond. Math. Soc, 1933; 8(1): pp. 71–79. doi:10.1112/jlms/s1-8.1.71

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