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The norming set of a bilinear form on a certain normed space $\mathbb{R}^2$

Abstract. In this paper we classify the norming set of a bilinear form on the plane with a certain norm whose unit ball has only four extreme points. We obtain the results of [6, 8] as corollary.
Key words: extreme points, bilinear forms, norming points, norming sets

Анотація. У цій статті ми класифікуємо нормуючу множину білінійної форми на площині з певною нормою, одинична куля якої має лише чотири екстремальні точки. Отримуємо результати [6, 8] як наслідок.
Ключові слова: екстремальні точки, білінійні форми, нормуючі точки, нормуючі множини

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1. Introduction

In 1961 Bishop and Phelps [2] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}, n \geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{(x_1,\cdots,x_n)\in S_E\times\cdots\times S_E} |T(x_1,\cdots,x_n)|$. $\mathcal{L}_s(^nE)$

denote the closed subspace of all continuous symmetric n-linear forms on E. An element $(x_1,\ldots,x_n)\in E^n$ is called a *norming point* of T if $\|x_1\|=\cdots=\|x_n\|=1$ and $|T(x_1,\ldots,x_n)|=\|T\|$.

For $T\in\mathcal{L}(^nE)$, we define

$$\mathrm{Norm}(T)=\Big\{(x_1,\ldots,x_n)\in E^n:(x_1,\ldots,x_n)\text{ is a norming point of }T\Big\}.$$

Norm(T) is called the *norming set* of T. Notice that $(x_1,\ldots,x_n)\in\mathrm{Norm}(T)$ if and only if $(\epsilon_1x_1,\ldots,\epsilon_nx_n)\in\mathrm{Norm}(T)$ for some $\epsilon_k=\pm1$ $(k=1,\ldots,n)$. Indeed, if $(x_1,\ldots,x_n)\in\mathrm{Norm}(T)$, then

$$|T(\epsilon_1x_1,\ldots,\epsilon_nx_n)|=|\epsilon_1\cdots\epsilon_nT(x_1,\ldots,x_n)|=|T(x_1,\ldots,x_n)|=\|T\|,$$

which shows that $(\epsilon_1x_1,\ldots,\epsilon_nx_n)\in\mathrm{Norm}(T)$. If $(\epsilon_1x_1,\ldots,\epsilon_nx_n)\in\mathrm{Norm}(T)$ for some $\epsilon_k=\pm1$ $(k=1,\ldots,n)$, then

$$(x_1,\ldots,x_n)=\Big(\epsilon_1(\epsilon_1x_1),\ldots,\epsilon_n(\epsilon_nx_n)\Big)\in\mathrm{Norm}(T).$$

The following examples show that $\mathrm{Norm}(T)=\emptyset$ or an infinite set.

Examples. (a) Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big)=\sum_{i=1}^{\infty}\frac{1}{2^i}x_iy_i\in\mathcal{L}_s(^2c_0).$$

We claim that $\mathrm{Norm}(T)=\emptyset$. Obviously, $\|T\|=1$. Assume that $\mathrm{Norm}(T)\neq\emptyset$. Let $\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big)\in\mathrm{Norm}(T)$. Then,

$$1=\Big|T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big)\Big|\leq\sum_{i=1}^{\infty}\frac{1}{2^i}|x_i|\ |y_i|\leq\sum_{i=1}^{\infty}\frac{1}{2^i}=1,$$

which shows that $|x_i|=|y_i|=1$ for all $i\in\mathbb{N}$. Hence, $(x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\notin c_0$. This is a contradiction. Therefore, $\mathrm{Norm}(T)=\emptyset$.

(b) Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big)=x_1y_1\in\mathcal{L}_s(^2c_0).$$

Then,

$$\begin{aligned}\mathrm{Norm}(T)\ &=\ \Big\{\Big((\pm1,x_2,x_3,\ldots),\ (\pm1,y_2,y_3,\ldots)\Big)\in c_0\times c_0:|x_j|\leq1,\\ &\qquad |y_j|\leq1\text{ for }j\geq2\Big\}.\end{aligned}$$

A mapping $P:E\to\mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E\times\cdots\times E$ such that $P(x)=L(x,\ldots,x)$ for every $x\in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^n E)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^n E)$, we define

$$\text{Norm}(P) = \Big\{x \in E : x \text{ is a norming point of } P\Big\}.$$

Norm(P) is called the *norming set* of P. Notice that Norm$(P) = \emptyset$ or a finite set or an infinite set.

Kim [7] classified Norm(P) for every $P \in \mathcal{P}(^2\ell^2_\infty)$, where $\ell^2_\infty = \mathbb{R}^2$ with the supremum norm.

If Norm$(T) \neq \emptyset$, $T \in \mathcal{L}(^n E)$ is called a *norm attaining* n-linear form and if Norm$(P) \neq \emptyset$, $P \in \mathcal{P}(^n E)$ is called a *norm attaining* n-homogeneoue polynomial.(See [3])

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

It seems to be natural and interesting to study about Norm(T) for $T \in \mathcal{L}(^n E)$. For $m \in \mathbb{N}$, let $\ell^m_p := \mathbb{R}^m$ with the the ℓ_p-norm $(1 \leq p \leq \infty)$. Notice that if $E = \ell^m_1$ or ℓ^2_∞ and $T \in \mathcal{L}(^n E)$, Norm$(T) \neq \emptyset$ since S_E is compact. Kim [6, 8, 9, 10] classified Norm(T) for every $T \in \mathcal{L}_s(^2\ell^2_\infty), \mathcal{L}(^2\ell^2_\infty), \mathcal{L}(^2\ell^2_1), \mathcal{L}_s(^2\ell^3_1)$ or $\mathcal{L}_s(^3\ell^2_1)$. Kim [11] classified Norm(T) for every $T \in \mathcal{L}(^2\mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ denotes the plane with the hexagonal norm of weight $0 < w < 1$ endowed with $\|(x, y)\|_{h(w)} = \max\Big\{|y|,\ |x| + (1 - w)|y|\Big\}$.

Let $d_*(1, w)^2 = \mathbb{R}^2$ with the octagonal norm of weight $0 < w < 1$ endowed with $\|(x, y)\|_{d_*(1,w)} = \max\Big\{|x|, |y|, \frac{|x|+|y|}{1+w}\Big\}$. Kim [12] classified the set

$$NA(\mathcal{L}(^2d_*(1, w)^2))(Z_1, Z_2) := \{T \in S_{\mathcal{L}(^2d_*(1,w)^2)} : T \text{ attains its norm at } (Z_1, Z_2)\} \text{ for given unit vectors } Z_1, Z_2 \in d_*(1, w)^2.$$

Throughout the paper, we let $\mathbb{R}^2_{\|\cdot\|}$ denote the plane with the norm such that $\{\pm A, \pm B\}$ is the set of all extreme points of its unit ball with $A \neq B$. We let $\Omega = \{(A, A), (A, B), (B, A), (B, B)\}$.

In this paper we classify the norming set of a bilinear form on $\mathbb{R}^2_{\|\cdot\|}$ in terms of Ω. We obtain the results of [6, 8] as corollary.

2. Results

Let $T((x_1, y_1),\ (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dy_1x_2 \in \mathcal{L}(^2\mathbb{R}^2_{\|\cdot\|})$ for some $a, b, c, d \in \mathbb{R}$.

Let

$$\begin{aligned}
W_1 &= \{(tA + (1 - t)B, t'A + (1 - t')B) : 0 \leq t, t' \leq 1\},\\
W_2 &= \{(tA + (1 - t)B, -t'A + (1 - t')B) : 0 \leq t, t' \leq 1\},\\
W_3 &= \{(-tA + (1 - t)B, -t'A + (1 - t')B) : 0 \leq t, t' \leq 1\},\\
W_4 &= \{(-tA + (1 - t)B, -t'A + (1 - t')B) : 0 \leq t, t' \leq 1\}.
\end{aligned}$$

Theorem 1. *Let* $T \in \mathcal{L}({}^2\mathbb{R}^2_{\|\cdot\|})$. *Then*

$$\|T\| = \max\{|T(A,A)|, |T(B,A)|, |T(A,B)|, |T(B,B)|\}.$$

Proof. By the Krein-Milman Theorem, $S_{\mathbb{R}^2_{\|\cdot\|}}$ is the closed convex hull of the set $\{\pm A, \pm B\}$. Let $(X,Y) \in S_{\mathbb{R}^2_{\|\cdot\|}} \times S_{\mathbb{R}^2_{\|\cdot\|}}$. Without loss of generality we may assume that $(X,Y) \in \bigcup_{1\leq j\leq 4} W_j$. Let $M = \max\{|T(A,A)|, |T(B,A)|, |T(A,B)|, |T(B,B)|\}$. Obviously, $M \leq \|T\|$. By the bilinearity of T, it follows that for $0 \leq t, t^{'} \leq 1$,

$$\begin{aligned} |T(X,Y)| &\leq tt^{'}|T(A,A)| + (1-t)t^{'}|T(B,A)| + t(1-t^{'})|T(A,B)| \\ &+ (1-t)(1-t^{'})|T(B,B)| \leq M, \end{aligned}$$

which shows that $\|T\| \leq M$. Therefore, $\|T\| = M$.

Lemma 1. *Let* $n \in \mathbb{N}$ *and* $w_j, t_j \in \mathbb{R}$ *for* $j = 1, \ldots, n$ *be such that* $|w_j| \leq 1, 0 \leq t_j \leq 1$ *and* $\sum_{j=1}^n t_j = 1$. *Suppose that* $1 = \Big|\sum_{j=1}^n t_j w_j\Big|$. *If* $|w_{j_0}| < 1$ *for some* $j_0 \in \{1, \ldots, n\}$, *then* $t_{j_0} = 0$.

Proof. Assume the contrary. It follows that

$$\begin{aligned} 1 &= \Big|\sum_{j=1}^n t_j w_j\Big| \leq t_{j_0}|w_{j_0}| + \sum_{1\leq j\neq j_0\leq n}^{n} t_j|w_j| \\ &< t_{j_0} + \sum_{1\leq j\neq j_0\leq n}^{n} t_j|w_j| \text{ (because } t_{j_0} > 0 \text{ and } |w_{j_0}| < 1) \\ &\leq t_{j_0} + \sum_{1\leq j\neq j_0\leq n}^{n} t_j = 1, \end{aligned}$$

which is a contradiction. Therefore, we complete the proof.

Let

$$\Omega = \{(A,A), (A,B), (B,A), (B,B)\}$$

and

$$l_1 = T(A,A), l_2 = T(B,A), l_3 = T(A,B), l_4 = T(B,B).$$

We are in position to prove the main result of this paper.

Theorem 2. *Let* $T \in \mathcal{L}({}^2\mathbb{R}^2_{\|\cdot\|})$ *with* $\|T\| = 1$.
Then the following assertions hold:

Case 1. $|\operatorname{Norm}(T) \cap \Omega| = 1$

Then

$$\operatorname{Norm}(T) = \{(\pm X, \pm Y) : (X,Y) \in \operatorname{Norm}(T) \cap \Omega\}.$$

Case 2. $|\,\mathrm{Norm}(T)\cap\Omega| = 2$

Suppose that $\mathrm{Norm}(T)\cap\Omega = \{(A,A),(B,B)\}$ *or* $\{(A,B),\ (B,A)\}$. *Then*

$$\mathrm{Norm}(T) = \{(\pm X, \pm Y) : (X,Y) \in \mathrm{Norm}(T)\cap\Omega\}.$$

Suppose that $\mathrm{Norm}(T)\cap\Omega = \{(A,A),(A,B)\}$.
If $l_1l_3 = 1,$ *then*

$$\mathrm{Norm}(T) \;=\; \Big\{(\pm A, \pm(tA + (1-t)B)) : 0 \le t \le 1\Big\}.$$

If $l_1l_3 = -1,$ *then*

$$\mathrm{Norm}(T) \;=\; \Big\{(\pm A, \pm(-tA + (1-t)B)) : 0 \le t \le 1\Big\}.$$

Suppose that $\mathrm{Norm}(T)\cap\Omega = \{(B,B),(B,A)\}$.
If $l_2l_4 = 1,$ *then*

$$\mathrm{Norm}(T) \;=\; \Big\{(\pm B, \pm(tB + (1-t)A)) : 0 \le t \le 1\Big\}.$$

If $l_2l_4 = -1,$ *then*

$$\mathrm{Norm}(T) \;=\; \Big\{(\pm B, \pm(-tB + (1-t)A)) : 0 \le t \le 1\Big\}.$$

Suppose that $\mathrm{Norm}(T)\cap\Omega = \{(A,A),(B,A)\}$.
If $l_1l_2 = 1,$ *then*

$$\mathrm{Norm}(T) \;=\; \Big\{(\pm(tA + (1-t)B), \pm A) : 0 \le t \le 1\Big\}.$$

If $l_1l_2 = -1,$ *then*

$$\mathrm{Norm}(T) \;=\; \Big\{(\pm(-tA + (1-t)B), \pm A) : 0 \le t \le 1\Big\}.$$

Suppose that $\mathrm{Norm}(T)\cap\Omega = \{(B,B),(A,B)\}$.
If $l_3l_4 = 1,$ *then*

$$\mathrm{Norm}(T) \;=\; \Big\{(\pm(tB + (1-t)A), \pm B) : 0 \le t \le 1\Big\}.$$

If $l_3l_4 = -1,$ *then*

$$\mathrm{Norm}(T) \;=\; \Big\{(\pm(-tB + (1-t)A), \pm B) : 0 \le t \le 1\Big\}.$$

Case 3. $|\,\mathrm{Norm}(T)\cap\Omega| = 3$

Suppose that $(A,A) \notin \mathrm{Norm}(T)\cap\Omega$.

If $l_2l_4 = l_3l_4 = 1,$ *then*

$$\mathrm{Norm}(T) = \Big\{(\pm B, \pm(tA + (1-t)B)), (\pm(tA + (1-t)B), \pm B) : 0 \le t \le 1\Big\}.$$

If $l_2l_4 = -l_3l_4 = 1,$ *then*

$$\mathrm{Norm}(T) = \Big\{(\pm B, \pm(tA + (1-t)B)), (\pm(-tA + (1-t)B), \pm B) : 0 \le t \le 1\Big\}.$$

If $-l_2l_4 = l_3l_4 = 1,$ *then*

$$\mathrm{Norm}(T) = \Big\{(\pm B, \pm(-tA + (1-t)B)), (\pm(tA + (1-t)B), \pm B) : 0 \le t \le 1\Big\}.$$

If $l_2l_4 = l_3l_4 = -1,$ *then*

$$\mathrm{Norm}(T) = \Big\{(\pm B, \pm(-tA + (1-t)B)), (\pm(-tA + (1-t)B), \pm B) : 0 \le t \le 1\Big\}.$$

Suppose that $(B, B) \notin \mathrm{Norm}(T) \cap \Omega.$

If $l_2l_1 = l_3l_1 = 1,$ *then*

$$\mathrm{Norm}(T) = \Big\{(\pm A, \pm(tA + (1-t)B)), (\pm(tA + (1-t)B), \pm A) : 0 \le t \le 1\Big\}.$$

If $l_2l_1 = -l_3l_1 = 1,$ *then*

$$\mathrm{Norm}(T) = \Big\{(\pm A, \pm(tA - (1-t)B)), (\pm(tA + (1-t)B), \pm A) : 0 \le t \le 1\Big\}.$$

If $-l_2l_1 = l_3l_1 = 1,$ *then*

$$\mathrm{Norm}(T) = \Big\{(\pm A, \pm(tA + (1-t)B)), (\pm(tA - (1-t)B), \pm A) : 0 \le t \le 1\Big\}.$$

If $l_2l_1 = l_3l_1 = -1,$ *then*

$$\mathrm{Norm}(T) = \Big\{(\pm A, \pm(tA - (1-t)B)), (\pm(tA - (1-t)B), \pm A) : 0 \le t \le 1\Big\}.$$

Suppose that $(A, B) \notin \mathrm{Norm}(T) \cap \Omega.$

If $l_1l_2 = l_2l_4 = 1,$ *then*

$$\mathrm{Norm}(T) = \Big\{(\pm B, \pm(tA + (1-t)B)), (\pm(tA + (1-t)B), \pm A) : 0 \le t \le 1\Big\}.$$

If $l_1l_2 = -l_2l_4 = 1,$ *then*

$$\mathrm{Norm}(T) = \Big\{(\pm B, \pm(-tA + (1-t)B)), (\pm(tA + (1-t)B), \pm A) : 0 \le t \le 1\Big\}.$$

If $-l_1l_2 = l_2l_4 = 1,$ *then*

$$\mathrm{Norm}(T) = \Big\{(\pm B, \pm(tA + (1-t)B)), (\pm(-tA + (1-t)B), \pm A) : 0 \le t \le 1\Big\}.$$

If $l_1l_2 = l_2l_4 = -1$, *then*

$$\text{Norm}(T) = \Big\{(\pm B, \pm(-tA + (1-t)B)), (\pm(-tA + (1-t)B), \pm A) : 0 \le t \le 1\Big\}.$$

Suppose that $(B, A) \notin \text{Norm}(T) \cap \Omega$.
If $l_1l_3 = l_3l_4 = 1$, *then*

$$\text{Norm}(T) = \Big\{(\pm A, \pm(tA + (1-t)B)), (\pm(tA + (1-t)B), \pm B) : 0 \le t \le 1\Big\}.$$

If $l_1l_3 = -l_3l_4 = 1$, *then*

$$\text{Norm}(T) = \Big\{(\pm A, \pm(tA - (1-t)B)), (\pm(tA + (1-t)B), \pm B) : 0 \le t \le 1\Big\}.$$

If $-l_1l_3 = l_3l_4 = 1$, *then*

$$\text{Norm}(T) = \Big\{(\pm A, \pm(tA + (1-t)B)), (\pm(tA - (1-t)B), \pm B) : 0 \le t \le 1\Big\}.$$

If $l_1l_3 = l_3l_4 = -1$, *then*

$$\text{Norm}(T) = \Big\{(\pm A, \pm(tA - (1-t)B)), (\pm(tA - (1-t)B), \pm B) : 0 \le t \le 1\Big\}.$$

Case 4. $|\,\text{Norm}(T) \cap \Omega| = 4$

If $(l_1, l_2, l_3, l_4) = (1, 1, 1, 1)$ *or* $(-1, -1, -1, -1)$, *then*

$$\text{Norm}(T) = \Big\{(\pm(tA + (1-t)B), \pm(t'A + (1-t')B)) : 0 \le t, t' \le 1\Big\}.$$

If $(l_1, l_2, l_3, l_4) = (1, 1, 1, -1)$ *or* $(-1, -1, -1, 1)$, *then*

$$\begin{aligned}\text{Norm}(T) = \Big\{&(\pm A, \pm(tA + (1-t)B), (\pm(tA + (1-t)B, \pm A),\\ &(\pm B, \pm(tA - (1-t)B), (\pm(tA - (1-t)B, \pm B) : 0 \le t \le 1\Big\}.\end{aligned}$$

If $(l_1, l_2, l_3, l_4) = (1, 1, -1, 1)$ *or* $(-1, -1, 1, -1)$, *then*

$$\begin{aligned}\text{Norm}(T) = \Big\{&(\pm A, \pm(tA - (1-t)B), (\pm(tA + (1-t)B, \pm A),\\ &(\pm B, \pm(tA + (1-t)B), (\pm(tA - (1-t)B, \pm B) : 0 \le t \le 1\Big\}.\end{aligned}$$

If $(l_1, l_2, l_3, l_4) = (1, -1, 1, 1)$ *or* $(-1, 1, -1, -1)$, *then*

$$\begin{aligned}\text{Norm}(T) = \Big\{&(\pm A, \pm(tA + (1-t)B), (\pm(tA - (1-t)B, \pm A),\\ &(\pm B, \pm(tA - (1-t)B), (\pm(tA + (1-t)B, \pm B) : 0 \le t \le 1\Big\}.\end{aligned}$$

If $(l_1, l_2, l_3, l_4) = (1, -1, -1, 1)$ *or* $(-1, 1, 1, -1)$, *then*

$$\begin{aligned}\text{Norm}(T) &= \Big\{(\pm A, \pm(tA-(1-t)B), (\pm(tA-(1-t)B, \pm A), \\ &\quad (\pm B, \pm(tA-(1-t)B), (\pm(tA-(1-t)B, \pm B) : 0 \le t \le 1\Big\}.\end{aligned}$$

If $(l_1, l_2, l_3, l_4) = (1, -1, 1, -1)$ *or* $(-1, 1, -1, 1)$, *then*

$$\begin{aligned}\text{Norm}(T) &= \Big\{(\pm A, \pm(tA+(1-t)B), (\pm(tA-(1-t)B, \pm A), \\ &\quad (\pm B, \pm(tA+(1-t)B), (\pm(tA-(1-t)B, \pm B) : 0 \le t \le 1\Big\}.\end{aligned}$$

If $(l_1, l_2, l_3, l_4) = (1, 1, -1, -1)$ *or* $(-1, -1, 1, 1)$, *then*

$$\begin{aligned}\text{Norm}(T) &= \Big\{(\pm A, \pm(tA-(1-t)B), (\pm(tA+(1-t)B, \pm A), \\ &\quad (\pm B, \pm(tA-(1-t)B), (\pm(tA+(1-t)B, \pm B) : 0 \le t \le 1\Big\}.\end{aligned}$$

If $(l_1, l_2, l_3, l_4) = (1, -1, -1, -1)$ *or* $(-1, -1, -1, 1)$, *then*

$$\begin{aligned}\text{Norm}(T) &= \Big\{(\pm A, \pm(tA-(1-t)B), (\pm(tA-(1-t)B, \pm A), \\ &\quad (\pm B, \pm(tA+(1-t)B), (\pm(tA+(1-t)B, \pm B) : 0 \le t \le 1\Big\}.\end{aligned}$$

Proof. Let $(X, Y) \in \text{Norm}(T)$. Without loss of generality we may assume that $(X, Y) \in \bigcup_{1 \le j \le 4} W_j$.

Let $(X, Y) \in W_1$ with $X = tA + (1-t)B, Y = t^{'}A + (1-t^{'})B$ for some $0 \le t, t^{'} \le 1$. It follows that

$$\begin{aligned}(1)\quad 1 &= |T(X,Y)| = |T(tA+(1-t)B, t^{'}A+(1-t^{'})B)| \\ &= \Big|tt^{'}T(A,A) + (1-t)t^{'}T(B,A) + t(1-t^{'})T(A,B) \\ &+ (1-t)(1-t^{'})T(B,B)\Big|.\end{aligned}$$

Let $(X, Y) \in W_2$ with $X = tA + (1-t)B, Y = -t^{'}A + (1-t^{'})B$ for some $0 \le t, t^{'} \le 1$. It follows that

$$\begin{aligned}(2)\quad 1 &= |T(X,Y)| = |T(tA+(1-t)B, -t^{'}A+(1-t^{'})B)| \\ &= \Big|-tt^{'}T(A,A) - (1-t)t^{'}T(B,A) + t(1-t^{'})T(A,B) \\ &+ (1-t)(1-t^{'})T(B,B)\Big|.\end{aligned}$$

Let $(X, Y) \in W_3$ with $X = -tA + (1-t)B, Y = -t^{'}A + (1-t^{'})B$ for some $0 \le t, t^{'} \le 1$. It follows that

$$\begin{aligned}(3)\quad 1 &= |T(X,Y)| = |T(-tA+(1-t)B, -t^{'}A+(1-t^{'})B)| \\ &= \Big|tt^{'}T(A,A) - (1-t)t^{'}T(B,A) - t(1-t^{'})T(A,B) \\ &+ (1-t)(1-t^{'})T(B,B)\Big|.\end{aligned}$$

Let $(X,Y) \in W_4$ with $X = -tA + (1-t)B, Y = t^{'}A + (1-t^{'})B$ for some $0 \leq t, t^{'} \leq 1$. It follows that

$$\begin{aligned}(4)\quad 1 &= |T(X,Y)| = |T(-tA + (1-t)B, t^{'}A + (1-t^{'})B)| \\ &= \Big| - tt^{'}T(A,A) + (1-t)t^{'}T(B,A) - t(1-t^{'})T(A,B) \\ &+ (1-t)(1-t^{'})T(B,B)\Big|.\end{aligned}$$

Case 1. $|\,\mathrm{Norm}(T) \cap \Omega| = 1$

Suppose that $\mathrm{Norm}(T) \cap \Omega = \{(A,A)\}$.

By $(1)-(4)$ and Lemma 1, $(1-t)t^{'} = t(1-t^{'}) = (1-t)(1-t^{'}) = 0$. So $t = t^{'} = 1$ and $\mathrm{Norm}(T) = \{(\pm A, \pm A)\}$.

Suppose that $\mathrm{Norm}(T) \cap \Omega = \{(A,B)\}$.

By $(1)-(4)$ and Lemma 1, $tt^{'} = (1-t)t^{'} = (1-t)(1-t^{'}) = 0$. So $t = 1, t^{'} = 0$ and $\mathrm{Norm}(T) = \{(\pm A, \pm B)\}$.

Suppose that $\mathrm{Norm}(T) \cap \Omega = \{(B,A)\}$.

By $(1)-(4)$ and Lemma 1, $tt^{'} = t(1-t^{'}) = (1-t)(1-t^{'}) = 0$. So $t = 0, t^{'} = 1$ and $\mathrm{Norm}(T) = \{(\pm B, \pm A)\}$.

Suppose that $\mathrm{Norm}(T) \cap \Omega = \{(B,B)\}$.

By $(1)-(4)$ and Lemma 1, $tt^{'} = (1-t)t^{'} = t(1-t^{'}) = 0$. So $t = t^{'} = 0$ and $\mathrm{Norm}(T) = \{(\pm B, \pm B)\}$.

Therefore, $\mathrm{Norm}(T) = \{(\pm X, \pm Y) : (X,Y) \in \mathrm{Norm}(T) \cap \Omega\}$.

Case 2. $|\,\mathrm{Norm}(T) \cap \Omega| = 2$

Suppose that $\mathrm{Norm}(T) \cap \Omega = \{(A,A),(B,B)\}$ or $\{(A,B),\ (B,A)\}$.

Notice that $\{(\pm X, \pm Y) : (X,Y) \in \mathrm{Norm}(T) \cap \Omega\} \subseteq \mathrm{Norm}(T)$.

Let $\mathrm{Norm}(T) \cap \Omega = \{(A,A),(B,B)\}$. By $(1)-(4)$ and Lemma 1, $(1-t)t^{'} = 0$. If $t = 1$, then $X = A$ and $1 = |t^{'}T(A,A) + (1-t^{'})T(A,B)|$. By Lemma 1, $1 - t^{'} = 0$ because $|T(A,B)| < 1$. So $Y = A$. Hence, $(X,Y) = (A,A)$.

If $t^{'} = 0$, then $Y = B$ and $1 = |tT(A,B) + (1-t)T(B,B)|$. By Lemma 1, $t = 0$ because $|T(A,B)| < 1$. So $X = B$. Hence, $(X,Y) = (B,B)$.

Therefore, $\mathrm{Norm}(T) = \{(\pm A, \pm A), (\pm B, \pm B)\}$.

Let $\mathrm{Norm}(T) \cap \Omega = \{(A,B),(B,A)\}$. By $(1)-(4)$ and Lemma 1, $tt^{'} = 0$. If $t = 0$, then $X = B$ and $1 = |t^{'}T(B,A) + (1-t^{'})T(B,B)|$. By Lemma 1, $1 - t^{'} = 0$ because $|T(B,B)| < 1$. So $Y = A$. Hence, $(X,Y) = (B,A)$.

If $t^{'} = 0$, then $Y = B$ and $1 = |tT(A,B) + (1-t)T(B,B)|$. By Lemma 1, $1 - t = 0$ because $|T(B,B)| < 1$. So $X = A$. Hence, $(X,Y) = (A,B)$.

Thus $\mathrm{Norm}(T) = \{(\pm A, \pm B), (\pm B, \pm A)\}$.

Therefore, $\mathrm{Norm}(T) = \{(\pm X, \pm Y) : (X,Y) \in \mathrm{Norm}(T) \cap \Omega\}$.

Suppose that $\mathrm{Norm}(T) \cap \Omega = \{(A,A),(A,B)\}$.

Let $l_1 l_3 = 1$. Notice that

$$(*)\quad \Big\{(\pm A, \pm(tA + (1-t)B)) : 0 \leq t \leq 1\Big\} \subseteq \mathrm{Norm}(T).$$

By (1) – (4) and Lemma 1, $(1-t)t^{'} = (1-t)(1-t^{'}) = 0$. Then $t = 1$ and so $X = A$. By $(*)$, $\mathrm{Norm}(T) = \Big\{(\pm A, \pm(tA + (1-t)B)) : 0 \le t \le 1\Big\}$.

Let $l_1 l_3 = -1$. Notice that

$$(**) \quad \Big\{(\pm A, \pm(-tA + (1-t)B)) : 0 \le t \le 1\Big\} \subseteq \mathrm{Norm}(T).$$

By (1) – (4) and Lemma 1, $(1-t)s = (1-t)(1-s) = 0$. Then $t = 1$ and so $X = A$. By $(**)$, $\mathrm{Norm}(T) = \Big\{(\pm A, \pm(-tA + (1-t)B)) : 0 \le t \le 1\Big\}$.

Suppose that $\mathrm{Norm}(T) \cap \Omega = \{(B, B), (B, A)\}$.

Changing A and B by B and A and by the above result, It follows that if $l_2 l_4 = 1$, then $\mathrm{Norm}(T) = \Big\{(\pm B, \pm(tB + (1-t)A)) : 0 \le t \le 1\Big\}$ and that if $l_2 l_4 = -1$, then $\mathrm{Norm}(T) = \Big\{(\pm B, \pm(-tB + (1-t)A)) : 0 \le t \le 1\Big\}$.

Suppose that $\mathrm{Norm}(T) \cap \Omega = \{(A, A), (B, A)\}$.

Let $l_1 l_2 = 1$. Then $\Big\{(\pm(tA + (1-t)B), \pm A) : 0 \le t \le 1\Big\} \subseteq \mathrm{Norm}(T)$. By (1) – (4) and Lemma 1, $t(1-t^{'}) = (1-t)(1-t^{'}) = 0$. Then $t^{'} = 1$ and so $Y = A$. Hence, $\mathrm{Norm}(T) = \Big\{(\pm(tA + (1-t)B), \pm A) : 0 \le t \le 1\Big\}$.

Let $l_1 l_2 = -1$. Then $\Big\{(\pm(-tA + (1-t)B), \pm A) : 0 \le t \le 1\Big\} \subseteq \mathrm{Norm}(T)$. By (1) – (4) and Lemma 1, $t(1-t^{'}) = (1-t)(1-t^{'}) = 0$. Then $t^{'} = 1$ and so $Y = A$. Hence, $\mathrm{Norm}(T) = \Big\{(\pm(-tA + (1-t)B), \pm A) : 0 \le t \le 1\Big\}$.

Suppose that $\mathrm{Norm}(T) \cap \Omega = \{(B, B), (A, B)\}$.

Changing A and B by B and A and by the above result, It follows that if $l_3 l_4 = 1$, then $\mathrm{Norm}(T) = \Big\{(\pm(tB + (1-t)A), \pm B) : 0 \le t \le 1\Big\}$ and that if $l_3 l_4 = -1$, then $\mathrm{Norm}(T) = \Big\{(\pm(-tB + (1-t)A), \pm B) : 0 \le t \le 1\Big\}$.

Case 3. $|\,\mathrm{Norm}(T) \cap \Omega| = 3$

Suppose that $(A, A) \notin \mathrm{Norm}(T) \cap \Omega$.

By (1) – (4) and Lemma 1, $tt^{'} = 0$.

Let $l_2 l_4 = l_3 l_4 = 1$. Notice that $\Big\{(\pm B, \pm(tA + (1-t)B)), (\pm(tA + (1-t)B), \pm B) : 0 \le t \le 1\Big\} \subseteq \mathrm{Norm}(T)$.

If $t = 0$, then $X = B$ and $1 = |t^{'}T(B, A) + (1-t^{'})T(B, B)|$. So $Y = t^{'}A + (1-t^{'})B$ for every $0 \le t^{'} \le 1$.

If $t^{'} = 0$, then $Y = B$ and $1 = |tT(A, B) + (1-t)T(B, B)|$. So $X = tA + (1-t)B$ for every $0 \le t \le 1$.

Hence, $\mathrm{Norm}(T) = \Big\{(\pm B, \pm(tA + (1-t)B)), (\pm(tA + (1-t)B), \pm B) : 0 \le t \le 1\Big\}$.

By analogous arguments, the other cases follow. We omit them.

Suppose that $(B, B) \notin \mathrm{Norm}(T) \cap \Omega$.

Changing A and B by B and A and by the above result, It follows.

Suppose that $(A, B) \notin \mathrm{Norm}(T) \cap \Omega$.

By (1) – (4) and Lemma 1, $t(1-t^{'}) = 0$.

Let $l_1 l_2 = -l_2 l_4 = 1$. Notice that $\Big\{(\pm B, \pm(-tA + (1-t)B)), (\pm(tA + (1-t)B), \pm A) : 0 \leq t \leq 1\Big\} \subseteq \mathrm{Norm}(T)$.

If $t = 0$, then $X = B$ and $1 = |-t^{'}T(B, A) + (1-t^{'})T(B, B)|$. So $Y = -t^{'}A + (1-t^{'})B$ for every $0 \leq t^{'} \leq 1$.

If $t^{'} = 1$, then $Y = A$ and $1 = |tT(A, A) + (1-t)T(B, A)|$. So $X = tA + (1-t)B$ for every $0 \leq t \leq 1$. Thus $\mathrm{Norm}(T) = \Big\{(\pm B, \pm(-tA + (1-t)B)), (\pm(tA + (1-t)B), \pm A) : 0 \leq t \leq 1\Big\}$.

By analogous arguments, the other cases follow.

Suppose that $(B, A) \notin \mathrm{Norm}(T) \cap \Omega$.

Changing A and B by B and A and by the above result, It follows.

Case 4. $|\,\mathrm{Norm}(T) \cap \Omega| = 4$

Let $(l_1, l_2, l_3, l_4) = (1, -1, -1, 1)$ or $(-1, 1, 1, -1)$. Then $(1-t)t^{'} = t(1-t^{'}) = 0$ or $tt^{'} = (1-t^{'})(1-t) = 0$. Notice that

$$\Big\{(\pm A, \pm(tA - (1-t)B), (\pm(tA - (1-t)B, \pm A), (\pm B, \pm(tA - (1-t)B), (\pm(tA - (1-t)B, \pm B) : 0 \leq t \leq 1\Big\} \subseteq \mathrm{Norm}(T).$$

If $t = 0$, then $X = B$ and $1 = |-t^{'}T(B, A) + (1-t^{'})T(B, B)|$. So $Y = -t^{'}A + (1-t^{'})B$ for every $0 \leq t^{'} \leq 1$.

If $t = 1$, then $X = A$ and $1 = |-t^{'}T(A, A) + (1-t^{'})T(A, B)|$. So $Y = -t^{'}A + (1-t^{'})B$ for every $0 \leq t^{'} \leq 1$.

If $t^{'} = 0$, then $Y = B$ and $1 = |-tT(A, B) + (1-t)T(B, B)|$. So $X = -tA + (1-t)B$ for every $0 \leq t \leq 1$.

If $t^{'} = 1$, then $Y = A$ and $1 = |-tT(A, A) + (1-t)T(B, A)|$. So $X = -tA + (1-t)B$ for every $0 \leq t \leq 1$. Thus

$$\mathrm{Norm}(T) = \Big\{(\pm A, \pm(tA - (1-t)B), (\pm(tA - (1-t)B, \pm A), (\pm B, \pm(tA - (1-t)B), (\pm(tA - (1-t)B, \pm B) : 0 \leq t \leq 1\Big\}.$$

By analogous arguments, the other cases follow. We complete the proof.

Remark 1. Let $\mathbb{R}^2_{\|\cdot\|} = \ell^2_\infty$ be the plane with the supremum norm. Then $\{\pm(1, 1), \pm(1, -1)\}$ is the set of all extreme points of the unit ball of ℓ^2_∞. We obtain the results of [6, 8] as corollary of Theorem 2.

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