Res. Math. **32**(2), 2024, p. 53–69
doi:10.15421/242419
ISSN (Print) 2664-4991
ISSN (Online) 2664-5009

UDK 517.5

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Convergence criteria of branched continued fractions

Celebrating the 75th birthdays of Professors V. F. Babenko and V. O. Kofanov

Abstract. The convergence criteria of branched continued fractions with N branches of branching and branched continued fractions of the special form are analyzed. The classical theorems of convergence of continued fractions that have become the subject of multidimensional generalizations are formulated. The convergence conditions of branched continued fractions of the general form with positive elements are reviewed. The problem the solution of which caused changes in the structure of such branched continued fractions is formulated. A multidimensional generalization of the convergence criterion of branched continued fractions of the special form is stated. A multidimensional generalization of Worpitzky's and van Vleck's convergence theorems, the Śleszyński-Pringsheim theorem for the considered types of branched continued fractions are considered. The obtained multidimensional analogs of the theorems are analyzed, and other conditions of convergence, in particular, of branched continued fractions with real elements, multidimensional Leighton's and Wall's theorems, and others are given.

Key words: continued fraction, branched continued fraction, branched continued fraction of the special form, convergence.

Анотація. Проведено аналіз критеріїв збіжності гіллястих ланцюгових дробів з N гілками розгалуження і гіллястих ланцюгових дробів спеціального вигляду. Сформульовано класичні ознаки збіжності неперервних дробів, які стали предметом багатовимірних узагальнень. Проведено аналіз ознак збіжності гіллястих ланцюгових дробів загального вигляду з додатними елементами. Сформульовано проблему, для вирішення якої потрібно було змінити саму структуру таких гіллястих ланцюгових дробів. Сформульовано багатовимірне узагальнення критерію збіжності гіллястих ланцюгових дробів спеціального вигляду. Розглянуто багатовимірне узагальнення ознаки збіжності Воропіцького, Ван Флека, Слешинського-Принцгейма для розглядуваних типів гіллястих ланцюгових дробів. Проаналізовано отримані багатовимірні аналоги теорем, наведено інші ознаки збіжності зокрема гіллястих ланцюгових дробів з дійсними елементами, багатовимірні ознаки Лейтона-Уола та інші.

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Ключові слова: неперервний дріб, гіллястий ланцюговий дріб, гіллястий ланцюговий дріб спеціального вигляду, збіжність.

MSC2020: 11A55, 11J70, 30B70, 40A15

Continued fraction is an effective tool for constructing fractional rational approximations of analytic functions. Different types of functional continued fractions are considered C-, S-, g-, J-, T- fractions, etc. Their approximants coincide with diagonal over- or under-diagonal approximants of the Padé table [9, 34, 43, 48, 53, 59]. In contrast to polynomial and trigonometric approximations [40, 44, 57], rational approximations (especially with non-fixed zeros of the denominator) are not yet well studied [42].

Often, convergence criteria of continued fractions are formulated as convergence sets. These convergence sets can be obtained for functional fractions by imposing certain restrictions on the fraction coefficients and variables. The study of the convergence of continued fractions with numerical elements is the main task in the analytic theory of continued fractions. The classical and most commonly used convergence theorems of continued fractions are the Seidel-Stern criterion, the necessary Stern-Stolz conditions, the sufficient theorems of convergence of Worpitzky's, and van Vleck's, the Śleszyński-Pringsheim, parabolic theorems, and convergence conditions of periodic and limit-periodic continued fractions (see [59]).

Branched continued fractions (BCFs) are multidimensional generalizations of continued fractions. Different types of functional BCFs are used to construct fractional rational approximations of functions of many variables. However, these approximations have nothing to do with multivariate Padé approximations. In this paper, we propose to analyze the obtained multidimensional generalizations of the classical convergence theorems of continued fractions for BCFs with N branches of branching and BCFs with independent variables. The latter one with fixed values of the variables are called BCFs of the special form.

1. Some convergence criteria of continued fractions

A rigorous definition of continued fractions is stated in the monographs [43, 59]. Let us have two sequences of complex numbers $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $a_n \neq 0,\ n \geq 1$. A continued fraction is the sequence $\{f_n\}_{n=0}^{\infty}$, where

$$f_0 = b_0,\ f_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n}}}},\ n \geq 1. \tag{1.1}$$

Formally, a continued fraction can be written in the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n + \ddots}}}}. \tag{1.2}$$

To write (1.2) or its approximant f_n compactly, it can be used the notation

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{a_k}{b_k}, \ f_n = b_0 + \mathop{\mathrm{D}}_{k=1}^{n} \frac{a_k}{b_k}, \ n \geq 1.$$

There are different interpretations of the concept of convergence of continued fractions:

- classical convergence: the continued fraction (1.2) converges if there is a finite limit $\lim\limits_{n\to\infty} f_n = f, \ f \in \mathbb{C}$;
- convergence in a wide sense (by Perron) [53]: the continued fraction (1.2) converges if there exists a limit $\lim\limits_{n\to\infty} f_n$ (possibly equal to ∞).

It is also investigated the general convergence [48].

Let us formulate classical criteria for convergence of continued fractions.

Theorem 1 (L. Seidel, M. A. Stern). *The continued fraction* (1.2)*, where* $a_k = 1, \ b_k > 0, \ k \geq 1$*, converges if and only if the series* $\sum\limits_{k=1}^{\infty} b_k$ *diverges.*

Theorem 2 (M. A. Stern, O. Stolz, H. von Koch). *The continued fraction* (1.2)*, where* $a_k = 1, \ b_k \in \mathbb{C}, \ k \geq 1$*, diverges if the series* $\sum\limits_{k=1}^{\infty} |b_k|$ *converges. There are finite limits for even and odd approximants.*

Theorem 3 (J. Worpitzky). *The continued fraction*

$$\left(1 + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{a_k}{1}\right)^{-1} \tag{1.3}$$

with complex partial numerators converges if $|a_k| \leq 1/4, \ k \geq 1$*, where the constant* $1/4$ *is the maximum possible. The values of the continued fraction and its approximants are in the domain* $|z - 4/3| \leq 2/3$*, which cannot be reduced.*

Theorem 4 (W. Leighton, H. Wall). *The continued fraction* (1.3) *with complex partial denominators converges if* $|a_{2k-1}| \leq 1/4, \ |a_{2k}| \geq 25/4, \ k \geq 1$.

Circular sets of convergence similar to the one in Leighton-Wall theorem have been studied by W. J. Thron, L. J. Lange, V. Cowling, J. Mc Laughlin, Nancy J. Wyshinski [33, 46, 49, 58].

Theorem 5 (E. B. Van Vleck). *The continued fraction* (1.2)*, where* $a_k = 1,\ b_k \in \mathcal{G},\ k \geq 1,\ \mathcal{G} = \{z \in \mathbb{C} : z \neq 0,\ |\arg z| < \pi/2 - \varepsilon, \}, \varepsilon > 0,$ *converges if the series* $\sum\limits_{k=1}^{\infty} |b_k|$ *diverges. The values of the continued fraction and all its approximants lie in the domain* $\mathcal{G}$*. There exists a limit of even and odd approximants.*

Theorem 6 (J. V. Śleszyński, A. Pringsheim). *The continued fraction* (1.2)*, where* $a_k, b_k \in \mathbb{C},\ k \geq 1$ *converges if* $|b_k| \geqslant |a_k| + 1,\ k \geq 1$*. The values of the continued fraction and all its approximants lie in the domain* $|z - b_0| \leq 1$.

Theorem 7 (Parabolic Theorem). *The continued fraction* (1.3)*, where* $a_k \in \mathcal{P},\ k \geq 1,\ \mathcal{P} = \{z \in \mathbb{C} : |z| - \Re(z) \leq 1/2\}$*, converges if there exists a number* k *such that* $a_k = 0$*, or all* $a_k \neq 0$*, and the series* $\sum\limits_{k=1}^{\infty} |b_k|$ *diverges, where* $b_0 = 1,\ a_k = (b_k b_{k-1})^{-1},\ k \geq 1$*. The values of the continued fraction and all its approximants are in the region* $|z - 1| \leq 1,\ z \neq 0$.

Various generalizations and modifications of these classical convergence theorems of continued fractions have been established [34, 48].

2. Criteria for convergence of BCFs

The study of branched continued fractions with N branches of branching $N \in \mathbb{N},\ N \geq 2$ was initiated by V. Ya. Skorobohatko [56]. The foundation of the analytical theory of BCFs was laid in the works of his students P. I. Bodnarchuk, D. I. Bodnar, Kh. Yo. Kuchminska, M. O. Nedashkovsky, M. S. Siavavko and their students [19, 31, 37, 45, 50, 54, 55].

Let $i(k) = (i_1, i_2, \ldots, i_k),\ 1 \leq i_p \leq N,\ p = \overline{1, k}$, be the shorten notation of multiindex. Similarly, $j(r) = (j_1, j_2, \ldots, j_r)$. Consider the sequences of complex numbers $\{a_{i(k)}\}, \{b_{j(r)}\},\ k \geq 1, r \geq 0$, with $b_{j(0)} = b_0$.

A BCF with N branches of branching is a sequence $\{f_n\},\ n \geq 0$, where

$$f_0 = b_0,\ f_n = b_0 + \sum_{i_1=1}^{N} \cfrac{a_{i(1)}}{b_{i(1)} + \sum\limits_{i_2=1}^{N} \cfrac{a_{i(2)}}{b_{i(2)} + \ddots + \sum\limits_{i_n=1}^{N} \cfrac{a_{i(n)}}{b_{i(n)}}}}, n \geq 1.$$

The infinite BCF can be formally written in the form

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{a_{i(k)}}{b_{i(k)}}. \tag{2.1}$$

Let $\mathcal{I}_k = \left\{i(k) : 1 \leq i_p \leq N;\ p = \overline{1, k}\right\},\ k \geq 1,\ \mathcal{I} = \bigcup\limits_{k=1}^{\infty} \mathcal{I}_k$.

In contrast to continued fractions, the conditions $a_{i(k)} \neq 0,\ i(k) \in \mathcal{I}$, are not suffisient for avoiding the uncertainty $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}, \infty - \infty$ while calculating f_n.

A key point in the theory of continued fractions is the existence of recurrence relations for the canonical numerators and denominators of approximants [59]. If $f_n = P_n/Q_n$, then, for example,

$$P_n = b_n P_{n-1} + a_n P_{n-2},\ n \geq 1,\ P_0 = b_0,\ P_{-1} = 1.$$

There are no such relations for BCFs.

The BCF (2.1) converges if a finite limit of its approximants f_n exists. The BCF (2.1) converges absolutely if the series $\sum\limits_{n=1}^{\infty} |f_{n+1} - f_n|$ converges.

An important role in the study of the convergence of the BCF was played by the formula for the difference between the approximants through the tails of the approximants of the BCFs [29].

Let

$$Q_{i(n)}^{(n)} = b_{i(n)},\ i(n) \in \mathcal{I}_n,\ Q_{i(k)}^{(n)} = b_{i(k)} + \sum_{i_{k+1}=1}^{N} \frac{a_{i(k+1)}}{Q_{i(k+1)}^{(n)}},\ i(k) \in \mathcal{I}_k,\ 0 \leq k < n,$$

with $Q_{i(0)}^{(n)} = f_n$, be the tails of $f_n,\ n \geq 1$.

Assuming that $m < n$ and all tails of f_n and f_m are nonzero, we have

$$f_n - f_m = (-1)^m \sum_{i_1, i_2, \ldots, i_{m+1}=1}^{N} \frac{\prod\limits_{k=1}^{m+1} a_{i(k)}}{\prod\limits_{k=1}^{m+1} Q_{i(k)}^{(n)} \prod\limits_{k=1}^{m} Q_{i(k)}^{(m)}}.$$

Let us state the theorems of convergence of a BCF with positive elements. Let $\alpha_k = \min\{b_{i(k)}, i(k) \in \mathcal{I}_k\},\ \beta_k = \max\{b_{i(k)}, i(k) \in \mathcal{I}_k\},\ k \geq 1$.

Theorem 8. *The BCF* (2.1)*, where* $a_{i(k)} = 1,\ b_{i(k)} > 0,\ i(k) \in \mathcal{I}$*, diverges if the series* $\sum\limits_{k=1}^{\infty} \beta_k$ *converges.*

The sufficiency of divergence of the series $\sum\limits_{k=1}^{\infty} \alpha_k$ for convergence of this BCF remains unproved for over 50 years. It is proved that, for example, if the series $\sum\limits_{k=1}^{\infty} \alpha_k \alpha_{k+1}$ is divergent, then the BCF (2.1) where all $a_{i(k)} = 1$ converges. More general theorems are established in [21, 30].

It was the problem that led to the simplification of the structure of the BCFs. BCFs with independent variables appeared [23]. When the values of the variables are fixed, such fractions are called BCFs of the special form. O. Baran proposed the notation for such fractions [10]

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)}}, \tag{2.2}$$

where $b_0, a_{i(k)}, b_{i(k)}$ are complex numbers, $i(k) \in \mathcal{J}_k$,

$$\mathcal{J}_k = \{i(k) = (i_1, i_2, \dots, i_k) \ : 1 \le i_k \le i_{k-1} \le ... \le i_0\}, \ k \ge 1,$$

$i_0 = N$ is a fixed positive integer, also $\mathcal{J} = \bigcup_{k=1}^{\infty} \mathcal{J}_k$.

The theory of BCFs with independent variables has been rapidly developed recently in the papers of R. I. Dmytryshyn, T. M. Antonova, D. I. Bodnar, I. B. Bilanyk, and others [2, 4, 6, 11, 13, 16, 14, 18, 25, 26, 27, 28, 35, 36, 39]. One of the reasons for such situation is the efficiency of these BCFs for approximation of functions of several variables, in particular, for the construction of expansion of the relations of Horn's, Appell's, Lauricella–Saran's hypergeometric functions [3, 7, 8, 38].

The first theorem of convergence of the BCF (2.2), where $N = 2$ and all $a_{i(k)} = 1, b_{i(k)} > 0$, is established in [23], which is a necessary and sufficient condition for the convergence of this fraction (analogous to the Seidel-Stern's theorem).

In [15], a multidimensional generalization of Seidel-Stern's criterion for an arbitrary $N, N \ge 2$ was established.

Let us define the set of multiindices for each $m, \ 2 \le m \le N$,

$$\mathcal{J}_n^{(m)} = \{i(n) = (i_1, i_2, \dots, i_n) \ : m \le i_n \le i_{n-1} \le ... \le i_0\}, \ n \ge 1, \tag{2.3}$$

where $i_0 = N$, and let us use the notation

$$m[s] = \underbrace{(m, m, ..., m)}_{s}; \ \underline{m-1}[s] = \underbrace{(m-1, m-1, ..., m-1)}_{s}; \ s = 1, 2, \dots \ .$$

Let us recursively define continued fractions

$$b_{i(r)}^{(m-1)} = b_{i(r)}^{(m-2)} + \mathop{\mathrm{D}}_{s=1}^{\infty} \frac{1}{b_{i(r),\underline{m-1}[s]}^{(m-2)}}, \ i(r) \in \mathcal{J}_r^{(m)}, \ m = \overline{2, N}, \ r \ge 1,$$

with initial conditions $b_{i(k)}^{(0)} = b_{i(k)}, \ i(k) \in \mathcal{J}_k, \ k \ge 1$.

Theorem 9. *The BCF* (2.2)*, where all* $a_{i(k)} = 1, \ b_{i(k)} > 0$*, converges if and only if for each* $m, \ 1 \le m \le N$*, the series* $\sum_{p=1}^{\infty} b_{m[p]}^{(m-1)}$ *diverge, and for each* $m, \ 1 \le m \le N$*, and each multiindex* $i(n), \ i(n) \in \mathcal{J}_n^{(m+1)}, \ n \ge 1$*, the series* $\sum_{p=1}^{\infty} b_{i(n),m[p]}^{(m-1)}$ *diverge.*

Since the elements of the series mentioned in this theorem are not easy to calculate, the following effective sufficient criterion of convergence was established [15].

Theorem 10. *The BCF* (2.2)*, where all* $a_{i(k)} = 1,\ b_{i(k)} > 0$*, converges if for each* $m,\ 1 \le m \le N$*, the series* $\sum\limits_{p=1}^{\infty} b_{m[p]}$*, diverge and for each* $m,\ 1 \le m \le N-1$*, and each multiindex* $i(n),\ i(n) \in \mathcal{J}_n^{(m+1)}$ *the series* $\sum\limits_{p=1}^{\infty} b_{i(n),m[p]}$ *diverge.*

In addition to BCFs with positive elements, the conditions for the convergence of BCFs with real elements are investigated [5, 22, 41]. Besides the convergence, Gladun also studied the stability to perturbations of infinite BCFs.

Theorem 11. *[5] Let the BCF*

$$\operatorname*{D}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{(-1)^{k-1} a_{i(k)}}{1}, \tag{2.4}$$

where all $a_{i(k)} \ge 0$*, satisfies the conditions*

$$\sum_{i_{2k}=1}^{N} a_{i(2k)} \le 1 - \rho_{i(2k-1)}, \quad \sum_{i_{2k+1}=1}^{N} a_{i(2k+1)}/\rho_{i(2k+1)} \le \rho_{i(2k)} - 1,$$

where $0 < \rho_{i(2k-1)} \le 1,\ \rho_{i(2k)} \ge 1,\ i_{(2k-1)}, i_{(2k)} \in \mathcal{I},\ k \ge 1$*, are real numbers and* $\prod\limits_{k=1}^{\infty} \eta_k = 0$*, where* $\eta_{2k} = 1 - \max^{-1}\{\rho_{i(2k)}\},\ \eta_{2k-1} = \max^{-1}\{\rho_{i(2k-1)}\} - 1,\ k \ge 1.$

Then the BCF (2.4) *converges and the following truncation error bounds hold* $|f - f_n| \le (1 + \eta_1) \sum\limits_{i_1=1}^{N} a_{i(1)} \prod\limits_{k=1}^{n} \eta_k.$

A multidimensional generalization of Worpitzky's theorem firstly was established for BCFs with N branches of branching [19].

Theorem 12. *Let the BCF*

$$\left(1 + \operatorname*{D}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{a_{i(k)}}{1}\right)^{-1}, \tag{2.5}$$

where $a_{i(k)} \in \mathbb{C},\ i(k) \in \mathcal{I}$*, satisfy the conditions*

$$\left|a_{i(k)}\right| \le t(1-t)/N,\ 0 \le t \le 1/2,\ i(k) \in \mathcal{I}.$$

Then

1) the BCF (2.5) *converge;*

2) the following truncation error bound holds

$$|f-f_m| \le \frac{(1-2t)t^m}{(1-t)\left[(1-t)^{m+1}-t^{m+1}\right]}, \text{ if } 0 \le t < 1/2, \tag{2.6}$$

and

$$|f-f_m| \le 2/(m+1), \text{ if } t=1/2, \tag{2.7}$$

where f is the value of BCF (2.5)*;*

3) the best value set is the circle $\left|z-1/(1-t^2)\right| \le t/(1-t^2)$*;*

4) the boundary constant $1/(4N)$ $(t=1/2)$ *is unimprovable.*

For a BCF of the special form

$$\left(1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{a_{i(k)}}{1}\right)^{-1} \tag{2.8}$$

we can weaken the conditions by requiring that $\left|a_{i(k)}\right| \le t(1-t)/i_{k-1},\ i(k) \in \mathcal{J}_k,\ k \ge 1$, while preserving the inequalities (2.6) and (2.7). It is possible not to change the conditions by requiring that $\left|a_{i(k)}\right| \le t(1-t)/N,\ i(k) \in \mathcal{J}_k,\ k \ge 1$. Then the truncation error bounds are improved

$$|f-f_n| \le L_n K_n, \text{ if } 0 \le t < 1/2,$$

$$|f-f_n| \le 2L_n/(n+1), \text{ if } t=1/2,$$

where K_n is the right-hand side of the inequality (2.6), $L_n = C_{N+n-1}^{N-1}N^{-n}$ [10]. The effect of considering BCFs of the special type instead of general BCFs is obvious.

R. Dmytryshyn established another Worpitzky-type theorem of convergence of BCF (2.8), different from the one stated above [36].

Theorem 13. *Let the elements $a_{i(k)}, i(k) \in \mathcal{J}_k, k \ge 1$, of the BCF*

$$\sum_{i_1=1}^{N}\frac{a_{i(1)}}{1}+\mathop{\mathrm{D}}_{k=2}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{a_{i(k)}}{1} \tag{2.9}$$

satisfy the conditions

$$\left|a_{i(k)}\right| \le q_{i(k)}^{i_k} q_{i(k-1)}^{i_k-1}\left(1-q_{i(k-1)}\right) \text{ for all } i(k) \in \mathcal{J}_k,\ k \ge 1, \tag{2.10}$$

where $\left\{q_{i(k)}\right\}_{i(k)\in\mathcal{J}_k,\ k\in\mathbb{N}_0}$ is a sequence of real constants such that $0 \le q_{i(k)} < 1$ for all $i(k) \in \mathcal{J}_k, k \ge 0$, or $0 < q_{i(k)} \le 1$ for all $i(k) \in \mathcal{J}_k,\ k \ge 0$. Then

1) the BCF with independent variables (2.9) *converges absolutely, and its value and the value of its approximants belong to the closed circle $|z| \le 1-q_0^N$;*

2) if the inequality (2.10) *holds $q_{i(k)} = 1/2$ for all $i(k) \in \mathcal{J}_k, k \ge 0$, then the closed circle $|z| \le 1-2^{-N}$ is the «best» value set of the BCF with independent variables* (2.9)*.*

The absolute convergence of the BCF with independent variables (2.9) was also proved in [2] under the conditions that

$$\left|a_{i(k)}\right| \leq t_{i(k)} \left(1 - \sum_{i_{k+1}=1}^{i_k} t_{i(k+1)}\right) \text{ for all } i(k) \in \mathcal{J}$$

where $t_{i(k)}$ for all $i(k) \in \mathcal{J}$ are nonnegative constants such that

$$\sum_{i_{k+1}=1}^{i_k} t_{i(k+1)} < 1, \text{ for all } i(k) \in \mathcal{J}.$$

There is a close result to Worpitzky's theorem, it is Leighton-Wall convergence theorem of continued fractions [47]. The idea of its proof is to construct a continued fraction based on the approximants $\{f_n\}$ of (1.3), but arranged in a different order $f_2, f_1, f_4, f_3, \ldots, f_n, f_{n-1}, \ldots$. Leighton-Wall theorem is established by applying the Worpitzky's theorem to the obtained fraction. This approach cannot be generalized to BCFs. The same formulation of the multidimensional Leighton-Wall theorem, namely «the BCF (2.5) converges if for some positive constants ε, M the conditions are satisfied

$$\left|a_{i(2k-1)}\right| \leq \varepsilon, \ \left|a_{i(2k)}\right| \geq M, \ k = 1, 2, \ldots, \ i(p) \in \mathcal{I}, \ p \geq 1.\text{»} \tag{2.11}$$

is incorrect. Y. Boltarovych showed that for any $\varepsilon > 0$ and $M > 0$ one can construct a BCF that satisfies (2.11) and is divergent [32]. Therefore, the theorem can not be generalized to BCFs if the conditions of (2.11) are satisfied.

O. Baran established an analog of the Leighton-Wall theorem for the BCF of the special form

$$\mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{1} \tag{2.12}$$

with complex partial numerators [11]. Let $l = l(i(k)) = \sum_{s=1}^{k} \delta_{i_k}^{i_s}$, where $\delta_{i_k}^{i_s}$ is the Kronecker symbol. Let us divide the set of indices $\mathcal{J}$ into subsets that do not intersect in pairs $\mathcal{J} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$, where

$$\mathcal{E}_1 = \{i(k) : i(k) \in \mathcal{J}_k, \ l = 1, \ k \geq 1\},$$

$$\mathcal{E}_2 = \{i(k) : \{i(k) \in \mathcal{J}_k, \ l \text{ is even}, \ k \geq 2\},$$

$$\mathcal{E}_3 = \{i(k) : \{i(k) \in \mathcal{J}_k, \ l \text{ is odd}, \ l > 1, \ k \geq 3\}.$$

Theorem 14. *Let all elements of the BCF* (2.12), $a_{i(k)}$ *are complex numbers and the following conditions hold*

1) $N > 1$,

$$\left|c_{i(k)}\right| \leq r_1/(i_{k-1} - 1), \ \textit{if } i(k) \in \mathcal{E}_1,$$

$$\left|c_{i(k)}\right| \leq r, \text{ if } i(k) \in \mathcal{E}_3, \tag{2.13}$$

$$\left|c_{i(k)}\right| \geq (2+r_1)(1+r_1+r), \text{ if } i(k) \in \mathcal{E}_2, \tag{2.14}$$

2) $N = 1$, and for the elements $c_{i(k)}$ holds (2.13)*, if $i(k) \in \mathcal{E}_1 \cup \mathcal{E}_3$, or* (2.14)*, if $i(k) \in \mathcal{E}_2$, where $r_1 = 0$ for $N = 1$ and $0 < r_1 < (1-3r)/(1+r)$ for $N > 1, 0 < r < 1/3$. Then the BCF* (2.12) *converges and the truncation error bound holds*

$$|f - f_n| \leq MC_{N+n}^{N-1} q^{n+1},$$

where $M = 1 - r$ if $N = 1$ and $M = \max\limits_{1 \leq p \leq N} (r_1/r)^p$ if $N > 1$, $q = \sqrt{(2+r_1)\, r/(1-r_1-r)}$.

In particular, a multidimensional generalization of the theorem on twin convergence sets for continued fractions is considered in [13].

Theorem 15. *The BCF*

$$1 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{c_{i(k)}^2}{1}, \tag{2.15}$$

where $c_{i(k)} \in \mathbb{C}$, converges if

a) $N > 1$ and the elements of $c_{i(k)}$ satisfy the conditions

$$\left|c_{i(k)} \pm i\Gamma_{1,i_k}\right| \leq \xi_{1,i_k}, \ (\xi_{1,i_k} + |\Gamma_{1,i_k}|)^2 \leq (\rho_1 - \varepsilon_1)/(i_{k-1} - 1), \ i(k) \in \mathcal{E}_1,$$

$$\left|c_{i(k)} \pm i\Gamma_{3,i_k}\right| \leq \xi_{3,i_k}, \ (\xi_{3,i_k} + |\Gamma_{3,i_k}|)^2 \leq \rho - \varepsilon_3, \ i(k) \in \mathcal{E}_3, \tag{2.16}$$

$$\left|c_{i(k)} \pm i\Gamma_{2,i_k}\right| \geq \xi_{2,i_k}, (\xi_{2,i_k} - |\Gamma_{2,i_k}|)^2 \geq (2+\rho_1)(1+\rho_1+\rho+\varepsilon_2), i(k) \in \mathcal{E}_2, \tag{2.17}$$

or

b) $N = 1$ and the elements of $c_{i(k)}$ satisfy (2.16) *if $i(k) \in \mathcal{E}_1 \cup \mathcal{E}_3$, or* (2.17) *if $i(k) \in \mathcal{E}_2$, where $\rho_1 = 0$ for $N = 1$ and $\rho_1 > 0$ for $N > 1$, $\rho > 0$, $0 < \varepsilon_1 < \rho_1, 0 < \varepsilon_3 < \rho$, $\varepsilon_2 > 0, \Gamma_{j,s} \in \mathbb{C}, \xi_{j,s} > 0, j = \overline{1,3}$, $s = \overline{1,N}$.*

The results obtained by O. Baran are a certain strengthening and generalization of the results established for continued fractions. If we set $N = 1$, i.e., when the BCF degenerates into a continued fraction, then the convergence set can be wider for certain values of the parameters. In addition to the classical results, we obtain additional truncation error bounds for continued fractions.

Let us consider a multidimensional generalization of the Śleszyński-Pringsheim theorem [19].

Theorem 16. *The BCF* (2.1) *with complex elements satisfying the conditions*

$$\left|b_{i(k)}\right| \geq \left|a_{i(k)}\right| + N, \ i(k) \in \mathcal{I}, \tag{2.18}$$

converges absolutely and its best value set is the circle $|z - b_0| \leq N$.

Theorem 17. *The BCF* (2.1) *with complex elements satisfying the conditions*

$$\left|b_{i(k)}\right| \geq N \left|a_{i(k)}\right| + 1,\ i(k) \in \mathcal{I}, \tag{2.19}$$

converges absolutely and its best value set is the circle $|z - b_0| \leq 1$.

For the BCF of the special form (2.2) the conditions (2.18) and (2.19) can be replaced with the following conditions [12]

$$\left|b_{i(k)}\right| \geq \left|a_{i(k)}\right| + i_k,\ i(k) \in \mathcal{J},$$

and

$$\left|b_{i(k)}\right| \geq i_{k-1} \left|a_{i(k)}\right| + 1,\ i(k) \in \mathcal{J}.$$

The multidimensional generalization of Perron's theorem is the next theorem of convergence of the BCF.

Theorem 18. *The BCF* (2.1) *with complex elements satisfying the conditions*

$$\left|b_{i(k)}\right| \geq \sum_{i_{k+1}=1}^{N} \left|a_{i(k+1)}\right| + 1,\ i(k) \in \mathcal{J}, \tag{2.20}$$

converges absolutely. The value set is the circle $\{z : |z| \leq \sum_{i_1=1}^{N} |a_{i_1}|\}$.

This theorem was established by M. O. Nedashkovskyj [51] and extended to matrix BCFs (2.1) when $a_{i(k)},\ b_{i(k)}$ are square nondegenerate matrices for $i(k) \in \mathcal{I}$ [52]. The conditions (2.20) should be replaced with

$$\left\|b_{i(k)}^{-1}\right\| \leq \left(1 + \sum_{i_{k+1}=1}^{N} \left\|a_{i(k)}\right\|\right)^{-1},\ i(k) \in \mathcal{I}.$$

Let us consider the BCF

$$\mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}}. \tag{2.21}$$

By analogy with Theorem 1, we could assume that if the series $\sum_{k=1}^{\infty} \beta_k$ converges, where $\beta_k = \max\{|b_{i(k)}|,\ i(k) \in \mathcal{J}_k\},\ k \geq 1$, then this fraction is divergent. But it is not. Even if this condition is met, it is possible to choose a BCF so that it is convergent [20].

Using the multidimensional generalization of Seidel's criterion, the multidimensional analog of Van Fleck's theorem is established [17].

Theorem 19. *Let the partial denominators of the BCF of the special form* (2.21) *lie in the domain*

$$G(\varepsilon)=\{z\in\mathbb{C}: z\neq 0, |\arg z|<\pi/2-\varepsilon\}, \tag{2.22}$$

where ε is an arbitrary positive integer, $0<\varepsilon<\pi/2$. Then

1) every nth approximant f_n of the BCF (2.21) *lie in the domain* (2.22)*;*

2) there are finite limits of even and odd approximants;

3) the BCF (2.21) *converges if for each m, $1\leq m\leq N$, the series $\sum\limits_{p=1}^{\infty}\left|b_{m[p]}\right|$ diverge as well as for each m, $1\leq m\leq N-1$, and each multiindex $i(n)$, $i(n)\in\mathcal{J}_n^{(m+1)}$, the series $\sum\limits_{p=1}^{\infty}\left|b_{i(n),m[p]}\right|$ diverge.*

For the BCF (2.1), where all $a_{i(k)}=1$, $b_0=0$, a similar statement is true [19]. But condition 3) of the theorem should be replaced with the condition of divergence of the series $\sum\limits_{k=2}^{\infty}\alpha_k S_{k-1}$, where $\alpha_k=\{\min|b_{i(k)}|,\ i(k)\in\mathcal{J}_k\}$, $k\geq 2$ $S_k=\alpha_k+N^{-1}\alpha_{k-2}+N^{-2}\alpha_{k-4}+\cdots+N^{-[(k-1)/2]}\alpha_{k-2[(k-1)/2]}$. For $N=1$, the divergence of this series is equivalent to the divergence of the series $\sum\limits_{k=2}^{\infty}\alpha_k$.

For certain subsets of the angular domain (2.22), truncation error bounds have been established, in particular, if the elements of the BCF of the special form (2.21) satisfy the conditions

$$\left|\arg b_{i(k)}\right|\leq\theta,\ \theta<\pi/4,\ i(k)\in\mathcal{J},$$

$$\Re(b_{i(n)})\geq\delta,\ \Re(b_{1[s]})\geq\delta/s^{\beta},\ \Re(b_{i(n),1[s]})\geq\delta/s^{\beta},$$

$$s\geq 1,\ 0<\delta<1,\ 0\leq\beta\leq 1/2,\ i(n)\in\mathcal{J}_n^{(2)},$$

then

$$|f_m-f_{Nn}|<M\ln^{-1}\left(1+\frac{\alpha}{1-\beta}\left((n+1)^{1-\beta}-1\right)\right),\ m\geq Nn,\ n\in\mathbb{N},$$

where α and M are positive constants independent of n and m [14].

A separate area in the theory of convergence of continued fractions is the so-called parabolic theorems. The first proofs of this type were obtained by W. J. Thron, W. T. Scott, and H. S. Wall in 1942. For BCFs, the first ones were Theorems 3.22 and 3.23 [19]. Let us formulate one of these theorems for BCFs.

Theorem 20. *Let the elements of the BCF* (2.5) $a_{i(k)},\ i(k)\in\mathcal{I}$, *lie in the domain*

$$P_\varepsilon(\gamma)=\left\{z\in\mathbb{C}:|z|-Re\left(ze^{-2\gamma i}\right)\le(2N)^{-1}(1-\varepsilon)\,cos^2\gamma\right\},\qquad(2.23)$$

where γ, ε *are arbitrary small, real numbers* $(0<\varepsilon<1,\ -\pi/2<\gamma<\pi/2)$, N *is the number of branches of branching of the BCF* (2.5). *Then*

1) exist finite limits of the even and odd approximants of the BCF (2.5)*;*

2) the BCF (2.5) *is convergent if one of the following two conditions is satisfied: there exists a number* k *such that all* $a_{i(k)}=0,\ i(k)\in\mathcal{I}_k,\ k\ge1$, *or the series* $\sum_{k=1}^{\infty}\delta_k$ *is divergent, where* $\delta_k=\min\left(1/\left|a_{i(k)}\right|,\ i(k)\in\mathcal{I}_k\right),\ k\ge1$.

This research is continued in the works of O. Baran, D. Bodnar, T. Antonova, I. Bilanyk, R. Dmytryshyn [1, 13, 18, 24].

Theorem 21. *[1] Let there exist positive constants* ε,φ, $\varepsilon<1$ *and* $\varphi<\pi/(2(1+\varepsilon))$ *such that the elements of* $a_{i(k)}$ *for all* $i(k)\in\mathcal{J}_k,\ k\ge1$, *of the BCF* (2.12), *satisfy the conditions*

$$\sum_{i_k=1}^{i_{k-1}}\frac{\left|a_{i(k)}\right|-\mathrm{Re}\left(a_{i(k)}e^{-i\left(\varphi_{i(k-1)}+\varphi_{i(k)}\right)}\right)}{\cos\varphi_{i(k)}-p_{i(k)}}\le2(1-\varepsilon)p_{i(k-1)},\ i(k-1)\in\mathcal{J}_{k-1},\ k\ge1,$$

where $\varphi_{i(k)},p_{i(k)}$ *are some real numbers such that* $\left|\varphi_{i(k-1)}\right|\le\varphi$ *for all* $i(k)\in\mathcal{J}_k,\ k\ge0$, $p_0\ge0,\ 0\le p_{i(k)}\le(1-\varepsilon)\cos\varphi_{i(k)}$ *for all* $i(k)\in\mathcal{J}_k,\ k\ge1$, *and let* $\{f_n\}_{n\in\mathbb{N}}$ *be the sequence of its approximants. Then*

1) the approximants of the BCF (2.12) *are finite and lie in the half-plane* $\mathrm{Re}\left(\omega e^{-i\varphi_0}\right)\ge1-p_0$*;*

2) there exist finite limits on the sequences of even $\{f_{2n}\}_{n\in\mathbb{N}}$ *and odd* $\{f_{2n-1}\}_{n\in\mathbb{N}}$ *approximants of the BCF* (2.12).

Let us consider a two-dimensional BCF of the special form $(i_0=2)$

$$\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{a_{i(k)}}{b_{i(k)}},\qquad(2.24)$$

where $a_{i(k)},\ b_{i(k)}\in\mathbb{C},\ i(k)\in\mathcal{J}$. An analog of the Thron and Jones theorem for these fractions [18] is established.

Theorem 22. *Let the elements of the two-dimensional BCF of the special form* (2.24) *satisfy the conditions*

$$\left|a_{1[n]}\right|-\Re\left(a_{1[n]}e^{-i(\psi_n+\psi_{n-1})}\right)\le2p_{n-1}\left(\Re\left(b_{1[n]}e^{-i\psi_n}\right)-p_n\right),\ n=1,2,\ldots;$$

$$\left|a_{2[k],1[n]}\right|-\Re\left(a_{2[k],1[n]}e^{-i(\psi_k+\psi_{k-1})}\right)\le2s_{k,n-1}\left(\Re\left(b_{2[k],1[n]}e^{-i\psi_k}\right)-s_{k,n}\right),$$
$$n=1,2,\ldots,\ k=1,2,\ldots;$$

$$\Re\left(b_{2[n]}e^{-i(\arg a_{2[n],1}-\psi_n)}\right)\ge q_n,\ n=1,2,\ldots;$$

and for some $l_k,\ l_k \in \mathbb{Z}$,

$$\arg a_{2[k],1} + \arg a_{2[k+1],1} - \arg a_{2[k+1]} = \psi_k + \psi_{k+1} + 2\pi l_k,\ k = 1, 2, \dots,$$

where ψ_k *are real numbers,* $p_n,\ s_{k,n},\ q_n,\ n = 0, 1, \dots;\ k = 1.2, \dots,$ *are some positive constants such that each of the sequences*

$$\left\{\frac{a_{1[n]}}{p_n p_{n-1}}\right\}_{n=1}^{\infty},\ \left\{\frac{a_{2[k],1[n]}}{s_{k,n} s_{k,n-1}}\right\}_{n=1}^{\infty},\ k = 1, 2, \dots,\ \left\{\frac{a_{2[n]}}{q_n, q_{n-1}}\right\}_{n=1}^{\infty}$$

is bounded. Then the BCF (2.24) *converges.*

The parabolic theorems have been used to study the convergence of functional continued fractions and functional BCFs.

We do not pretend to present all the results concerning general and special types of BCFs. The emphasis is on the analogs of the classical results of continued fractions for the multidimensional cases.

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Received: 01.10.2024. *Accepted:* 10.10.2024