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V. F. Babenko, N. V. Parfinovych, D. S. Skorokhodov
Oles Honchar Dnipro National University, Dnipro 49045.
E-mails: babenko.vladislav@gmail.com, parfinovich@mmf.dnu.edu.ua, dmitriy.skorokhodov@gmail.com

Optimal recovery of operators in sequence spaces

Abstract. In this paper we solve the problem of optimal recovery of the operator $A_\alpha x = (\alpha_1 x_1, \alpha_2 x_2, \ldots)$ on the class $W_q^T = \{(t_1 h_1, t_2 h_2, \ldots) : \|h\|_{\ell_q} \le 1\}$, where $1 \le q < \infty$ and $t_1 \ge t_2 \ge \ldots \ge 0$, and $\alpha_1 t_1 \ge \alpha_2 t_2 \ge \ldots \ge 0$ are given, in the space ℓ_q. We solve this problem under assumption that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n t_n = 0$. Information available about a sequence $x \in W_q^T$ is provided either (i) by an element $y \in \mathbb{R}^n$, $n \in \mathbb{N}$, whose distance to the first n coordinates $(x_1, \ldots, x_n)$ of x in the space ℓ_p^n, $0 < p \le \infty$, does not exceed given $\varepsilon \ge 0$, or (ii) by a sequence $y \in \ell_p$ whose distance to x in the space ℓ_r does not exceed ε. We show that the optimal method of recovery in this problem is either operator Φ_m^* with some $m \in \mathbb{Z}_+$ ($m \le n$ in case $y \in \ell_p^n$), defined by

$$\Phi_m^*(y) = \left\{\alpha_1 y_1 \left(1 - \frac{\alpha_{m+1}^q t_{m+1}^q}{\alpha_1^q t_1^q}\right), \ldots, \alpha_m y_m \left(1 - \frac{\alpha_{m+1}^q t_{m+1}^q}{\alpha_m^q t_m^q}\right), 0, \ldots\right\},$$

where $y \in \mathbb{R}^n$ or $y \in \ell_p$ or convex combination $(1-\lambda)\Phi_{m+1}^* + \lambda \Phi_m^*$, or the operator A_α itself.

Key words: optimal recovery of operators, method of recovery, recovery with non-exact information, sequence spaces

Анотація. В цій роботі розв'язана задача найкращого відновлення оператора $A_\alpha x = (\alpha_1 x_1, \alpha_2 x_2, \ldots)$ на класі $W_q^T = \{(t_1 h_1, t_2 h_2, \ldots) : \|h\|_{\ell_q} \le 1\}$, де $1 \le q < \infty$, $t_1 \ge t_2 \ge \ldots \ge 0$ і $\alpha_1 t_1 \ge \alpha_2 t_2 \ge \ldots \ge 0$ – задані, в просторі ℓ_q. Ця задача розв'язана за умови $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n t_n = 0$. Інформацією про послідовність $x \in W_q^T$ виступає (i) елемент $y \in \mathbb{R}^n$, $n \in \mathbb{N}$, розташований на відстані не більше за задане $\varepsilon \ge 0$ від перших n координат $(x_1, \ldots, x_n)$ елемента x в просторі ℓ_p^n, $0 < p \le \infty$, або (ii) послідовність $y \in \ell_p$, що розташована на відстані не більше за ε від елементу x в просторі ℓ_r. Показано, що оптимальним методом відновлення в цій задачі є або оператор Φ_m^* для деякого $m \in \mathbb{Z}_+$ ($m \le n$ у випадку $y \in \ell_p^n$), означений рівністю

$$\Phi_m^*(y) = \left\{\alpha_1 y_1 \left(1 - \frac{\alpha_{m+1}^q t_{m+1}^q}{\alpha_1^q t_1^q}\right), \ldots, \alpha_m y_m \left(1 - \frac{\alpha_{m+1}^q t_{m+1}^q}{\alpha_m^q t_m^q}\right), 0, \ldots\right\},$$

де $y \in \mathbb{R}^n$, або $y \in \ell_p$, або опукла комбінація $(1-\lambda)\Phi_{m+1}^* + \lambda \Phi_m^*$, або сам оператор A_α.

Ключові слова: найкраще відновлення операторів, метод відновлення, відновлення за неточною інформацією, простори послідовностей

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1. Introduction

Let us consider the problem of optimal recovery of operators in in sequence spaces. These results are closely related to and somehow generalize the results in paper [3]. We refer the interested reader to this paper for the history of the topic and further references. In what follows we will be following notations from this paper.

Let X, Z be complex linear spaces, Y be a complex normed space, $A : X \to Y$ be an operator, in general non-linear, with domain $\mathcal{D}(A)$, $W \subset \mathcal{D}(A)$ be some class of elements. Denote by $\mathfrak{B}(Z)$ the set of non-empty subsets of Z, and let $I : \overline{\operatorname{span} W} \to \mathfrak{B}(Z)$ be a given mapping called *information.* When saying that information about element $x \in W$ is available we mean that some element $z \in I(x)$ is known. An arbitrary mapping $\Phi : Z \to Y$ is called *method of recovery* of operator A. Define *the error of method of recovery* Φ of operator A on the set W given information I:

$$\mathcal{E}(A, W, I, \Phi) = \sup_{x\in W} \sup_{z\in I(x)} \|Ax - \Phi(z)\|_Y . \tag{1.1}$$

The quantity

$$\mathcal{E}(A, W, I) = \inf_{\Phi:Z\to Y} \mathcal{E}(A, W, I, \Phi) \tag{1.2}$$

is called *the error of optimal recovery* of operator A on elements of class W given information I. Method Φ^* delivering inf in (1.2) (if any exists) is called *optimal.*

Note that results of the present work supplement and generalize results of paper [4] on optimal recovery of functions and its derivatives and paper [2].

The following proposition see (Corollary 1 in [3]) is a trivial yet effective lower estimate for the error of optimal recovery (1.2). Denote by θ_Z the null element of space Z and let I be some information mapping.

Corollary 1. *Let A be an odd operator, $\tilde{x} \in W$ be such that $-\tilde{x} \in W$ and $\theta_Z \in I(\tilde{x}) \cap I(-\tilde{x})$. Then*

$$\mathcal{E}(A, W, I) \geq \|A\tilde{x}\|_X.$$

Similar and related lower estimates were established in many papers (see, *e.g.*, [4, 1]).

2. Optimal recovery of operators in sequence spaces

In the rest of the paper we use the following notations. Let $1 \le p, q \le \infty$, ℓ_q be the standard space of sequences $x = \{x_k\}_{k=1}^\infty$, complex-valued in general, with corresponding norm $\|x\|_q$, and ℓ_q^n, $n \in \mathbb{N}$, be the spaces of finite sequences. Denote by θ the null element of ℓ_q and by θ^n the null element of ℓ_q^n.

For a given non-increasing sequence $t = \{t_k\}_{k=1}^\infty$ of non-negative numbers vanishing at infinity, consider bounded operator $T : \ell_q \to \ell_q$ defined as follows

$$Th := \{t_k h_k\}_{k=1}^\infty, \qquad h \in \ell_q,$$

and the class

$$W_q^T := \{x = Th \;:\; h \in \ell_q,\ \|h\|_q \le 1\}.$$

Let also the sequence $\alpha = \{\alpha_k\}_{k=1}^\infty$ of non-negative numbers be such that the sequence $\tau = \{\tau_k = \alpha_k t_k\}_{k=1}^\infty$ is non-increasing and is vanishing at infinity. Define the operator $A_\alpha : \ell_q \to \ell_q$ by the rule $Ax = (\alpha_1 x_1, \alpha_2 x_2, \ldots)$, $x \in \ell_q$.

In this section we will study the problem of optimal recovery of the operator A_α on the class W_q^T when information mapping I is given in one of the forms:

1. $Ix = I_{\overline{\varepsilon}}^n x = (x_1, \ldots, x_n) + B\,[\varepsilon_1] \times B\,[\varepsilon_n]$, where $n \in \mathbb{N}$, $\varepsilon_1, \ldots, \varepsilon_n \ge 0$ and $B[\varepsilon_j] = [-\varepsilon_j, \varepsilon_j]$;
2. $Ix = I_{\varepsilon,p}^n x = (x_1, \ldots, x_n) + B\left[\varepsilon, \ell_p^n\right]$, where $n \in \mathbb{N}$, $\varepsilon \ge 0$ and $B\left[\varepsilon, \ell_p^n\right]$ is the ball of radius ε in the space ℓ_p^n centered at θ^n;
3. $Ix = I_{\varepsilon,p} x = x + B\,[\varepsilon, \ell_p]$, where $\varepsilon \ge 0$ and $B\,[\varepsilon, \ell_p]$ is the ball of radius ε in the space ℓ_p centered at θ.

To simplify further notations, for $m \in \mathbb{N}$ and $q < \infty$, introduce the method of recovery $\Phi_m^* : \ell_p \to \ell_q$:

$$\Phi_m^*(a) = \left\{a_1\alpha_1\left(1 - \frac{\tau_{m+1}^q}{\tau_1^q}\right), \ldots, a_m\alpha_m\left(1 - \frac{\tau_{m+1}^q}{\tau_m^q}\right), 0, \ldots\right\}, \qquad a \in \ell_p,$$

that would be optimal in many situations. Also, we set $\Phi_0^*(a) := \theta$, $a \in \ell_p$.

In what follows we define $\sum_{k=1}^0 a_k := 0$ for numeric a_k's. In addition, for simplicity we assume that $t_k > 0$ for every $k \in \mathbb{N}$. Results in this paper remain true in the case when τ_k's (or t_k's) can attain zero value with the substitution of $1/\tau_k$ with $+\infty$ and τ_s/τ_k, $s \ge k$ with 1.

2.1. Information mapping $I_{\overline{\varepsilon}}^n(x) = (x_1, \ldots, x_n) + B[\varepsilon_1] \times \ldots \times B[\varepsilon_n]$

Theorem 1. *Let $n \in \mathbb{N}$, $1 \le q < \infty$ and $\varepsilon_1, \ldots, \varepsilon_n \ge 0$. If*

$$1 - \sum_{k=1}^n \frac{\varepsilon_k^q}{t_k^q} \ge 0,$$

we set $m = n$. Otherwise we choose $m \in \mathbb{Z}_+$, $m \le n$, to be such that

$$1 - \sum_{k=1}^m \frac{\varepsilon_k^q}{t_k^q} \ge 0 \qquad \text{and} \qquad 1 - \sum_{k=1}^{m+1} \frac{\varepsilon_k^q}{t_k^q} < 0.$$

Then

$$\mathcal{E}\left(A_\alpha, W_q^T, I_{\bar{\varepsilon}}^n\right) = \mathcal{E}\left(A_\alpha, W_q^T, I_{\bar{\varepsilon}}^n, \Phi_m^*\right) = \left(\tau_{m+1}^q + \sum_{k=1}^m \left(1 - \frac{\tau_{m+1}^q}{\tau_k^q}\right)\alpha_k^q \varepsilon_k^q\right)^{\frac{1}{q}}.$$

The proof of this theorem follows closely the proof of Theorem 1 in[3] with necessary elementary changes. We give the proof here for completeness.

Proof. Using convexity inequality, relations $|x_k - a_k| \le \varepsilon_k$, $k = 1, \dots, n$, and monotony of the sequence t, we obtain that, for $x = Th \in W_q^T$ and $a \in I_{\bar{\varepsilon}}^n(x)$,

$$\|A_\alpha x - \Phi_m^*(a)\|_q^q = \sum_{k=1}^m \left|\alpha_k x_k - \alpha_k a_k \left(1 - \frac{\tau_{m+1}^q}{\tau_k^q}\right)\right|^q + \sum_{k=m+1}^\infty |\alpha_k x_k|^q$$

$$= \sum_{k=1}^m \left|\left(1 - \frac{\tau_{m+1}^q}{\tau_k^q}\right)\alpha_k(x_k - a_k) + \frac{\tau_{m+1}^q}{\tau_k^q}\alpha_k x_k\right|^q + \sum_{k=m+1}^\infty \tau_k^q |h_k|^q$$

$$\le \sum_{k=1}^m \left(\left(1 - \frac{\tau_{m+1}^q}{\tau_k^q}\right)\alpha_k^q |x_k - a_k|^q + \frac{\tau_{m+1}^q}{t_k^q}|x_k|^q\right) + \tau_{m+1}^q \sum_{k=m+1}^\infty |h_k|^q$$

$$= \sum_{k=1}^m \left(1 - \frac{\tau_{m+1}^q}{\tau_k^q}\right)\alpha_k^q \varepsilon_k^q + \sum_{k=1}^m \tau_{m+1}^q |h_k|^q + \tau_{m+1}^q \sum_{k=m+1}^\infty |h_k|^q$$

$$\le \sum_{k=1}^m \left(1 - \frac{\tau_{m+1}^q}{\tau_k^q}\right)\alpha_k^q \varepsilon_k^q + \tau_{m+1}^q.$$

To obtain the lower estimate, we choose

$$u_k := \frac{\varepsilon_k}{t_k}, \quad k = 1, \dots, m, \qquad \text{and} \qquad u_{m+1} := \left(1 - \sum_{k=1}^m \frac{\varepsilon_k^q}{t_k^q}\right)^{1/q},$$

and consider $h^* = (u_1, \dots, u_{m+1}, 0, \dots) \in l_q$. It is clear that $Th^* \in W_q^T$, as $\|h^*\|_q \le 1$. Furthermore, by the choice of number m we have that $\theta \in I_{\bar{\varepsilon}}^n(Th^*)$. Hence, by Corollary 1,

$$\left(\mathcal{E}\left(A_\alpha, W_q^T, I_{\bar{\varepsilon}}^n\right)\right)^q \ge \|A_\alpha(Th^*)\|_q^q = \sum_{k=1}^m \alpha_k^q t_k^q u_k^q + \alpha_{m+1}^q t_{m+1}^q u_{m+1}^q$$

$$= \sum_{k=1}^m \alpha_k^q \varepsilon_k^q + \tau_{m+1}^q \left(1 - \sum_{k=1}^m \frac{\varepsilon_k^q}{t_k^q}\right) = \tau_{m+1}^q + \sum_{k=1}^m \alpha_k^q \varepsilon_k^q \left(1 - \frac{\tau_{m+1}^q}{\tau_k^q}\right),$$

which finishes the proof.

2.2. Information mapping $I_{\varepsilon,p}^n(x) = (x_1, \dots, x_n) + B\left[\varepsilon, \ell_p^n\right]$

We consider three cases separately: $p = \infty$, $p \le q$ and $p > q$.

2.2.1 Case $p = \infty$

Setting $\varepsilon_1 = \ldots = \varepsilon_n = \varepsilon$, we obtain from Theorem 1 the following corollary.

Theorem 2. *Let $n \in \mathbb{N}$, $1 \le q < \infty$ and $\varepsilon \ge 0$. If*

$$1 - \varepsilon^q \sum_{k=1}^{n} \frac{1}{t_k^q} \ge 0$$

then we set $m = n$. Otherwise we choose $m \in \mathbb{Z}_+$, $m \le n$, to be such that

$$1 - \varepsilon^q \sum_{k=1}^{m} \frac{1}{t_k^q} \ge 0 \qquad \text{and} \qquad 1 - \varepsilon^q \sum_{k=1}^{m+1} \frac{1}{t_k^q} < 0.$$

Then

$$\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,\infty}^n\right) = \mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,\infty}^n, \Phi_m^*\right)$$
$$= \left(\tau_{m+1}^q + \varepsilon^q \sum_{k=1}^{m} \alpha_k^q \left(1 - \frac{\tau_{m+1}^q}{\tau_k^q}\right)\right)^{\frac{1}{q}}.$$

2.2.2 Case $0 < p \le q$

Theorem 3. *Let $n \in \mathbb{N}$, $1 \le q < \infty$ and $0 < p \le q$. Let $r \in \{1, \ldots, n\}$ be such that*

$$\alpha_r^q \left(1 - \frac{\tau_{n+1}^q}{\tau_r^q}\right) = \max_{k=1,\ldots,n} \alpha_k^q \left(1 - \frac{\tau_{n+1}^q}{\tau_k^q}\right).$$

If $\varepsilon \in [0, t_r]$ then

$$\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n\right) = \mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n, \Phi_n^*\right) = \left(\tau_{n+1}^q + \alpha_r^q \varepsilon^q \left(1 - \frac{\tau_{n+1}^q}{\tau_r^q}\right)\right)^{1/q},$$

and if $\varepsilon \ge t_1$ then $\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n\right) = \mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n, \Phi_0^*\right) = \tau_1$.

Proof. First, consider the case $\varepsilon \in [0, t_r]$. For $x = Th \in W_q^T$, $\|h\|_q \le 1$, and $a \in I_{\varepsilon,p}^n(x)$, similarly to the proof of Theorem 1 we have

$$\|A_\alpha x - \Phi_n^*(a)\|_q^q$$
$$\le \sum_{k=1}^{n} \left(\left(1 - \frac{\tau_{n+1}^q}{\tau_k^q}\right)\alpha_k^q |x_k - a_k|^q + \frac{\tau_{n+1}^q}{\tau_k^q}\alpha_k^q |x_k|^q\right) + \tau_{n+1}^q \sum_{k=n+1}^{\infty} |h_k|^q$$
$$= \sum_{k=1}^{n} \alpha_k^q \left(1 - \frac{\tau_{n+1}^q}{\tau_k^q}\right)\left(|x_k - a_k|^p\right)^{q/p} + \tau_{n+1}^q \sum_{k=1}^{\infty} |h_k|^q$$
$$\le \alpha_r^q \left(1 - \frac{\tau_{n+1}^q}{\tau_r^q}\right)\left(\sum_{k=1}^{n} |x_k - a_k|^p\right)^{q/p} + \tau_{n+1}^q \le \alpha_r^q \left(1 - \frac{\tau_{n+1}^q}{\tau_r^q}\right)\varepsilon^q + \tau_{n+1}^q.$$

Now, we prove the lower estimate for $\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n\right)$. Let u_r and u_{n+1} be such that $t_r u_r = \varepsilon$ and $u_r^q + u_{n+1}^q = 1$, *i.e.* $u_r = \varepsilon/t_r$ and $u_{n+1}^q = 1 - \varepsilon^q/t_r^q$. Set $h^* := (0, \dots, 0, u_r, 0, \dots, 0, u_{n+1}, 0, \dots)$ with u_r and u_{n+1} on positions r and $n+1$, respectively. Obviously, $\|h^*\|_q \le 1$ and $\theta \in I_{\varepsilon,p}^n(Th^*)$. By Corollary 1,

$$\left(\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n\right)\right)^q \ge \|A_\alpha(Th^*)\|_q^q = \alpha_r^q t_r^q u_r^q + \alpha_{n+1}^q t_{n+1}^q u_{n+1}^q$$
$$= \alpha_r^q \varepsilon^q + \tau_{n+1}^q \left(1 - \frac{\varepsilon^q}{t_r^q}\right) = \tau_{n+1}^q + \alpha_r^q \varepsilon^q \left(1 - \frac{\tau_{n+1}^q}{\tau_r^q}\right).$$

Finally, consider the case $\varepsilon > t_1$. For $x = Th \in W_q^T$ and $a \in I_{\varepsilon,p}^n(x)$,

$$\|A_\alpha x - \Phi_0^*(a)\|_q^q = \|A_\alpha(Th)\|_q^q = \sum_{n=1}^\infty \alpha_n^q t_n^q |h_n|^q \le \tau_1^q \sum_{n=1}^\infty |h_n|^q \le \tau_1^q.$$

Taking $h^* := (1, 0, \dots)$, it is clear that $\theta \in I_{\varepsilon,p}^n(Th^*)$ and by Corollary 1,

$$\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n\right) \ge \|A_\alpha(Th^*)\|_q = \alpha_1 t_1 = \tau_1.$$

Theorem is proved.

2.2.3 Case $1 \le q < p < \infty$

We introduce some preliminary notations. For $m = 1, \dots, n$, define

$$\delta_{j,m} := \left(1 - \frac{\tau_{m+1}^q}{\tau_j^q}\right)^{\frac{p}{p-q}}, \quad j = 1, \dots, m,$$

and set $c_1 := t_1$ and, for $m \ge 2$,

$$c_m := \left(\sum_{j=1}^m \alpha_j^{\frac{pq}{p-q}} \delta_{j,m}\right)^{1/p} \left(\sum_{j=1}^m \frac{\alpha_j^{\frac{pq}{p-q}} \delta_{j,m}^{q/p}}{\tau_j^q}\right)^{-1/q}. \tag{2.1}$$

The sequence $\{c_m\}_{m=1}^n$ is non-increasing. Indeed, let $\delta_{j,m}(\xi) := \left(1 - \frac{\xi\tau_m^q + (1-\xi)\tau_{m+1}^q}{\alpha_j^q t_j^q}\right)^{\frac{p}{p-q}}$ and consider the function

$$g(\xi) := \left(\sum_{j=1}^m \alpha_j^{\frac{pq}{p-q}} \delta_{j,m}(\xi)\right)^{1/p} \left(\sum_{j=1}^m \frac{\alpha_j^{\frac{pq}{p-q}} \delta_{j,m}^{q/p}(\xi)}{\tau_j^q}\right)^{-1/q}, \qquad \xi \in [0,1].$$

Differentiating g and applying the Cauchy-Swartz inequality we have

$$g'(\xi) = \frac{\tau_{m+1}^q - \tau_m^q}{p-q} \left(\sum_{j=1}^m \alpha_j^{\frac{pq}{p-q}} \delta_{j,m}(\xi)\right)^{\frac{1}{p}-1} \left(\sum_{j=1}^m \frac{\alpha_j^{\frac{pq}{p-q}} \delta_{j,m}^{q/p}(\xi)}{\tau_j^q}\right)^{-1/q-1} \times$$
$$\times \left(\left(\sum_{j=1}^m \frac{\alpha_j^{\frac{pq}{p-q}} \delta_{j,m}^{q/p}(\xi)}{\tau_j^q}\right)^2 - \left(\sum_{j=1}^m \alpha_j^{\frac{pq}{p-q}} \delta_{j,m}(\xi)\right)\left(\sum_{j=1}^m \frac{\alpha_j^{\frac{pq}{p-q}} \delta_{j,m}^{2q/p-1}(\xi)}{\tau_j^{2q}}\right)\right) \ge 0$$

Hence, $c_{m+1} = g(0) \le g(1) = c_m$.

For convenience, for $\lambda \in [0,1]$ denote $\tau_{m,\lambda}^q := (1-\lambda)\tau_{m+1}^q + \lambda\tau_m^q$.

Theorem 4. *Let $n \in \mathbb{N}$ and $1 \le q < p < \infty$.*

1. If $\varepsilon \le c_n$ then

$$\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n\right) = \mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n, \Phi_n^*\right)$$

$$= \left(\tau_{n+1}^q + \varepsilon^q \left(\sum_{j=1}^n \alpha_j^{\frac{pq}{p-q}} \delta_{j,n}\right)^{\frac{p-q}{p}}\right)^{\frac{1}{q}};$$

2. If $\varepsilon \in (c_n, c_1]$ then there exist $m \in \{1,\dots,n-1\}$ such that $\varepsilon \in (c_{m+1}, c_m]$ and $\lambda = \lambda(\varepsilon) \in [0,1)$ such that

$$\varepsilon = \left(\sum_{j=1}^m \alpha_j^{\frac{pq}{p-q}} \delta_{j,m}(\lambda)\right)^{\frac{1}{p}} \left(\sum_{j=1}^m \frac{\alpha_j^{\frac{pq}{p-q}} \delta_{j,m}^{q/p}(\lambda)}{\tau_j^q}\right)^{-1/q}. \tag{2.2}$$

Then

$$\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n\right) = \mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n, \Phi_{m,\lambda}^*\right)$$

$$= \left(\tau_{m,\lambda}^q + \varepsilon^q \left(\sum_{j=1}^m \alpha_j^{\frac{pq}{p-q}} \delta_{j,m}(\lambda)\right)^{\frac{p-q}{p}}\right)^{\frac{1}{q}},$$

where $\Phi_{m,\lambda}^* = \lambda\Phi_m^* + (1-\lambda)\Phi_{m+1}^*$.

3. If $\varepsilon > c_1$ then $\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n\right) = \mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n, \Phi_0^*\right) = \tau_1$.

Proof. Let $m \in \{0,\dots,n\}$, $\lambda \in [0,1]$ and Φ be either Φ_n^*, or Φ_0^*, or $\Phi_{m,\lambda}^*$. For $x \in W_q^T$ and $a \in I_{\varepsilon,p}^n(x)$,

$$\|A_\alpha x - \Phi(a)\|_q^q \le \sum_{k=1}^m \left|\alpha_k \left(1 - \frac{\tau_{m,\lambda}^q}{\tau_k^q}\right)(x_k - a_k) + \frac{\tau_{m,\lambda}^q}{\tau_k^q}\alpha_k x_k\right|^q + \sum_{k=m+1}^\infty \alpha_k^q |x_k|^q$$

$$\le \sum_{k=1}^m \left(1 - \frac{\tau_{m,\lambda}^q}{\tau_k^q}\right)\alpha_k^q\, |x_k - a_k|^q + \sum_{k=1}^m \tau_{m,\lambda}^q |h_k|^q + \sum_{k=m+1}^\infty \tau_k^q |h_k|^q.$$

Using the Hölder inequality with parameters $p/(p-q)$ and p/q to estimate the first term and inequality $\tau_k^q \le \tau_{m,\lambda}^q$, $k = m+1, m+2, \dots$, we obtain

$$\|A_\alpha x - \Phi(a)\|_q^q \le \left\{\sum_{k=1}^m \alpha_k^{\frac{pq}{p-q}} \delta_{j,m}(\lambda)\right\}^{1-\frac{q}{p}} \left\{\sum_{k=1}^m |x_k - a_k|^p\right\}^{\frac{q}{p}} + \tau_{m,\lambda}^q \sum_{k=1}^\infty |h_k|^q$$

$$\le \left\{\sum_{k=1}^m \alpha_k^{\frac{pq}{p-q}} \delta_{j,m}(\lambda)\right\}^{1-\frac{q}{p}} \varepsilon^q + \tau_{m,\lambda}^q,$$

which proves the estimate from above.

Now, we turn to the proof of lower estimate. First, let $\varepsilon \le c_n$, and define

$$u_k := \frac{\varepsilon\, \alpha_k^{\frac{q}{p-q}} \delta_{k,n}^{1/p}}{t_k} \left(\sum_{j=1}^n \alpha_j^{\frac{pq}{p-q}} \delta_{j,n} \right)^{-1/p}, \quad k = 1, \ldots, n,$$

and

$$u_{n+1} := \left(1 - \sum_{k=1}^n u_k^q \right)^{1/q}.$$

Consider $h^* := (u_1, \ldots, u_{n+1}, 0, \ldots)$. Evidently, u_{n+1} is well-defined as

$$\sum_{k=1}^n u_k^q = \varepsilon^q \left(\sum_{j=1}^n \delta_{j,n} \right)^{-q/p} \sum_{k=1}^n \frac{\alpha_k^{\frac{pq}{p-q}} \delta_{k,n}^{q/p}}{\tau_k} = \frac{\varepsilon^q}{c_n^q} \le 1,$$

$\|h^*\|_q = 1$ and $\theta \in I_{\varepsilon,p}^n(Th^*)$ as $\sum_{k=1}^n t_k^p h_k^p = \varepsilon^p$. Hence, by Corollary 1,

$$\begin{aligned}
\left(\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n\right)\right)^q &\ge \|A_\alpha(Th^*)\|_q^q \\
&= \varepsilon^q \sum_{k=1}^n \alpha_k^{\frac{pq}{p-q}} \delta_{k,n}^{q/p} \left(\sum_{j=1}^n \alpha_j^{\frac{pq}{p-q}} \delta_{j,n} \right)^{-\frac{q}{p}} + \tau_{n+1}^q \\
&\quad - \tau_{n+1}^q \sum_{k=1}^n \varepsilon^q \frac{\alpha_k^{\frac{pq}{p-q}} \delta_{k,n}^{q/p}}{\tau_k^q} \left(\sum_{j=1}^n \alpha_j^{\frac{pq}{p-q}} \delta_{j,n} \right)^{-\frac{q}{p}} \\
&= \varepsilon^q \left(\sum_{j=1}^n \alpha_j^{\frac{pq}{p-q}} \delta_{j,n} \right)^{-\frac{q}{p}} \sum_{k=1}^n \alpha_k^{\frac{pq}{p-q}} \delta_{k,n}^{q/p} \left(1 - \frac{\tau_{n+1}^q}{\tau_k^q} \right) + \tau_{n+1}^q \\
&= \tau_{n+1}^q + \varepsilon^q \cdot \left(\sum_{k=1}^n \alpha_k^{\frac{pq}{p-q}} \left(1 - \frac{\tau_{n+1}^q}{\tau_k^q} \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}}.
\end{aligned}$$

Next, let $m \in \{1, 2, \ldots, n-1\}$ be such that $c_{m+1} < \varepsilon \le c_m$ and $\lambda = \lambda_\varepsilon \in [0, 1)$ be defined by (2.2). Set

$$u_k := \frac{\varepsilon \alpha_k^{\frac{q}{p-q}} \delta_{k,m}^{1/p}(\lambda)}{t_k} \left(\sum_{j=1}^m \alpha_k^{\frac{pq}{p-q}} \delta_{j,m}(\lambda) \right)^{-1/p}, \quad k = 1, \ldots, m,$$

and consider $h^* = (u_1, \ldots, u_m, 0, \ldots)$. Clearly, $\|h^*\|_q = 1$ and $\theta \in I_{\varepsilon,p}^n(Th^*)$. Using Corollary 1, we obtain the desired lower estimate for $\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n\right)$.

Finally, let $\varepsilon > c_1$. Consider $h^* := (1, 0, 0, \ldots)$. Since $c_1 = t_1$, we have $\theta \in I_{\varepsilon,p}^n(Th^*)$. By Corollary 1, $\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}^n\right) \ge \|A_\alpha(Th^*)\|_q = \alpha_1^q t_1^q = \tau_1^q$.

2.3. Information mapping $I(x) = I_{\varepsilon,p}(x) := x + B\left[\varepsilon, \ell_p\right]$

First, let $p = \infty$. As a limiting case of Theorem 2 we have.

Theorem 5. *Let* $1 \le q < \infty$ *and* $\varepsilon \ge 0$. *Choose* $m \in \mathbb{Z}_+$ *to be such that*

$$1 - \varepsilon^q \sum_{k=1}^{m} \frac{1}{t_k^q} \ge 0 \qquad and \qquad 1 - \varepsilon^q \sum_{k=1}^{m+1} \frac{1}{t_k^q} < 0.$$

Then

$$\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,\infty}\right) = \mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,\infty}, \Phi_m^*\right)$$
$$= \left(\tau_{m+1}^q + \varepsilon^q \sum_{k=1}^{m} \alpha_k^q \left(1 - \frac{\tau_{m+1}^q}{\tau_k^q}\right)\right)^{\frac{1}{q}}.$$

Next, let $1 \le q < \infty$, $0 < p \le q$. For $n, r \in \mathbb{N}$, define

$$A_{r,n} := \alpha_r^q \left(1 - \frac{\tau_{n+1}^q}{\tau_r^q}\right),$$

and denote by r_n the largest $r \in \{1, \ldots, n\}$ such that

$$A_{r,n} = \max_{k=1,\ldots,n} A_{k,n}.$$

Note that the sequence $\{r_n\}_{n=1}^\infty$ is non-decreasing. Indeed, otherwise if $r_n > r_{n+1}$ for some $n \in \mathbb{N}$ then we have

$$\alpha_{r_n}^q \left(1 - \frac{\tau_{n+1}^q}{\tau_{r_n}^q}\right) \ge \alpha_{r_{n+1}}^q \left(1 - \frac{\tau_{n+1}^q}{\tau_{r_{n+1}}^q}\right)$$

and

$$\alpha_{r_n}^q \left(1 - \frac{\tau_{n+2}^q}{\tau_{r_n}^q}\right) < \alpha_{r_{n+1}}^q \left(1 - \frac{\tau_{n+2}^q}{\tau_{r_{n+1}}^q}\right),$$

hence

$$\left(1 - \frac{\tau_{n+1}^q}{\tau_{r_n}^q}\right)\left(1 - \frac{\tau_{n+2}^q}{\tau_{r_{n+1}}^q}\right) > \left(1 - \frac{\tau_{n+1}^q}{\tau_{r_{n+1}}^q}\right)\left(1 - \frac{\tau_{n+2}^q}{\tau_{r_n}^q}\right),$$

or equivalently

$$\tau_{r_n}^q > \tau_{r_{n+1}}^q.$$

However this contradicts to the assumption that the sequence $\{\tau_n\}_{n=1}^\infty$ is non-increasing.

Theorem 6. *Let* $1 \le q < \infty$ *and* $0 < p \le q$. *If* $\varepsilon = t_{r_n}$ *for some* $n \in \mathbb{N}$ *then* $\mathcal{E}\left(A, W_q^T, I_{\varepsilon,p}\right) = \mathcal{E}\left(A, W_q^T, I_{\varepsilon,p}, \Phi_n^*\right) = \tau_{r_n}$. *If* $0 \le \varepsilon \le \lim_{n\to\infty} t_{r_n}$ *then* $\mathcal{E}\left(A, W_q^T, I_{\varepsilon,p}\right) = \mathcal{E}\left(A, W_q^T, I_{\varepsilon,p}, A\right) = \alpha_r \varepsilon$, *where* $r \in \mathbb{N}$ *is such that* $r_n = r$ *for every sufficiently large* n. *Finally, if* $\varepsilon > t_1$ *then* $\mathcal{E}\left(A, W_q^T, I_{\varepsilon,p}\right) = \mathcal{E}\left(A, W_q^T, I_{\varepsilon,p}, \Phi_0^*\right) = \tau_1$.

Proof. In case $\varepsilon = t_{r_n}$ or $\varepsilon \geq t_1$ the assertion of the theorem follows easily from Theorem 3. Assume now that $\lim_{n\to\infty} t_{r_n} = \mu > 0$. Since $\lim_{n\to\infty} t_n = 0$ and r_n is non-decreasing we conclude that there exists $r \in \mathbb{N}$ and $N \in \mathbb{N}$ such that $r_n = r$ for every $n > N$. For $x = Th \in W_q^T$, $\|h\|_q \leq 1$, and $a \in I_{\varepsilon,p}(x)$,

$$\|Ax - A(a)\|_q \leq \sup_{k\in\mathbb{N}} \alpha_k \cdot \|x - a\|_q \leq \sup_{k\in\mathbb{N}} \alpha_k \cdot \varepsilon.$$

Let us show that $\sup_{k\in\mathbb{N}} \alpha_k = \alpha_r$. Assume to the contrary that there exists $k \in \mathbb{N}$ such that $\alpha_k > \alpha_r$. Then there exists $n > \max\{N, k, r\}$ such that

$$\alpha_k^q \left(1 - \frac{\tau_{n+1}^q}{\tau_k^q}\right) > \alpha_r^q \left(1 - \frac{\tau_{n+1}^q}{\tau_r^q}\right) = A_{r,n} \geq \alpha_k^q \left(1 - \frac{\tau_{n+1}^q}{\tau_k^q}\right).$$

The above contradiction proves the above estimate $\mathcal{E}(A, W_q^T, I_{\varepsilon,p}, A) \leq \alpha_r \varepsilon$. Clearly, the element $h^* = (0, \ldots, 0, \frac{\varepsilon}{t_r}, 0, \ldots)$ with non-negative element appearing on the position r gives the desired lower estimate.

Now, let $q < p < \infty$. Define the sequence $\{c_n\}_{n=1}^\infty$ using formulas (2.1). It is not difficult to verify that $\{c_n\}_{n=1}^\infty$ is non-increasing and tend to 0 as $n \to \infty$. Indeed, since $\lim_{n\to\infty} \tau_n = 0$ and $q/p < 1$,

$$\lim_{n\to\infty} c_n \leq \lim_{N\to\infty} \tau_N \limsup_{n\to\infty} \left(\sum_{j=1}^N \alpha_j^{\frac{pq}{p-q}} \delta_{j,n}\right)^{1/p} \left(\sum_{j=N+1}^n \alpha_j^{\frac{pq}{p-q}} \delta_{j,n}\right)^{-1/q} = 0.$$

Theorem 7. *Let $1 \leq q < p < \infty$. If $\varepsilon \in (0, c_1]$ then there exists $m \in \mathbb{N}$ such that $\varepsilon \in (c_{m+1}, c_m]$ and $\lambda = \lambda(\varepsilon) \in [0, 1)$ such that*

$$\varepsilon = \left(\sum_{j=1}^m \alpha_j^{\frac{pq}{p-q}} \delta_{j,n}(\lambda)\right)^{1/p} \left(\sum_{j=1}^m \frac{\alpha_j^{\frac{pq}{p-q}} \delta_{j,n}^{q/p}(\lambda)}{\tau_j^q}\right)^{-1/q}. \tag{2.3}$$

Then

$$\begin{aligned}\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}\right) &= \mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}, \Phi_{m,\lambda}^*\right) \\ &= \left(\tau_{m,\lambda}^q + \varepsilon^q \left(\sum_{j=1}^m \alpha_j^{\frac{pq}{p-q}} \delta_{j,m}(\lambda)\right)^{\frac{p-q}{p}}\right)^{\frac{1}{q}}.\end{aligned}$$

where the method $\Phi_{m,\lambda}^*$ is defined in Theorem 3. Otherwise, if $\varepsilon > c_1$ then $\mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}\right) = \mathcal{E}\left(A_\alpha, W_q^T, I_{\varepsilon,p}, \Phi_0^*\right) = \tau_1$.

2.4. Applications

Let H be a complex Hilbert space with orthonormal basis $\{\varphi_n\}_{n=1}^\infty$, $\{t_k\}_{k=1}^\infty$ be a non-increasing sequence; $T : \ell_2 \to \ell_2$ be an operator mapping sequence

$x = (x_1, x_2, \dots)$ into sequence $Tx = (t_1x_1, t_2x_2, \dots)$, and $A_\alpha : \ell_2 \to \ell_2$ be an operator mapping sequence $x = (x_1, x_2, \dots)$ into the sequence $A_\alpha x = (\alpha_1 x_1, \alpha_2 x_2, \dots)$. Consider the class

$$\mathcal{W}^T := \left\{ x = \sum_{n=1}^{\infty} t_n c_n \varphi_n \ : \ \sum_{n=1}^{\infty} |c_n|^2 \le 1 \right\},$$

the operator

$$\mathcal{A}_\alpha x = \sum_{n=1}^{\infty} \alpha_n x_n \varphi_n, \qquad x = \sum_{n=1}^{\infty} x_n \varphi_n \in H,$$

and information operator $\mathcal{I}_{p,\varepsilon} : H \to \ell_p$, with $0 < p \le \infty$, mapping an element $x = \sum_{n=1}^{\infty} x_n\varphi_n$ into the set $\mathcal{I}_{p,\varepsilon} x = (x_1, x_2, \dots) + B[\varepsilon, \ell_p] \in \ell_p$. Due to isomorphism between ℓ_2 and H, under notations of Section 2 we have

$$\mathcal{E}\left(\mathcal{A}_\alpha, \mathcal{W}^T, \mathcal{I}_{\varepsilon,p}\right) = \mathcal{E}\left(A_\alpha, W_2^T, I_{\varepsilon,p}\right). \tag{2.4}$$

Moreover, methods of recovery $F^*_{m,\lambda} := \mathfrak{A} \circ \Phi^*_{m,\lambda}$ are optimal, where $\mathfrak{A} : \ell_2 \to H$ is the natural isomorphism between ℓ_2 and H: $\mathfrak{A}(x_1, x_2, \dots) = \sum_{n=1}^{\infty} x_n \varphi_n$. Remark that $F_{m,\lambda}$ are triangular methods of recovery that play an important role in the theory of ill-posed problems (see, *e.g.* [5, Theorem 2.1] and references therein).

Consider an important case when $t_{2m-1} = t_{2m} = m^{-\mu}$ and $\alpha_{2m-1} = \alpha_{2m} = m^\gamma$, $m \in \mathbb{N}$, with some fixed $\mu > 0$ and $\gamma \in [0, \mu)$. It corresponds *e.g.*, to the space $H = L_2(\mathbb{T})$ of square integrable functions defined on a period $\mathcal{T}$ with zero mean, the class $\mathcal{W}^T = W_2^\mu(\mathbb{T})$ of functions having L_2-bounded Weyl derivative of order μ and the Weyl fractional differentiation operator $\mathcal{A}_\alpha = \frac{d^\gamma}{dx^\gamma}$ of order γ. Due to equality (2.4), Theorems 5, 6 and 7 allow finding the exact value of $\mathcal{E}(\frac{d^\gamma}{dx^\gamma}, W_2^\mu(\mathbb{T}), \mathcal{I}_{\varepsilon,p})$ for all or some values of ε. Let us also establish sharp asymptotical behavior of this quantity as $\varepsilon \to 0^+$.

First, consider the case $2 < p < \infty$. It is not difficult to see that

$$\lim_{n\to\infty} n^{-1-\gamma\frac{2p}{p-2}} \sum_{j=1}^{n} \alpha_j^{\frac{2p}{p-2}} \delta_{j,n}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=1}^{n} \frac{\left(\frac{j}{2}\right)^{\gamma\frac{2p}{p-2}}}{(n+1)^{\gamma\frac{2p}{p-2}}} \left(1 - \frac{j^{(\mu-\gamma)2}}{(n+1)^{(\mu-\gamma)q}}\right)^{\frac{p}{p-2}}$$

$$= \int_0^1 \left(\frac{t}{2}\right)^{\gamma\frac{p2}{p-2}} \left(1 - t^{(\mu-\gamma)2}\right)^{\frac{p}{p-2}} dt = \frac{B\left(\frac{\gamma p}{(p-2)(\mu-\gamma)} + \frac{1}{(\mu-\gamma)2}, \frac{p}{p-2} + 1\right)}{(\mu-\gamma)2 \cdot 2^{\gamma\frac{2p}{p-2}}} =: B_1$$

and

$$\lim_{n\to\infty} n^{-1-\gamma\frac{2p}{p-2}-(\mu-\gamma)2}\sum_{j=1}^{n}\frac{\alpha_j^{\frac{2p}{p-2}}}{\tau_j^2}\delta_{j,n}^{2/p}$$
$$=\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=1}^{n}\frac{\left(\frac{j}{2}\right)^{\gamma\frac{2p}{p-2}+(\mu-\gamma)2}}{(n+1)^{\gamma\frac{2p}{p-2}+(\mu-\gamma)2}}\left(1-\frac{j^{(\mu-\gamma)2}}{(n+1)^{(\mu-\gamma)2}}\right)^{\frac{2}{p-2}}$$
$$=\int_0^1\left(\frac{t}{2}\right)^{\gamma\frac{2p}{p-2}+(\mu-\gamma)2}\left(1-t^{(\mu-\gamma)2}\right)^{\frac{2}{p-2}}dt$$
$$=\frac{B\left(\frac{\gamma p}{(p-2)(\mu-\gamma)}+\frac{1}{(\mu-\gamma)2}+1,\frac{2}{p-q}+1\right)}{(\mu-\gamma)2\cdot 2^{\gamma\frac{2p}{p-2}+(\mu-\gamma)2}}$$
$$=: B_2,$$

where $B(\alpha,\beta)$ is the Euler beta-function. Hence,

$$\lim_{n\to\infty} n^{\mu+\frac{1}{2}-\frac{1}{p}}c_n = B_1^{1/p}B_2^{-1/2}.$$

Selecting $n=n_\varepsilon\in\mathbb{N}$ and $\lambda\in[0,1)$ such that equation (2.3) is satisfied, we see that $c_{n_\varepsilon+1}\le\varepsilon<c_{n_\varepsilon}$ and from the above relation obtain that

$$\lim_{\varepsilon\to 0^+} n_\varepsilon^{\mu+\frac{1}{2}-\frac{1}{p}}\varepsilon = B_1^{1/p}B_2^{-1/2}.$$

By Theorem 7,

$$\lim_{\varepsilon\to+0}\varepsilon^{-\frac{\mu-\gamma}{\mu+\frac{1}{2}-\frac{1}{p}}}\mathcal{E}\left(\frac{d^\gamma}{dx^\gamma},W_2^\mu(\mathbb{T}),\mathcal{I}_{\varepsilon,p}\right)$$
$$=\lim_{\varepsilon\to+0}\varepsilon^{-\frac{\mu-\gamma}{\mu+\frac{1}{2}-\frac{1}{p}}}\left(\left(\frac{n_\varepsilon}{2}\right)^{-2}+\varepsilon^2\cdot n_\varepsilon^{\frac{p-2}{p}+\gamma 2}\cdot B_1^{\frac{p-2}{p}}\right)^{\frac{1}{2}}$$
$$=\left(2^2B_1^{\frac{2}{p(\mu+\frac{1}{2}-\frac{1}{p})}}B_2^{-\frac{1}{\mu+\frac{1}{2}-\frac{1}{p}}}+B_1^{\frac{2(\gamma+\frac{1}{2}-\frac{1}{q})}{p(\mu+\frac{1}{2}-\frac{1}{p})}+\frac{p-2}{p}}B_2^{-\frac{\gamma+\frac{1}{2}-\frac{1}{p}}{\mu+\frac{1}{2}-\frac{1}{p}}}\right)^{\frac{1}{2}}.$$

This provides sharp asymptotics behaviour for $\mathcal{E}\left(\frac{d^\gamma}{dx^\gamma},W_2^\mu(\mathbb{T}),\mathcal{I}_{\varepsilon,p}\right)$ as $\varepsilon\to 0^+$.

Similar arguments are applicable for $p=\infty$, in which case $1/p$ should be replaced with 0 (see Theorem 5) and for $p\in(0,q]$ (see Theorem 6).

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