Researches in Mathematics https://vestnmath.dnu.dp.ua

Res. Math. **32**(2), 2024, p. 70–87 ISSN (Print) 2664-4991
doi:10.15421/242420 ISSN (Online) 2664-5009

UDK 517.5

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Approximation of functions by linear methods in weighted Orlicz type spaces with variable exponent [1]

Celebrating the 75th birthdays of Professors V. F. Babenko and V. O. Kofanov

Abstract. The approximation properties of various classical methods of linear summation of Fourier series in weighted spaces of Orlicz type with variable exponent are considered. In particular, in terms of approximation by such methods the constructive characterizations for classes of functions whose moduli of smoothness do not exceed some majorant are obtained.
Key words: direct approximation theorem, inverse approximation theorem, modulus of smoothness, spaces with variable exponent, Zygmund means, Abel-Poisson means, Taylor-Abel-Poisson means, de la Vallée Poussin means

Анотація. Розглянуто апроксимаційні властивості різних класичних лінійних методів підсумовування рядів Фур'є у зважених просторах типу Орліча зі змінним показником. Зокрема, у термінах апроксимації такими методами отримано конструктивні характеристики для класів функцій, модулі гладкості яких не перевищують деякої мажоранти.
Ключові слова: пряма теорема наближення, обернена теорема наближення, модуль гладкості, простори зі змінною експонентою, середні Зигмунда, середні Абеля-Пуассона, середні Тейлора-Абеля-Пуассона, середні Валле-Пуссена

MSC2020: Pri 41A65, Sec 26A15, 41A25

1. Introduction

The paper is devoted to the study of the approximation properties of the classical linear summation methods of Fourier series. Such methods represent

[1]This work was partially supported by a grant from the Simons Foundation (1290607, A. Shidlich).

a significant area of research within approximation theory. In particular, this is due to the fact that most of these methods naturally give rise to a corresponding set of approximations. These topics are well studied in classical function spaces such as Lebesgue spaces L_p, spaces of continuous functions C, etc. However, there are relatively few papers dealing with similar issues in Banach spaces of Orlicz type. This is especially true for direct and inverse approximation theorems by linear summation methods. Here, we study such methods in weighted Orlicz type spaces $\mathcal{S}_{\mathbf{p},\,\mu}$ with variable exponent. The spaces $\mathcal{S}_{\mathbf{p},\,\mu}$ are defined in the following way. Let $\mathbf{p} = \{p_k\}_{k=-\infty}^{\infty}$ be a sequence of positive numbers such that

$$1 \le p_k \le K, \quad k = 0, \pm 1, \pm 2, \ldots, \ (k \in \mathbb{Z}), \tag{1.1}$$

where K is a positive number, and $\mu = \{\mu_k\}_{k=-\infty}^{\infty}$ be a sequence of nonnegative numbers. Let $\mathcal{S}_{\mathbf{p},\,\mu}$ be the space of all 2π-periodic Lebesgue summable functions f ($f \in L_1$) such that the following quantity (which is also called the Luxemburg norm of f) is finite:

$$\|f\|_{\mathbf{p},\,\mu} := \inf\left\{a > 0 : \ \sum_{k\in\mathbb{Z}} \mu_k |\widehat{f}(k)/a|^{p_k} \le 1\right\}. \tag{1.2}$$

where $\widehat{f}(k) := [f]^{\widehat{}}(k) = (2\pi)^{-1} \int_0^{2\pi} f(t) \mathrm{e}^{-\mathrm{i}kt} \mathrm{d}t$, $k \in \mathbb{Z}$, are the Fourier coefficients of f.

The spaces $\mathcal{S}_{\mathbf{p},\,\mu}$ defined in this way are the Banach spaces. They were considered in [1]. In particular, direct and inverse approximation theorems in terms of the best approximations of functions and moduli of fractional smoothness are proved for the spaces $\mathcal{S}_{\mathbf{p},\,\mu}$ in [1]. In case when $p_k = p$ and $\mu_k = 1$, $k \in \mathbb{Z}$, $p \ge 1$, they coincide with the well-known spaces $\mathcal{S}^p$ [27]. In $\mathcal{S}^p$, approximative properties of linear summation methods of Fourier series were studied in [25]. In [9], approximative properties of linear summation methods were studied in the Orlicz spaces $\mathcal{S}_M$. The purpose of this paper is to continue this study in the Orlicz type spaces. In particular, in the spaces $\mathcal{S}_{\mathbf{p},\,\mu}$, we obtain direct and inverse approximation theorems for the Zygmund, Abel-Poisson, Taylor-Abel-Poisson, de la Vallée Poussin methods which relate the approximation properties of these methods to differential properties of functions.

2. Preliminaries

Let $\mathcal{T}_n$, $n = 0, 1, \ldots$, be the set of all trigonometric polynomials $\tau_n(x) := \sum_{|k|\le n} c_k \mathrm{e}^{\mathrm{i}kx}$ of the order n, where c_k are arbitrary complex numbers. Let also

$$E_n(f)_{\mathbf{p},\,\mu} := \inf_{\tau_{n-1}\in\mathcal{T}_n} \|f - \tau_n\|_{\mathbf{p},\,\mu}$$

denotes the best approximation of a function $f \in \mathcal{S}_{\mathbf{p},\,\mu}$ by the trigonometric polynomials $\tau_n \in \mathcal{T}_n$ in $\mathcal{S}_{\mathbf{p},\,\mu}$.

Further, for any function $f \in L_1$ with the Fourier series of the form

$$S[f](x) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx},$$

consider the following linear transformations:

$$S_n(f)(x) := \sum_{k=-n}^{n} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}, \quad n = 0, 1, \ldots,$$

$$Z_n^{(s)}(f)(x) := \sum_{k=-n}^{n} \left(1 - \left(\frac{|k|}{n+1}\right)^s\right)\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}, \quad s > 0,$$

$$V_{n,m}(f)(x) = \sum_{k=-n}^{n} \lambda_{|k|,m}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx},$$

where $m = 0, 1, \ldots, n$, $m = m(n)$, and

$$\lambda_{k,m} = \begin{cases} 1, & k = 0, 1, \ldots, n-m, \\ 1 - \frac{k-n+m}{m+1}, & k = n-m+1, \ldots, n, \end{cases}$$

$$P_{\varrho,s}(f)(x) := \sum_{k\in\mathbb{Z}} \varrho^{|k|^s}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}, \quad s > 0, \ \varrho \in [0, 1),$$

and

$$A_{\varrho,r}(f)(x) := \sum_{k\in\mathbb{Z}} \lambda_{|k|,r}(\varrho)\widehat{f}_k\mathrm{e}^{\mathrm{i}kx}, \tag{2.1}$$

where for $k = 0, 1, \ldots, r-1$, the numbers $\lambda_{k,r}(\varrho) \equiv 1$ and

$$\lambda_{k,r}(\varrho) := \sum_{j=0}^{r-1} \binom{k}{j}(1-\varrho)^j\varrho^{k-j}, \quad k = r, r+1, \ldots, \quad \varrho \in [0, 1]. \tag{2.2}$$

The expressions $S_n(f)$, $Z_n^{(s)}(f)$, $V_{n,m}(f)$ and $P_{\varrho,s}(f)$ are called the partial Fourier sum, the Zygmund sum, the de la Vallée Poussin sums and the generalised Abel-Poisson sum of the function f, respectively.

If $s = 1$, then the sum $Z_n^{(s)}(f)$ coincides with the Fejér sum of f, i.e.,

$$Z_n^{(1)}(f)(x) = \sigma_n(f)(x) := \frac{1}{n+1}\sum_{k=0}^{n} S_k(f)(x) = \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}.$$

Also note that $V_{n,0}(f) = S_n(f)$ and $V_{n,n}(f) = \sigma_n(f)$.

The expression $A_{\varrho,r}(f)$ is called the Taylor-Abel-Poisson sum of the function f. Note that the transformation $A_{\varrho,r}$ can be considered as a linear operator on L_1 into itself. Indeed, for $k = 0, 1, \ldots, r-1$, the numbers $\lambda_{k,r}(\varrho) \equiv 1$ and

$$\sum_{j=0}^{r-1} \binom{k}{j}(1-\varrho)^j\varrho^{k-j} \le rq^k k^{r-1}, \text{ where } q = \max\{1-\varrho, \varrho\},$$

and hence, for any $f \in L_1$ and $0 < \varrho < 1$, the series on the right-hand side of (2.1) is majorized by the convergent series $2r\|f\|_{L_1} \sum_{k=r}^{\infty} q^k k^{r-1}$.

The operators $A_{\varrho,r}$ were first studied in [24], where in the terms of these operators, the author gives the structural characteristic of Hardy-Lipschitz classes of one variable functions, holomorphic in the unit disc in the complex plane. These operators possess the main properties of the Abel-Poisson and Taylor operators, but they can also be adapted to the smoothness properties of functions of arbitrarily large order. Their approximative properties were also considered in [25, 20, 21, 22]. In general case, the operators $P_{\varrho,s}$ were perhaps first considered as the aggregates of approximation of functions of one variable in [6, 7]. In the case when $r = s = 1$, the operators $A_{\varrho,1}$ and $P_{\varrho,1}$ coincide with each other and generate the Abel-Poisson summation method of Fourier series. The problem of approximation of 2π-periodic functions by the classical linear methods (methods of Zygmund, Abel-Poisson, Fejér, de la Vallée Poussin) in Lebegue spaces L_p and spaces of continuous functions C has a long history, full of many results (see, for example, the monographs [4, 29, 8, 28]). In the Orlicz type spaces such methods were studied in [15, 11, 2, 3, 13, 14] etc.

3. Derivatives and moduli of smoothness

Let $\psi = \{\psi(k)\}_{k\in\mathbb{Z}}$ be a numerical sequence whose members are not all zero and

$$\mathscr{Z}(\psi) := \{k \in \mathbb{Z} : \psi(k) = 0\}.$$

In what follows, assume that the number of elements of the set $\mathscr{Z}(\psi)$ is finite.

If for a function $f \in L_1$, there exists a function $g \in L_1$ with the Fourier series of the form

$$S[g](x) = \sum_{k\in\mathbb{Z}\setminus\mathscr{Z}(\psi)} \widehat{f}(k)e^{\mathrm{i}kx}/\psi(k), \tag{3.1}$$

then we say that for the function f, there exists ψ-derivative g, for which we use the notation $g = f^{\psi}$.

This definition of ψ-derivative is adapted to the needs of the research described in this paper and it is not fundamentally different from the established concept of ψ-derivative of A.I. Stepanets [28, Ch. XI].

In the paper, we consider ψ-derivatives defined by the sequences of the following two forms: 1) $\psi(k) = |k|^{-s}$, $k \in \mathbb{Z}$, $s > 0$, and 2) $\psi(k) = 0$ for $|k| \le r-1$ and $\psi(k) = (|k|-r)!/(|k|!)$ for $|k| \ge r$, where $r \in \mathbb{N}$. In the first case, for ψ-derivative of f, we use the notation $f^{(s)}$ and in the second case, we use the notation $f^{[r]}$. If $r = 0$ then we set $f^{(0)} = f^{[0]} = f$. Also note that $f^{(1)} = f^{[1]}$.

Denote by $P(f)(\varrho, x)$, $0 \le \varrho < 1$, the Poisson integral (the Poisson operator) of f, i.e.,

$$P(f)(\varrho, x) := \frac{1}{2\pi}\int_0^{2\pi} f(t)P(\varrho, x-t)dt, \tag{3.2}$$

where $P(\varrho,t)=\frac{1-\varrho^2}{|1-\varrho e^{it}|^2}$ is the Poisson kernel.

According to the decomposition of the Poisson kernel in powers of ϱ, for any function $f\in L_1$, its Poisson integral $P(f)(\varrho,x)$, with $\varrho\in[0,1)$ and $x\in\mathbb{T}$ can be written in the form

$$P(f)\,(\varrho,x)=\sum_{k\in\mathbb{Z}}\varrho^{|k|}\widehat{f}_k e^{ikx}. \tag{3.3}$$

In the terms of Poisson integrals, we give the following interpretation of the derivative $f^{[r]}$:

$$P(f^{[r]})(\varrho,x)=\varrho^r\frac{\partial^r}{\partial\varrho^r}P(f)(\varrho,x),\quad \varrho\in[0,1), \tag{3.4}$$

and by virtue of the well-known theorem on radial limit values of the Poisson integral (see, e.g., [23]), for almost all $x\in\mathbb{T}$

$$f^{[r]}(x)=\lim_{\varrho\to 1-}\frac{\partial^r}{\partial\varrho^r}P(f)(\varrho,x).$$

The modulus of smoothness of $f\in\mathcal{S}_{\mathbf{p},\,\mu}$ of the index $\alpha>0$ is defined by

$$\omega_\alpha(f,\delta)_{\mathbf{p},\,\mu}:=\sup_{|h|\le\delta}\|\Delta_h^\alpha f\|_{\mathbf{p},\,\mu}=\sup_{|h|\le\delta}\Big\|\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}f(x-jh)\Big\|_{\mathbf{p},\,\mu},$$

where $\delta>0$, $\binom{\alpha}{0}:=1$, $\binom{\alpha}{j}=\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-j+1)/j!$, $j\in\mathbb{N}$.

Let ω be a function defined on the interval $[0,1]$. For $\alpha>0$, we set

$$\mathcal{S}_{\mathbf{p},\,\mu}H_\omega^\alpha:=\left\{f\in\mathcal{S}_{\mathbf{p},\,\mu}:\quad \omega_\alpha(f,\delta)_{\mathbf{p},\,\mu}=\mathcal{O}(\omega(\delta)),\quad \delta\to 0+\right\}.$$

Let $\mathscr{B}$ and $\mathscr{B}_s$, $s\in\mathbb{N}$, denote the sets of all continuous strictly increasing functions $\omega(t)$, $t\in[0,1]$, with $\omega(0)=0$ satisfying the following conditions (3.5) and (3.6), respectively:

$$\sum_{v=n+1}^{\infty}v^{-1}\,\omega(v^{-1})=\mathcal{O}[\omega(n^{-1})],\quad n\to\infty, \tag{3.5}$$

and

$$\sum_{v=1}^{n}v^{s-1}\,\omega(v^{-1})=\mathcal{O}[n^s\omega(n^{-1})],\quad n\to\infty. \tag{3.6}$$

Conditions (3.5) and (3.6) are well-known (see, for example, [5]).

Remark 1. From condition (3.6) it follows that $\liminf\limits_{\delta\to 0+}(\delta^{-s}\omega(\delta))>0$ or that for any $r\ge s$, the quantity $(1-\varrho)^{r-s}\omega(1-\varrho)\ge C_0(1-\varrho)^r$ as $\varrho\to 1-$.

Hereinafter, $C_0, C_1, \dots$ are some positive constants that do not depend on the values that are parameters in this consideration (in the considered case, independent of the variable ϱ).

4. The main results.

4.1. Approximations of functions by the Zygmund and Fejér methods and the best approximations

Here and below, we assume that $\mathbf{p} = \{p_k\}_{k=-\infty}^{\infty}$ and $\mu = \{\mu_k\}_{k=-\infty}^{\infty}$ are sequences of nonnegative numbers such that condition (1.1) holds.

Theorem 1. *Assume that $f \in L_1$, $s > 0$ and $\omega \in \mathscr{B}$. The following statements are equivalent:*

1) $\|S_n(f^{(s)})\|_{\mathbf{p},\mu} = \mathcal{O}(n^s\omega(n^{-1})), \quad n \to \infty;$
2) $\left\| f - Z_n^{(s)}(f) \right\|_{\mathbf{p},\mu} = \mathcal{O}(\omega(n^{-1})), \quad n \to \infty;$
3) $f \in \mathcal{S}_{\mathbf{p},\mu} H^s_\omega$.

Furthermore, if one of the statements 1), 2) or 3) is satisfied, then

4) $E_n(f)_{\mathbf{p},\mu} = \mathcal{O}(\omega(n^{-1})), \quad n \to \infty;$

If $\omega \in \mathscr{B} \cap \mathscr{B}_s$, then the statements 1)–4) *are equivalent.*

Note that in the case when $s \in \mathbb{N}$ and $\omega \in \mathscr{B}_s$, the relation 1) of Theorem 1 is equivalent to the corresponding relation for the derivative $f^{[s]}$:

$$\|S_n(f^{[s]})\|_{\mathbf{p},\mu} = \mathcal{O}(n^s\omega(n^{-1})), \quad n \to \infty. \tag{4.1}$$

Indeed, by the definition $0 = |\widehat{f^{[s]}}(k)| \le |\widehat{f^{(s)}}(k)|$ for $|k| < s$, and for $|k| \ge s$

$$|\widehat{f^{[s]}}(k)| = |k|(|k|-1)\cdot \ldots \cdot (|k|-s+1)\widehat{f}(k) \le |k|^s|\widehat{f}(k)| = |\widehat{f^{(s)}}(k)|.$$

Therefore, if the statement 1) of Theorem 1 holds, then

$$\|S_n(f^{[s]})\|_{\mathbf{p},\mu} \le \|S_n(f^{(s)})\|_{\mathbf{p},\mu} = \mathcal{O}(n^s\omega(n^{-1})), \quad n \to \infty.$$

On the other hand, for $|k| \ge s$, we have

$$|\widehat{f^{[s]}}(k)| = |k|^s \cdot \Big(1 - \frac{1}{|k|}\Big) \cdot \ldots \cdot \Big(1 - \frac{s-1}{|k|}\Big)|\widehat{f}(k)| \ge \frac{|k|^s}{s^s}|\widehat{f}(k)| = s^{-s}|\widehat{f^{(s)}}(k)|.$$

Therefore, taking into account Remark 1, we see that relation (4.1) yields the statement 1):

$$\|S_n(f^{(s)})\|_{\mathbf{p},\mu} \le \|S_{s-1}(f^{(s)})\|_{\mathbf{p},\mu} + \Bigg\| \sum_{s \le |k| \le n} |k|^s \widehat{f}(k) \mathrm{e}^{\mathrm{i}kx} \Bigg\|_{\mathbf{p},\mu}$$

$$\le \|S_{s-1}(f^{(s)})\|_{\mathbf{p},\mu} + s^s\|S_n(f^{[s]})\|_{\mathbf{p},\mu} = \mathcal{O}(n^s\omega(n^{-1})), \quad n \to \infty.$$

Hence, the following assertion is valid:

Theorem 2. *Assume that $f\in L_1$, $s\in\mathbb{N}$ and $\omega\in\mathscr{B}\cap\mathscr{B}_s$. The following statements are equivalent:*

1) $\|S_n(f^{\{s\}})\|_{\mathbf{p},\,\mu}=\mathcal{O}(n^s\omega(n^{-1}))$, $\quad n\to\infty$, *where $f^{\{s\}}$ is one of the derivatives $f^{[s]}$ or $f^{(s)}$;*

2) $\Big\|f-Z_n^{(s)}(f)\Big\|_{\mathbf{p},\,\mu}=\mathcal{O}(\omega(n^{-1}))$, $\quad n\to\infty$;

3) $f\in\mathcal{S}_{\mathbf{p},\,\mu}H^s_\omega$;

4) $E_n(f)_{\mathbf{p},\,\mu}=\mathcal{O}(\omega(n^{-1}))$, $\quad n\to\infty$.

In the case when $s=1$, we have $f^{(1)}=f^{[1]}$ and $Z_n^{(1)}(f)=\sigma_n(f)$.

Corollary 1. *Assume that $f\in L_1$ and $\omega\in\mathscr{B}\cap\mathscr{B}_1$. The following statements are equivalent:*

1) $\|S_n(f^{[1]})\|_{\mathbf{p},\,\mu}=\mathcal{O}(n\omega(n^{-1}))$, $\quad n\to\infty$;

2) $\|f-\sigma_n(f)\|_{\mathbf{p},\,\mu}=\mathcal{O}(\omega(n^{-1}))$, $\quad n\to\infty$;

3) $f\in\mathcal{S}_{\mathbf{p},\,\mu}H^1_\omega$;

4) $E_n(f)_{\mathbf{p},\,\mu}=\mathcal{O}(\omega(n^{-1}))$, $\quad n\to\infty$.

Before proving Theorem 1 let us make a few comments. Note that in the proposed assertions, the implication 1) $\Rightarrow$ 3) gives a condition for any function $f\in L_1$ to belong to the set $\mathcal{S}_{\mathbf{p},\,\mu}H^s_\omega$ and, in particular, to the space $\mathcal{S}_{\mathbf{p},\,\mu}$.

The equivalence 2) $\Leftrightarrow$ 3) is the statement of the type direct and inverse theorem for Zygmund and Fejér method [8].

Since for any $f\in\mathcal{S}_{\mathbf{p},\,\mu}$ the following relation holds [1]:

$$E_n(f)_{\mathbf{p},\,\mu}=\|f-S_n(f)\|_{\mathbf{p},\,\mu}=\Big\|\sum_{|k|>n}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}, \tag{4.2}$$

the implications 1) $\Rightarrow$ 4) are the statements of the Sunouchi type [26].

In [16, 17, 18, 19], Móricz investigated properties of 2π-periodic functions represented by Fourier series, which convergent absolutely. In particular, in [16] and [19], he found the conditions under which such functions satisfy the Lipshitz and Zygmund condition respectively.

In the cases where $p_k=p$, $\mu_k=1$ and $\omega(t)=t^\beta$, the implication 1) $\Rightarrow$ 3) of Corollary 1 ($\beta\in(0,1)$) coincides with the statements (i) of Theorem 1 [16] and the implication 1) $\Rightarrow$ 3) of Theorem 1 ($\beta\in(0,2)$) coincides with the statements (i) of Theorem 1 [17].

In [25] and [9], some statements similar to those in this subsection were proved for spaces $\mathcal{S}^p$ and Orlicz spaces $\mathcal{S}_M$. Here, in the proof of Theorems 1 and 4 we mostly use the scheme similar to [25, 9], modifying it to take into account the peculiarities of the spaces $\mathcal{S}_{\mathbf{p},\,\mu}$

Proof of Theorem 1. Implication 1) $\Rightarrow$ 2). For any $n\in\mathbb{N}$, we have

$$\Big\|f-Z_n^{(s)}(f)\Big\|_{\mathbf{p},\,\mu}\le(n+1)^{-s}\Big\|\sum_{|k|\le n}|k|^s\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}+\Big\|\sum_{|k|>n}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}. \tag{4.3}$$

Therefore, if relation 1) holds, then

$$(n+1)^{-s}\Big\|\sum_{|k|\le n}|k|^s\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}=(n+1)^{-s}\Big\|\sum_{|k|\le n}\widehat{f^{(s)}}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}=$$

$$=(n+1)^{-s}\|S_n(f^{(s)})\|_{\mathbf{p},\,\mu}=\mathcal{O}(\omega(n^{-1})),\quad n\to\infty. \tag{4.4}$$

To estimate the second term in (4.3), fix an integer $N>n$ and apply the Abel transformation,

$$\Big\|\sum_{n<|k|\le N}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}=\Big\|\sum_{n<|k|\le N}|k|^{-s}\widehat{f^{(s)}}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}$$

$$=\Big\|\sum_{j=n+1}^{N-1}\Big(\frac{1}{j^s}-\frac{1}{(j+1)^s}\Big)\sum_{|k|\le j}\widehat{f^{(s)}}(k)\mathrm{e}^{\mathrm{i}kx}$$

$$+N^{-s}\sum_{|k|\le N}\widehat{f^{(s)}}(k)\mathrm{e}^{\mathrm{i}kx}-(n+1)^{-s}\sum_{|k|\le n}\widehat{f^{(s)}}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}.$$

Then

$$\Big\|\sum_{n<|k|\le N}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}\le s\sum_{j=n+1}^{N-1}j^{-s-1}\|S_j(f^{(s)})\|_{\mathbf{p},\,\mu}$$

$$+N^{-s}\|S_N(f^{(s)})\|_{\mathbf{p},\,\mu}+(n+1)^{-s}\|S_n(f^{(s)})\|_{\mathbf{p},\,\mu}.$$

If relation 1) holds, then there exist a number $C_1>0$ such that for all integers $N>n$,

$$\Big\|\sum_{n<|k|\le N}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}\le C_1\Big(\sum_{j=n+1}^{N-1}\omega(j^{-1})/j+\omega(N^{-1})+\omega(n^{-1})\Big)$$

$$\le C_1\Big(\sum_{j=n+1}^{\infty}\omega(j^{-1})/j+2\omega(n^{-1})\Big).$$

In view of the condition (3.5), this yields that

$$\Big\|\sum_{|k|>n}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}=\mathcal{O}(\omega(n^{-1})),\quad n\to\infty. \tag{4.5}$$

Combining relations (4.3)–(4.5), we get the relation 2). Furthermore, since $\omega(\delta)\to 0$ as $\delta\to 0+$, then from 2) it follows that $f\in\mathcal{S}_{\mathbf{p},\,\mu}$.

2) $\Rightarrow$ 3). For any $\alpha>0$, $h\in\mathbb{R}$ and $k\in\mathbb{Z}$, we have

$$[\Delta_h^\alpha f]^\wedge(k)=\Big[\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}f(\cdot-jh)\Big]^\wedge(k)$$

$$= \widehat{f}(k)\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}\mathrm{e}^{-\mathrm{i}kjh} = (1-\mathrm{e}^{-\mathrm{i}kh})^\alpha \widehat{f}(k). \qquad (4.6)$$

Let us set $n := [1/\delta] - 1$. Then for any $|h| \le \delta$ and $|k| \le n$, we have

$$|[\Delta_h^s f]\widehat{\ }(k)| = |1-\mathrm{e}^{-\mathrm{i}kh}|^s|\widehat{f}(k)| = \Big|2\sin\frac{hk}{2}\Big|^s|\widehat{f}(k)|$$

$$\le \delta^s|k|^s|\widehat{f}(k)| \le (n+1)^{-s}|k|^s|\widehat{f}(k)|$$

and $|[\Delta_h^s f]\widehat{\ }(k)| \le |\widehat{f}(k)|$ when $|k| > n$. Let $a_1 := \|f - Z_n^{(s)}(f)\|_{\mathbf{p},\mu}$. Then

$$\sum_{k\in\mathbb{Z}}\mu_k\Big(|[\Delta_h^s f]\widehat{\ }(k)|/a_1)^{p_k} \le \sum_{|k|\le n}\mu_k\Big((n+1)^{-s}|k|^s|\widehat{f}(k)|/a_1)^{p_k}$$

$$+\sum_{|k|>n}\mu_k\Big(|\widehat{f}(k)|/a_1)^{p_k} \le 1.$$

Therefore, for any $|h| \le \delta$,

$$\|\Delta_h^s f\|_{\mathbf{p},\mu} \le \|f - Z_n^{(s)}(f)\|_{\mathbf{p},\mu} = \mathcal{O}(\omega(n^{-1})) = \mathcal{O}(\omega(\delta)), \quad \delta \to 0+,$$

and hence $f \in \mathcal{S}_{\mathbf{p},\mu}H_\omega^s$.

3) $\Rightarrow$ 1). Setting $h_n := \pi/n$, $n \in \mathbb{N}$, and $a_2 := (n/2)^s\|\Delta_{h_n}^s f\|_{\mathbf{p},\mu}$, by virtue of the inequality $th_n \le \pi\sin(th_n/2)$, which is valid for all $t \in [0,n]$, we see that

$$\sum_{|k|\le n}\mu_k\Big(|\widehat{f}^{(s)}(k)|/a_2\Big)^{p_k} = \sum_{|k|\le n}\mu_k\Big(h_n^s|k|^s|\widehat{f}(k)|/(a_2h_n^s)\Big)^{p_k}$$

$$\le \sum_{|k|\le n}\mu_k\Big(\pi^s\Big|\sin\frac{kh_n}{2}\Big|^s|\widehat{f}(k)|/(a_2h_n^s)\Big)^{p_k}$$

$$\le \sum_{k\in\mathbb{Z}}\mu_k\Big(\Big|2\sin\frac{kh_n}{2}\Big|^s\frac{|\widehat{f}(k)|}{\|\Delta_{h_n}^s f\|_{\mathbf{p},\mu}}\Big)^{p_k} \le 1.$$

Thus,

$$\|S_n(f^{(s)})\|_{\mathbf{p},\mu} \le (n/2)^s\|\Delta_{h_n}^s f\|_{\mathbf{p},\mu}$$

$$\le (n/2)^s\omega_s(f,\pi/n)_{\mathbf{p},\mu} = \mathcal{O}(n^s\omega(n^{-1})), \quad n \to \infty.$$

The implication 1) $\Rightarrow$ 4) follows from relations (4.2) and (4.5). The implication 4) $\Rightarrow$ 3) for $\omega \in \mathscr{B}_s$, follows from the following assertion, which was actually proved in [1]:

Theorem A [1, Theorem 6.1]. *Assume that $\omega(t)$, $t \in [0,1]$, is a continuous strictly increasing function, $\omega(0) = 0$, and $f \in \mathcal{S}_{p,\mu}H_s^\omega$, $s > 0$. Then the best approximation $E_n(f)_{\mathbf{p},\mu}$ satisfies the relation* 4) *of Theorem 1. On the other hand, if $\omega \in \mathscr{B}_s$, then any function $f \in \mathcal{S}_{p,\mu}$ satisfying* 4) *belongs to the set $\mathcal{S}_{p,\mu}H_s^\omega$.*

□

Note that in the case where $\omega(t) = t^\beta$, $\beta > 0$, and all $\mu_k = p_k = 1$ the equivalence of the relations 1) and (4.5) was also proved in [16, Lemma 1].

Analyzing the proof of Theorem 1, we see that condition (3.5) was only used for the estimate (4.5). Therefore, taking into account Theorem A and relation (4.2), we can formulate the following corollary from Theorem 1:

Corollary 2. *Assume that $\omega(t)$, $t \in [0, 1]$, is a continuous strictly increasing function, $\omega(0) = 0$, and $s > 0$. If $f \in \mathcal{S}_{\mathbf{p},\mu} H^s_\omega$, then the statements 1), 2) and 4) hold. On the other hand, if $f \in \mathcal{S}_{p,\mu}$ and $\omega \in \mathscr{B}_s$, then the statements 2), 3) and 4) are equivalent and each of them gives the statement 1).*

4.2. Approximation of functions by the de la Vallée Poussin method

Theorem 3. *Assume that $f \in L_1$, $s > 0$, and n and $m = m(n)$ are arbitrary nonnegative integers, $m \le n$. If $\omega \in \mathscr{B}$ and*

$$\lim_{n\to\infty} \frac{m}{n} = 0 \quad or \quad 0 < C_2 \le \frac{m}{n}, \tag{4.7}$$

then the statement

1) $\|S_n(f^{(s)})\|_{\mathbf{p},\mu} = \mathcal{O}(n^s \omega(n^{-1}))$, $\quad n \to \infty$,

gives

2) $\|f - V_{n,m}(f)\|_{\mathbf{p},\mu} = \mathcal{O}(\omega(n^{-1}))$, $\quad n \to \infty$.

If $\omega \in \mathscr{B}_s$, then 2) $\Rightarrow$ 1). If $\omega \in \mathscr{B} \cap \mathscr{B}_s$, then 1) and 2) are equivalent.

Proof of Theorem 3. First of all, note that in the case $\omega \in \mathscr{B}_s$, implication 2) $\Rightarrow$ 1) follows from the definition of the quantity $E_n(f)_{\mathbf{p},\mu}$ and Corollary 2.

Let us prove the implication 1) $\Rightarrow$ 2). For any $n \in \mathbb{N}$, we have

$$\|f - V_{n,m}(f)\|_{\mathbf{p},\mu} = \Big\| \sum_{n-m+1\le|k|\le n} \frac{|k| - n + m}{m+1} \widehat{f}(k) e^{ikx} + \sum_{|k|>n} \widehat{f}(k) e^{ikx} \Big\|_{\mathbf{p},\mu}$$

$$\le \Big\| \sum_{n-m+1\le|k|\le n} \frac{|k| - n + m}{m+1} \widehat{f}(k) e^{ikx} \Big\|_{\mathbf{p},\mu} + \Big\| \sum_{|k|>n} \widehat{f}(k) e^{ikx} \Big\|_{\mathbf{p},\mu} \tag{4.8}$$

We use the estimate (4.5) for the second term on the right-hand side of (4.8). If we have a similar estimate

$$M_2 := \Big\| \sum_{n-m+1\le|k|\le n} \frac{|k| - n + m}{m+1} \widehat{f}(k) e^{ikx} \Big\|_{\mathbf{p},\mu} = \mathcal{O}(\omega(n^{-1})),\ n \to \infty. \tag{4.9}$$

for the first term, then the statement 2) can simply be obtained by combining relations (4.8), (4.5) and (4.9).

To show the validity of (4.9) we use the following considerations. Set

$$\rho_{n,m} := \rho_{n,m}(f) = f - V_{n,m}(f).$$

For $n-m+1\le |k|\le n$ the Fourier coefficient modules

$$|\widehat{\rho}_{n,m}(k)| = \frac{|k|-n+m}{m+1}|\widehat{f}(k)| = \frac{|k|-n+m}{|k|^s(m+1)}|\widehat{f}^{(s)}(k)| \le \frac{|\widehat{f}^{(s)}(k)|}{(n-m+1)^s}.$$

Therefore, if $\lim\limits_{n\to\infty}\frac{m}{n}=0$, then relation (4.9) holds:

$$M_2 = \Big\| \sum_{n-m+1\le|k|\le n} \widehat{\rho}_{n,m}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu} \le \frac{\|S_n(f^{(s)})\|_{\mathbf{p},\,\mu}}{(n-m+1)^s} = \mathcal{O}(\omega(n^{-1})),\ n\to\infty.$$

Now let $0<C_2\le\frac{m}{n}$. Using the Abel transformation, we obtain

$$M_2 = \frac{1}{m+1}\Big\| \sum_{n-m+1\le|k|\le n} \frac{|k|-n+m}{|k|^s}\widehat{f}^{(s)}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}$$

$$= \frac{1}{m+1}\Big\| \sum_{j=n-m+1}^{n-1}\Big(\frac{j-n+m}{j^s} - \frac{j+1-n+m}{(j+1)^s}\Big)\sum_{|k|\le j}\widehat{f}^{(s)}(k)\mathrm{e}^{\mathrm{i}kx}$$

$$+\frac{m}{n^s}\sum_{|k|\le n}\widehat{f}^{(s)}(k)\mathrm{e}^{\mathrm{i}kx} - \frac{1}{(n-m+1)^s}\sum_{|k|\le n-m}\widehat{f}^{(s)}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|_{\mathbf{p},\,\mu}. \qquad (4.10)$$

For fixed $s>0$, n and m, $1\le m\le n$, consider the function $F(t)=\frac{t-n+m}{t^s}$, $t>0$. For any $t\in[j,j+1]\subset[n-m+1,n]$ its derivative

$$F'(t) = \frac{s(n-m)}{t^{s+1}} - \frac{s-1}{t^s} \le \frac{s(a_s n-m)}{j^{s+1}},$$

where $a_s:=\max\{1,\frac{1}{s}\}$. Therefore,

$$\left|\frac{j+1-n+m}{(j+1)^s} - \frac{j-n+m}{j^s}\right| = |F(j+1)-F(j)| \le \frac{s(a_s n-m)}{j^{s+1}},$$

and from (4.10) we obtain

$$M_2 \le \frac{s(a_s n-m)}{m+1}\sum_{j=n-m+1}^{n-1}\frac{\|S_j(f^{(s)})\|_{\mathbf{p},\,\mu}}{j^{s+1}}$$

$$+\frac{m}{n^s(m+1)}\|S_n(f^{(s)})\|_{\mathbf{p},\,\mu} + \frac{n-m}{m+1}\frac{\|S_{n-m}(f^{(s)})\|_{\mathbf{p},\,\mu}}{(n-m)^{s+1}}.$$

By virtue of the inequality $0<C_2\le\frac{m}{n}$, there exist a number $C_3>0$ such that

$$M_2 \le C_3\bigg(\sum_{j=n-m}^{n-1}\frac{\|S_j(f^{(s)})\|_{\mathbf{p},\,\mu}}{j^{s+1}} + \frac{\|S_n(f^{(s)})\|_{\mathbf{p},\,\mu}}{n^s}\bigg).$$

Since

$$\sum_{j=n-m}^{n-1} \frac{\|S_j(f^{(s)})\|_{\mathbf{p},\mu}}{j^{s+1}} = \sum_{i=n+1}^{n+m} \frac{\|S_{i-m-1}(f^{(s)})\|_{\mathbf{p},\mu}}{(i-m-1)^{s+1}}$$

$$\le \sum_{i=n+1}^{n+m} \frac{i^{s+1}}{(i-m-1)^{s+1}} \frac{\|S_i(f^{(s)})\|_{\mathbf{p},\mu}}{i^{s+1}}$$

$$\le \Big(1+\frac{m+1}{n-1}\Big)^{s+1} \sum_{i=n+1}^{\infty} \frac{\|S_i(f^{(s)})\|_{\mathbf{p},\mu}}{i^{s+1}}, \tag{4.11}$$

taking into account (3.5), we see that in this case relation (4.9) also holds:

$$M_2 \le C_4\Big(\sum_{i=n+1}^{\infty} i^{-1}\omega(i^{-1}) + \omega(n^{-1})\Big) = \mathcal{O}(\omega(n^{-1})),\ n\to\infty.$$

□

4.3. Approximation of functions by the generalised Abel-Poisson and the Taylor-Abel-Poisson methods

In the following theorem, we give the direct and inverse theorem of the approximation of functions by the linear operator $A_{\varrho,r}$ in the space $\mathcal{S}_{\mathbf{p},\mu}$ and constructive characteristics for classes of functions of $\mathcal{S}_{\mathbf{p},\mu}$ such that the moduli of smoothness of their generalized derivatives do not exceed majorants ω.

Theorem 4. *Assume that $f\in L_1$, $s,r\in\mathbb{N}$, $s\le r$ and $\omega\in\mathscr{B}\cap\mathscr{B}_s$. The following statements are equivalent:*

1) $\|f-A_{\varrho,r}(f)\|_{\mathbf{p},\mu} = \mathcal{O}((1-\varrho)^{r-s}\omega(1-\varrho)),\quad \varrho\to 1-;$

2) $\big\|P(f^{[r]})(\varrho,\cdot)\big\|_{\mathbf{p},\mu} = \mathcal{O}((1-\varrho)^{-s}\omega(1-\varrho)),\quad \varrho\to 1-;$

3) $\|S_n(f^{[r]})\|_{\mathbf{p},\mu} = \mathcal{O}(n^s\omega(n^{-1})),\quad n\to\infty;$

4) $f^{[r-s]}\in\mathcal{S}_{\mathbf{p},\mu}H^s_\omega.$

Let us note that the implication 2) ⇒ 3) is the statement of the Hardy-Littlewood type theorems [12].

Remark 2. In Remark 1 it is noted that from the condition $(\mathscr{B}_s)$ it follows that $(1-\varrho)^{r-s}\omega(1-\varrho)\ge C_0(1-\varrho)^r$ as $\varrho\to 1-$. Therefore, if the condition $(\mathscr{B}_s)$ is satisfied, then the quantity on the right-hand side of the relation in statement 1) decreases to zero as $\varrho\to 1-$ not faster, than the function $(1-\varrho)^r$. Also note that the relation $\|f-A_{\varrho,r}(f)\|_{\mathbf{p},\mu} = o\left((1-\varrho)^r\right)$, $\varrho\to 1-$, holds only in the trivial case when $f(x)=\sum_{|k|\le r-1}\widehat{f}_k \mathrm{e}^{\mathrm{i}kx}$, and in such case, the theorems are easily true. This is related to the fact that the linear approximation method, generated by the operator $A_{\varrho,r}$, is saturated with the saturation order $(1-\varrho)^r$ (see, for example [24, 22]).

Consider approximative properties of the sums $P_{\varrho,s}(f)$ in the space $\mathcal{S}_{\mathbf{p},\,\mu}$.

Let us prove that for any function $f \in \mathcal{S}_{\mathbf{p},\,\mu}$ such that the derivative $f^{(s)} \in \mathcal{S}_{\mathbf{p},\,\mu}$, the following relation holds as $\varrho \to 1-$:

$$\|f - P_{\varrho,s}(f)\|_{\mathbf{p},\,\mu} \sim \|f^{(s-1)} - P_{\varrho,1}(f^{(s-1)})\|_{\mathbf{p},\,\mu} \sim (1-\varrho)\|f^{(s)}\|_{\mathbf{p},\,\mu}. \tag{4.12}$$

For this, let us show that

$$\|f - P_{\varrho,s}(f)\|_{\mathbf{p},\,\mu} \sim (1-\varrho)\|f^{(s)}\|_{\mathbf{p},\,\mu}, \quad \varrho \to 1-. \tag{4.13}$$

The second relation in (4.12) is proved similarly.

For any $n \in \mathbb{N}$, we have $1-\varrho^n = (1-\varrho)(1+\varrho+\ldots+\varrho^{n-1})$. Then setting $b_1 := (1-\varrho)\|f^{(s)}\|_{\mathbf{p},\,\mu}$, we get for all $\varrho \in (0,1)$,

$$\sum_{k\in\mathbb{Z}} \mu_k \Big((1-\varrho^{|k|^s})|\widehat{f}(k)|/b_1\Big)^{p_k} \le \sum_{k\in\mathbb{Z}} \mu_k \Big((1-\varrho)|k|^s|\widehat{f}(k)|/b_1\Big)^{p_k} \le 1.$$

Therefore, $\|f - P_{\varrho,s}(f)\|_{\mathbf{p},\,\mu} \le (1-\varrho)\|f^{(s)}\|_{\mathbf{p},\,\mu}$.

On the other hand side, since $f^{(s)} \in \mathcal{S}_{\mathbf{p},\,\mu}$, then for any $\varepsilon > 0$ there exists a number $N \in \mathbb{N}$ such that for all $n \ge N$

$$\|S_n(f^{(s)})\|_{\mathbf{p},\,\mu} \ge \|f^{(s)}\|_{\mathbf{p},\,\mu} - \varepsilon/4$$

and by the definition of the norm

$$\sum_{|k|\le N} \mu_k \bigg(\frac{|k|^s|\widehat{f}(k)|}{\|f^{(s)}\|_{\mathbf{p},\,\mu} - \varepsilon/2}\bigg)^{p_k} \ge \sum_{|k|\le N} \mu_k \bigg(\frac{|k|^s|\widehat{f}(k)|}{\|S_n(f^{(s)})\|_{\mathbf{p},\,\mu} - \varepsilon/4}\bigg)^{p_k} > 1.$$

Choosing ϱ_0 such that for all $\varrho \in (\varrho_0, 1)$ and $|k| \le N$, the following inequality holds:

$$(\|f^{(s)}\|_{\mathbf{p},\,\mu} - \varepsilon/2)(1+\varrho+\ldots+\varrho^{|k|^s-1}) > |k|^s(\|f^{(s)}\|_{\mathbf{p},\,\mu} - \varepsilon)$$

we see that for such ϱ and $b_2 := (1-\varrho)(\|f^{(s)}\|_{\mathbf{p},\,\mu} - \varepsilon)$

$$\sum_{k\in\mathbb{Z}} \mu_k \Big((1-\varrho^{|k|^s})|\widehat{f}(k)|/b_2\Big)^{p_k} \ge \sum_{|k|\le N} \mu_k \Big((1-\varrho)(1+\varrho+\ldots+\varrho^{|k|^s-1})|\widehat{f}(k)|/b_2\Big)^{p_k}$$

$$= \sum_{|k|\le N} \mu_k \bigg(\frac{(1+\ldots+\varrho^{|k|^s-1})^{p_k}|\widehat{f}(k)|}{\|f^{(s)}\|_{\mathbf{p},\,\mu} - \varepsilon}\bigg)^{p_k} > \sum_{|k|\le N} \mu_k \bigg(\frac{|k|^s|\widehat{f}(k)|}{\|f^{(s)}\|_{\mathbf{p},\,\mu} - \varepsilon/2}\bigg)^{p_k} > 1.$$

Thus, for all $\varrho \in (\varrho_0, 1)$, we have $\|f - P_{\varrho,s}(f)\|_{\mathbf{p},\,\mu} \ge (1-\varrho)(\|f^{(s)}\|_{\mathbf{p},\,\mu} - \varepsilon)$ and hence relation (4.13) holds.

It is clear that

$$P_{\varrho,1}(f)(x) = A_{\varrho,1}(f)(x).$$

Therefore, applying Theorem 4 to the function $f = g^{(s-1)}$ with $r = 1$ and taking into account relation (4.12), we obtain the following result.

Theorem 5. *Assume that $f \in L_1$, $s \in \mathbb{N}$, and $\omega \in \mathscr{B} \cap \mathscr{B}_s$. The following statements are equivalent:*

1) $\|f - P_{\varrho,s}(f)\|_{\mathbf{p},\mu} = \mathcal{O}(\omega(1-\varrho)), \quad \varrho \to 1-;$

2) $\left\|P(f^{(s)})(\varrho,\cdot)\right\|_{\mathbf{p},\mu} = \mathcal{O}(\frac{\omega(1-\varrho)}{1-\varrho}), \quad \varrho \to 1-;$

3) $f^{(s-1)} \in \mathcal{S}_{\mathbf{p},\mu} H^1_\omega$.

In [25] and [9], statements similar to Theorem 4 and Theorem 5 in this subsection were proved for spaces $\mathcal{S}^p$ and Orlicz spaces $\mathcal{S}_M$.

Proof of Theorem 4. It is shown above that the Theorem 5 follows from Theorem 4. Therefore, it remains to prove the truth of Theorem 4.

1) $\Rightarrow$ 2). Since

$$\sum_{j=0}^{\nu} \binom{\nu}{j} (1-\varrho)^j \varrho^{\nu-j} = \big((1-\varrho)+\varrho\big)^\nu = 1, \ \nu = 0,1,\ldots, \tag{4.14}$$

then for $a_3 := \|f - A_{\varrho,r}(f)\|_{\mathbf{p},\mu}$, we have

$$1 \ge \sum_{|k|\ge r} \mu_k \Big(|1-\lambda_{|k|,r}(\varrho)||\widehat{f}(k)|/a_3\Big)^{p_k}$$

$$= \sum_{|k|\ge r} \mu_k \Bigg(\Bigg|1 - \sum_{j=0}^{r-1} \binom{|k|}{j} (1-\varrho)^j \varrho^{|k|-j}\Bigg| |\widehat{f}(k)|/a_3\Bigg)^{p_k}$$

$$= \sum_{|k|\ge r} \mu_k \Bigg(\sum_{j=r}^{|k|} \binom{|k|}{j} (1-\varrho)^j \varrho^{|k|-j} |\widehat{f}(k)|/a_3\Bigg)^{p_k}$$

$$\ge \sum_{|k|\ge r} \mu_k \Bigg(\binom{|k|}{r} (1-\varrho)^r \varrho^{|k|-r} |\widehat{f}(k)|/a_3\Bigg)^{p_k}. \tag{4.15}$$

On the other hand, by virtue of (3.4),

$$\|P(f^{[r]})(\varrho,\cdot)\|_{\mathbf{p},\mu} = \Big\|\varrho^r \frac{\partial^r}{\partial \varrho^r} P(f)(\varrho,\cdot)\Big\|_{\mathbf{p},\mu}$$

$$= \inf\Bigg\{a > 0 : \sum_{|k|\ge r} \mu_k \Bigg(r!\binom{|k|}{r} \varrho^{|k|} |\widehat{f}(k)|/a\Bigg)^{p_k} \le 1\Bigg\}.$$

Combining these relations and equality (3.4), we see that for $\varrho \to 1-$,

$$\|P(f^{[r]})(\varrho,\cdot)\|_{\mathbf{p},\mu} \le r!\varrho^r (1-\varrho)^{-r} \|f - A_{\varrho,r}(f)\|_{\mathbf{p},\mu} = \mathcal{O}((1-\varrho)^{-s}\omega(1-\varrho)).$$

2) $\Rightarrow$ 3). For $a_4 := \|P(f^{[r]})(\varrho,\cdot)\|_{\mathbf{p},\mu}$ and for any numbers $n > r$ and $\varrho \in [0,1)$, we have

$$1 \ge \sum_{|k|\ge r} \mu_k \Bigg(\binom{|k|}{r} \frac{r!\varrho^{|k|}|\widehat{f}(k)|}{a_4}\Bigg)^{p_k}$$

$$\geq \sum_{r\leq|k|\leq n}\mu_k\bigg(\varrho^n\binom{|k|}{r}\frac{r!|\widehat{f}(k)|}{a_4}\bigg)^{p_k}=\sum_{r\leq|k|\leq n}\mu_k\bigg(\frac{\varrho^n|\widehat{f}^{[r]}(k)|}{a_4}\bigg)^{p_k}.$$

This yields $\|S_n(f^{[r]})\|_{\mathbf{p},\,\mu}\leq\varrho^{-n}\|P(f^{[r]})(\varrho,\cdot)\|_{\mathbf{p},\,\mu}$ and putting $\varrho=1-1/n$ and taking into account statement 2), we see that

$$\|S_n(f^{[r]})\|_{\mathbf{p},\,\mu}\leq(1-1/n)^{-n}\mathcal{O}(n^s\omega(n^{-1}))=\mathcal{O}(n^s\omega(n^{-1})),\ \text{as } n\to\infty.$$

3) $\Rightarrow$ 4). Let us set $g:=f^{[r-s]}$. By the definition, for $|k|\geq r$, we have

$$|\widehat{f}^{[r]}(k)|=\frac{|k|!|\widehat{f}(k)|}{(|k|-r)!}=$$

$$|g^{[s]}(k)|\frac{(|k|-r+1)(|k|-r+2)\cdot\ldots\cdot(|k|-r+s)}{|k|(|k|-1)\cdot\ldots\cdot(|k|-s+1)}$$

$$\geq|g^{[s]}(k)|\Big(1-\frac{r-1}{|k|}\Big)^s\geq r^{-s}|g^{[s]}(k)|.$$

Therefore, taking into account Remark 1, we get

$$\|S_n(g^{[s]})\|_{\mathbf{p},\,\mu}\leq\|S_{r-1}(g^{[s]})\|_{\mathbf{p},\,\mu}+\Big\|\sum_{r\leq|k|\leq n}g^{[s]}(k)\mathrm{e}^{\mathrm{i}kx}\Big\|$$

$$\leq\|S_{r-1}(g^{[s]})\|_{\mathbf{p},\,\mu}+r^s\|S_n(f^{[r]})\|_{\mathbf{p},\,\mu}=\mathcal{O}(n^s\omega(n^{-1})),\quad n\to\infty.$$

By virtue of Theorem 2, we see that $\|g-Z_n^{(s)}(g)\|_M=\mathcal{O}(\omega(n^{-1}))$, $n\to\infty$, hence, $g=f^{[r-s]}\in\mathcal{S}_{\mathbf{p},\,\mu}$, $f\in\mathcal{S}_{\mathbf{p},\,\mu}$ and $f^{[r-s]}\in\mathcal{S}_{\mathbf{p},\,\mu}H^s_\omega$.

4) $\Rightarrow$ 3). If $g:=f^{[r-s]}$, then according to Proposition 2, we get

$$\|S_n(g^{[s]})\|_{\mathbf{p},\,\mu}=\mathcal{O}(n^s\omega(n^{-1})),\quad n\to\infty.$$

For $|k|<r$ we have $\widehat{f}^{[r]}(k)=0$ and for $|k|\geq r$,

$$\widehat{f}^{[r]}(k)|=\frac{|k|!}{(|k|-r)!}|\widehat{f}(k)|\leq\frac{|k|!}{(|k|-s)!}\frac{|k|!}{(|k|-r+s)!}|\widehat{f}(k)|=|g^{[s]}(k)|.$$

Thus

$$\|S_n(f^{[r]})\|_{\mathbf{p},\,\mu}\leq\|S_n(g^{[s]})\|_{\mathbf{p},\,\mu}=\mathcal{O}(n^s\omega(n^{-1})),\quad n\to\infty.$$

3) $\Rightarrow$ 1). From identity (4.14)), it follows that for any $\varrho\in[0,1]$,

$$\sum_{j=r}^{\nu}\binom{\nu}{j}(1-\varrho)^j\varrho^{\nu-j}\leq 1,\quad \nu\geq r.$$

This implies the relation

$$\sum_{|k|\geq r}\mu_k\Big(|1-\lambda_{|k|,r}(\varrho)|\frac{|\widehat{f}(k)|}{a_5}\Big)^{p_k}$$

$$=\sum_{|k|\ge r}\mu_k\left(\sum_{j=r}^{|k|}\binom{|k|}{j}(1-\varrho)^j\varrho^{|k|-j}\frac{|\widehat{f}(k)|}{a_5}\right)^{p_k}\le\sum_{|k|\ge r}\mu_k\left(\frac{|\widehat{f}(k)|}{a_5}\right)^{p_k}\le 1,$$

where $a_5 := \|f\|_{\mathbf{p},\mu}$, and therefore, we have $\|f - A_{\varrho,r}(f)\|_{\mathbf{p},\mu} \le \|f\|_{\mathbf{p},\mu} < \infty$. From this relation, we conclude that for any $\varepsilon > 0$ there exists the number n_0 such that for all $n > n_0$ and all $\varrho \in [0,1)$,

$$\|f - A_{\varrho,r}(f)\|_{\mathbf{p},\mu} \le \left\|\sum_{r\le|k|\le n}\sum_{j=r}^{|k|}\binom{|k|}{j}(1-\varrho)^j\varrho^{|k|-j}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\right\|_{\mathbf{p},\mu}+\varepsilon. \quad (4.16)$$

Let us use the following inequality

$$\sum_{j=r}^{\nu}\binom{\nu}{j}(1-\varrho)^j\varrho^{\nu-j}\le\binom{\nu}{r}(1-\varrho)^r \quad (4.17)$$

which is valid for all $\nu \ge r$ and $\varrho \in [0,1]$ (see, for example [25]). Putting $a_6 := (1-\varrho)^r\|S_n(f^{[r]})\|_{\mathbf{p},\mu}/r!$, we get

$$\sum_{r\le|k|\le n}\mu_k\left(\sum_{j=r}^{|k|}\binom{|k|}{j}(1-\varrho)^j\varrho^{|k|-j}\frac{|\widehat{f}(k)|}{a_6}\right)^{p_k}$$

$$\le\sum_{r\le|k|\le n}\mu_k\left((1-\varrho)^r\binom{|k|}{r}\frac{|\widehat{f}(k)|}{a_6}\right)^{p_k}\le 1.$$

Thus,

$$\left\|\sum_{r\le|k|\le n}\sum_{j=r}^{|k|}\binom{|k|}{j}(1-\varrho)^j\varrho^{|k|-j}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\right\|_{\mathbf{p},\mu}\le\frac{(1-\varrho)^r}{r!}\|S_n(f^{[r]})\|_{\mathbf{p},\mu}. \quad (4.18)$$

Combining relations (4.16)) and (4.18)) and putting $n := n_\varrho = [(1-\varrho)^{-1}]$, where $[\cdot]$ means the integer part of the number, we get

$$\|f - A_{\varrho,r}(f)\|_{\mathbf{p},\mu}\le\frac{(1-\varrho)^r}{r!}\|S_n(f^{[r]})\|_{\mathbf{p},\mu}+\varepsilon$$

$$=(1-\varrho)^r\mathcal{O}(n_\varrho^s\omega(n_\varrho^{-1}))+\varepsilon=\mathcal{O}((1-\varrho)^{r-s}\omega(1-\varrho))+\varepsilon,$$

as $\varrho \to 1-$. By virtue of arbitrary ε, from this relation it follows that the implication 3) $\Rightarrow$ 1) is true.

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Received: 04.06.2024. *Accepted:* 22.09.2024