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Kolmogorov-type inequalities for functions with asymmetric restrictions on the highest derivative

Abstract. For $k, r \in \mathbf{N}$, $k < r$; $q \geq 1$, $p > 0$; $\alpha, \beta > 0$ and for functions $x \in L^r_\infty(\mathbf{R})$ inequalities that estimate the norm $\|x^{(k)}_\pm\|_{L_q[a,b]}$ on an arbitrary segment $[a,b] \subset \mathbf{R}$ such that $x^{(k)}(a) = x^{(k)}(b) = 0$ via a local norm of the function $|||x^{\uparrow\downarrow}|||_p := \sup\left\{E_0(x)_{L_p[a,b]} : \pm x'(t) > 0 \; \forall t \in (a,b), \;\; a,b \in \mathbf{R}\right\}$, and the asymmetric norm $\|\alpha^{-1}x^{(r)}_+ + \beta^{-1}x^{(r)}_-\|_\infty$ of its highest derivative are proved, where $E_0(x)_{L_p([a,b])} := \inf\{\|x - c\|_{L_p([a,b])} : c \in \mathbf{R}\}$. As a consequence, generalizations of a number of well-known Kolmogorov-type inequalities are obtained.

Key words: Sharp Kolmogorov-type inequality, asymmetric case, local norm.

Анотація. Для $k, r \in \mathbf{N}$, $k < r$; $q \geq 1$, $p > 0$; $\alpha, \beta > 0$ і для функцій $x \in L^r_\infty(\mathbf{R})$ доведені точні нерівності, які оцінюють норму $\|x^{(k)}_\pm\|_{L_q[a,b]}$ на довільному відрізку $[a,b] \subset \mathbf{R}$, такому, що $x^{(k)}(a) = x^{(k)}(b) = 0$ через локальну норму функції $|||x^{\uparrow\downarrow}|||_p := \sup\left\{E_0(x)_{L_p[a,b]} : \pm x'(t) > 0 \; \forall t \in (a,b), \;\; a,b \in \mathbf{R}\right\}$, та несиметричну норму $\|\alpha^{-1}x^{(r)}_+ + \beta^{-1}x^{(r)}_-\|_\infty$ її старшої похідної, де $E_0(x)_{L_p([a,b])} := \inf\{\|x - c\|_{L_p([a,b])} : c \in \mathbf{R}\}$.
Як наслідок, отримано узагальнення ряду відомих нерівностей колмогоровського типу.

Key words: Точна нерівність колмогоровського типу, несиметричний випадок, локальна норма.

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1. Introduction. The symbol G will denote a segment $[a,b]$, the real axis $\mathbf{R}$ or the circle $\mathbf{T}$ realized as a segment $[0, 2\pi]$ with identified ends. We will consider the spaces $L_p(G)$ of measurable functions $x\colon G \to \mathbf{R}$ such that $\|x\|_p = \|x\|_{L_p(G)} < \infty$, where

$$\|x\|_p := \left(\int_G |x(t)|^p \, dt\right)^{1/p}, \text{ if } 1 \leq p < \infty,$$

$$\|x\|_\infty := \operatorname{vrai}\sup_{t\in G} |x(t)|.$$

Set $E_0(x)_p = E_0(x)_{L_p(G)} := \inf\{\|x-c\|_{L_p(G)} : c \in \mathbf{R}\}$.

Denote by $L^r_\infty(G)$ the space of functions $x \in L_\infty(G)$ having locally absolutely continuous derivatives up to $(r-1)$-th order, and such that $x^{(r)} \in L_\infty(G)$.

For $\alpha, \beta > 0$ and $x \in L_\infty(G)$ set

$$\|x\|_{\infty,\alpha,\beta} := \|\alpha x_+ + \beta x_-\|_\infty,$$

where $x_\pm(t) := \max\{x_\pm(t), 0\}$.

In many extremal problems of analysis, an important role is played by the Kolmogorov-type inequalities

$$\left\|x^{(k)}\right\|_q \le C \,\|x\|_p^\alpha \left\|x^{(r)}\right\|_s^{1-\alpha} \tag{1.1}$$

for differentiable functions on the axis, and for 2π-periodic functions, where $k, r \in \mathbf{N}$, $k < r; q, p, s \in [1,\infty]; \alpha \in (0,1)$.

Of the greatest interest are inequalities (1.1) with the maximum possible exponent α and the smallest possible constant C. A detailed bibliography of inequalities of this type can be found in [1]–[3]. Note that in [4] the question of the coincidence of sharp constants in the inequalities of the type (1.1) for periodic functions and non-periodic functions on the axis was investigated.

In a number of papers the method of local norms was used in the proofs of Kolmogorov-type inequalities. In particular, it was used to obtain the most general sharp Bernstein-type inequalities for polynomials and splines (see [5] and [6]), as well as to prove a theorem on sharp constants in inequalities for functions $x \in L^r_\infty(\mathbf{T})$ (see [7]). In addition, using this method for functions of small smoothness, an analogue of Ligun's inequality [8] was obtained for a sharp estimate of the quasinorms of derivatives in the spaces $L^r_q(\mathbf{T})$, $q \in [0,1)$ (see [9]). We also note the paper [10], in which, using the method of local norms, sharp constants in inequalities with different metrics for periodic functions were found. Related problems for functions with asymmetric constraints were considered in [11] – [16].

Let us give the necessary definitions. We set

$$|||x^\uparrow|||_p := \sup\left\{E_0(x)_{L_p[a,b]} \;:\; x'(t) > 0 \;\; \forall t \in (a,b), \;\; a, b \in \mathbf{R}\right\},$$

$$|||x^\downarrow|||_p := \sup\left\{E_0(x)_{L_p[a,b]} \;:\; x'(t) < 0 \;\; \forall t \in (a,b), \;\; a, b \in \mathbf{R}\right\}$$

and

$$|||x|||_p := \max\left\{|||x^\uparrow|||_p, \quad |||x^\downarrow|||_p\right\}.$$

Let further [17]

$$L(x)_p := \sup\left\{\|x\|_{L_p[a,b]} : \quad |x(t)| > 0 \quad \forall t \in (a,b) \quad a, b \in \mathbf{R}\right\}.$$

The symbol $\varphi_r^{\alpha,\beta}$, $r \in \mathbf{N}$, $\alpha, \beta > 0$, denotes the r-th 2π-periodic integral with zero mean over the period of the 2π-periodic function $\varphi_0^{\alpha,\beta}$ defined on the segment $[0, 2\pi]$ as follows: $\varphi_0^{\alpha,\beta}(0) = \varphi_0^{\alpha,\beta}(2\pi) := 0$, and

$$\varphi_0^{\alpha,\beta}(t) := \alpha, \quad t \in (0, 2\pi\beta/(\alpha+\beta)],$$

$$\varphi_0^{\alpha,\beta}(t) := -\beta, \quad t \in (2\pi\beta/(\alpha+\beta), 2\pi).$$

Note that $\varphi_r := \varphi_r^{1,1}$ is the Euler spline of the order r. Let's put $\varphi_{\lambda,r}^{\alpha,\beta}(t) := \lambda^{-r}\varphi_r^{\alpha,\beta}(\lambda t)$ for $\lambda > 0$. We'll need

Theorem A [15]. *Let $k, r \in \mathbf{R}$, $k < r$, $G = \mathbf{R}$ or $G = \mathbf{T}$ and $\alpha, \beta > 0$. Then for functions $x \in L_\infty^r(G)$ the following sharp inequality*

$$\|x_\pm^{(k)}\|_\infty \le \frac{\|(\varphi_{r-k}^{\alpha,\beta})_\pm\|_\infty}{|||\varphi_r^{\alpha,\beta}|||_\infty^{1-k/r}} |||x|||_\infty^{1-k/r} \|x^{(r)}\|_{\infty;\alpha^{-1},\beta^{-1}}^{k/r}, \tag{1.2}$$

holds. For $q \ge 1$ and functions $x \in L_\infty^r(\mathbf{T})$ the following sharp inequality

$$\|x_\pm^{(k)}\|_{L_q(\mathbf{T})} \le \frac{\|(\varphi_{r-k}^{\alpha,\beta})_\pm\|_{L_q(\mathbf{T})}}{|||\varphi_r^{\alpha,\beta}|||_\infty^{1-k/r}} |||x|||_\infty^{1-k/r} \|x^{(r)}\|_{\infty;\alpha^{-1},\beta^{-1}}^{k/r}. \tag{1.3}$$

holds. Equality in (1.2) is achieved for the functions $x(t) = a\varphi_{\lambda,r}^{\alpha,\beta}(t) + b$, $a > 0$, $b \in \mathbf{R}$, $\lambda > 0$ for $G = \mathbf{R}$, and $\lambda \in \mathbf{N}$ for $G = \mathbf{T}$. Equality in (1.3) is achieved for the functions $x(t) = a\varphi_{n,r}^{\alpha,\beta}(t) + b$, $a > 0$, $b \in \mathbf{R}$, $n \in \mathbf{N}$.

Inequalities (1.2) and (1.3) strengthen the Xörmander [18] and the Babenko [19] inequalities, respectively, in which $E_0(x)_\infty$ is replaced by the local norm $|||x|||_\infty$. It is clear that $|||x|||_\infty \le E_0(x)_\infty$, but it is easy to give examples of infinitely differentiable functions x for which the fraction $\frac{|||x|||_\infty}{E_0(x)_\infty}$ is arbitrarily small.

The proof of Theorem A in [15] is based on the following refinement of Kolmogorov's comparison theorem in the asymmetric case, necessary in the paper.

Theorem B. *Let $r \in \mathbf{N}$, $\alpha, \beta > 0$, $x \in L_\infty^r(G)$, $\left\|x^{(r)}\right\|_{\infty,\alpha^{-1},\beta^{-1}} \le 1$, and the number λ be chosen from the condition $|||x|||_\infty = |||\varphi_{\lambda,r}^{\alpha,\beta}|||_\infty$. Let further a segment $[a, b]$, where the function x increases (decreases), satisfy the condition $x'(t) \ne 0, t \in (a, b)$, and $x'(a) = 0$, if $a \ne -\infty$, $x'(b) = 0$, if $b \ne +\infty$, and a segment $[\xi, \eta]$, where the function $\varphi_{\lambda,r}^{\alpha,\beta}$ increases (resp. decreases), be such that $\varphi_{\lambda,r-1}^{\alpha,\beta}(\xi) = \varphi_{\lambda,r-1}^{\alpha,\beta}(\eta) = 0$, and let γ be the point of the local extremum of the function $\varphi_{\lambda,r-1}^{\alpha,\beta}$ on the interval (ξ, η).*

If for a point $t \in [a, b]$ the point $y \in [\xi, \eta]$ is chosen so that

$$|x(b) - x(t)| = |\varphi_{\lambda,r}^{\alpha,\beta}(\eta) - \varphi_{\lambda,r}^{\alpha,\beta}(y)|,$$

or so that

$$|x(t) - x(a)| = |\varphi_{\lambda,r}^{\alpha,\beta}(y) - \varphi_{\lambda,r}^{\alpha,\beta}(\xi)|,$$

then

$$|x'(t)| \le |\varphi^{\alpha,\beta}_{\lambda,r-1}(y)|.$$

In this paper, we obtain (Theorem 2) a generalization of inequality (1.3) to the classes $L^r_\infty(\mathbf{R})$ of non-periodic functions on the axis, in which the uniform local norm $|||x|||_\infty$ is replaced by the local norm $|||x|||_p$, $p > 0$. In addition, an inequality of the same type, which takes into account the number of intervals of constant sign of the derivative (Theorem 3), is proved. The central place in the proofs of these results is occupied by new inequalities for local norms (Theorem 1) obtained with the help of Theorem B.

2. Main results. We set

$$M\left(\frac{|||x|||_p}{|||\varphi^{\alpha,\beta}_{\lambda,r}|||_p}\right) := \max\left\{\frac{|||x^{\uparrow}|||_p}{|||(\varphi^{\alpha,\beta}_{\lambda,r})^{\uparrow}|||_p},\ \frac{|||x^{\downarrow}|||_p}{|||(\varphi^{\alpha,\beta}_{\lambda,r})^{\downarrow}|||_p}\right\}.$$

The symbol $r(x,t)$, $t > 0$, denotes the permutation (see, for example, [20, §1.3]) of the function $|x|$, $x \in L_1[a,b]$. Wherein we agree that $r(x,t) = 0$ for $t \ge b-a$.

Theorem 1. *Let $r, k \in \mathbf{N}$, $k < r$, $q \in [1,\infty]$, $p \in (0,\infty]$, $\alpha,\beta > 0$. Then for any function $x \in L^r_\infty(\mathbf{R})$ such that $|||x|||_p < \infty$, the following inequalities hold:*

$$L\left(x^{(k)}_\pm\right)_q \le L\left((\varphi^{\alpha,\beta}_{r-k})_\pm\right)_q \cdot M\left(\frac{|||x|||_p}{|||\varphi^{\alpha,\beta}_{r}|||_p}\right)^{\delta} \left\|x^{(r)}\right\|^{1-\delta}_{\infty,\alpha^{-1},\beta^{-1}}, \tag{2.1}$$

where $\delta = \frac{r-k+1/q}{r+1/p}$, and

$$|||x|||_\infty \le |||\varphi^{\alpha,\beta}_{r}|||_\infty \cdot M\left(\frac{|||x|||_p}{|||\varphi^{\alpha,\beta}_{r}|||_p}\right)^{\frac{r}{r+1/p}} \left\|x^{(r)}\right\|^{\frac{1/p}{r+1/p}}_{\infty,\alpha^{-1},\beta^{-1}}. \tag{2.2}$$

Inequalities (2.1) and (2.2) are sharp and turn into equality for the functions of the form $x(t) = a\varphi^{\alpha,\beta}_r(\lambda t + b)$; $\lambda, a > 0, b \in \mathbf{R}$.

Proof. In view of the homogeneity of inequalities (2.1) and (2.2), we can assume that

$$\|x^{(r)}\|_{\infty,\alpha^{-1},\beta^{-1}} = 1. \tag{2.3}$$

Let's choose λ from the condition

$$|||x|||_\infty = |||\varphi^{\alpha,\beta}_{\lambda,r}|||_\infty. \tag{2.4}$$

Let's prove that

$$M\left(\frac{|||x|||_p}{|||\varphi^{\alpha,\beta}_{\lambda,r}|||_p}\right) \ge 1. \tag{2.5}$$

Let us prove (2.5) under the assumption $|||x|||_\infty = |||x^\uparrow|||_\infty$ ((2.5) is proved similarly under the assumption $|||x|||_\infty = |||x^\downarrow|||_\infty$).

From the assumption $|||x|||_\infty = |||x^\uparrow|||_\infty$ it follows that for any $\varepsilon > 0$ there exists an interval $[a,b]$, where the function x strictly increases, and such that $x'(a) = x'(b) = 0$, and

$$E_0(x)_{L_\infty[a,b]} \ge |||x|||_\infty - \varepsilon. \tag{2.6}$$

We denote by $c_p = c_p(x)$ the constant of the best L_p-approximation of the function x on the segment $[a,b]$ i.e., such a constant that $E_0(x)_{L_p[a,b]} = \|x - c_p(x)\|_{L_p[a,b]}$. It is clear, that $x(t) - c_p(x)$ has a zero on $[a,b]$. We denote this zero by the letter z, so that

$$x(z) = c_p. \tag{2.7}$$

Let $[m,M]$ be an interval, where the spline $\varphi_{\lambda,r}^{\alpha,\beta}$ strictly increases, where $\varphi_{\lambda,r-1}^{\alpha,\beta}(m) = \varphi_{\lambda,r-1}^{\alpha,\beta}(M) = 0$. We choose points $u_1 = u_1(\varepsilon)$, $u_2 = u_2(\varepsilon)$, $u_1, u_2 \in [m,M]$ so that

$$\varphi_{\lambda,r}^{\alpha,\beta}(M) - \varphi_{\lambda,r}^{\alpha,\beta}(u_1) = x(b) - x(z) \tag{2.8}$$

and

$$\varphi_{\lambda,r}^{\alpha,\beta}(u_2) - \varphi_{\lambda,r}^{\alpha,\beta}(m) = x(z) - x(a). \tag{2.9}$$

It is clear that $u_2 < u_1$, while $u_2 = u_2(\varepsilon)$ increases, and $u_1 = u_1(\varepsilon)$ decreases, as $\varepsilon \to 0$. Moreover, $u_1 - u_2 \to 0$ for $\varepsilon \to 0$. Therefore, there is a point $u \in [m,M]$ such that $u_i \to u$ for $\varepsilon \to 0$, $i = 1,2$.

It follows from (2.8) and (2.9) that for any $t \in [z,b]$ (or $t \in [a,z]$) there exists a point $y \in [u_1, M]$ (or $y \in [m, u_2]$) such that

$$\varphi_{\lambda,r}^{\alpha,\beta}(M) - \varphi_{\lambda,r}^{\alpha,\beta}(y) = x(b) - x(t) \tag{2.10}$$

or

$$\varphi_{\lambda,r}^{\alpha,\beta}(y) - \varphi_{\lambda,r}^{\alpha,\beta}(m) = x(t) - x(a). \tag{2.11}$$

By virtue of Theorem B, for any such pair of points (t,y), the following inequality holds

$$|x'(t)| \le |\varphi_{\lambda,r-1}^{\alpha,\beta}(y)|, \tag{2.12}$$

while, obviously,

$$b - z \ge M - u_1, \quad z - a \ge u_2 - m. \tag{2.13}$$

From (2.8) — (2.12) we obtain the following inequalities:

$$x(b-s) - x(z) \ge \varphi_{\lambda,r}^{\alpha,\beta}(M-s) - \varphi_{\lambda,r}^{\alpha,\beta}(u_1) \ge 0, \quad s \in [0, M-u_1] \tag{2.14}$$

and

$$x(a+s) - x(z) \le \varphi_{\lambda,r}^{\alpha,\beta}(m+s) - \varphi_{\lambda,r}^{\alpha,\beta}(u_2) \le 0, \quad s \in [0, u_2 - m]. \tag{2.15}$$

From (2.7) and (2.13) — (2.15) we have

$$|||x^{\uparrow}|||_p^p \geq \|x - c_p\|_{L_p[a,b]}^p = \|x - x(z)\|_{L_p[a,b]}^p =$$

$$= \int_z^b |x(s) - x(z)|^p ds + \int_a^z |x(z) - x(s)|^p ds =$$

$$= \int_0^{b-z} |x(b-s) - x(z)|^p ds + \int_0^{z-a} |x(z) - x(a+s)|^p ds \geq$$

$$\geq \int_0^{M-u_1} |\varphi_{\lambda,r}^{\alpha,\beta}(M-s) - \varphi_{\lambda,r}^{\alpha,\beta}(u_1)|^p ds +$$

$$+ \int_0^{u_2-m} |\varphi_{\lambda,r}^{\alpha,\beta}(m+s) - \varphi_{\lambda,r}^{\alpha,\beta}(u_2)|^p ds = \int_{u_1}^{M} |\varphi_{\lambda,r}^{\alpha,\beta}(s) - \varphi_{\lambda,r}^{\alpha,\beta}(u_1)|^p ds +$$

$$+ \int_m^{u_2} |\varphi_{\lambda,r}^{\alpha,\beta}(s) - \varphi_{\lambda,r}^{\alpha,\beta}(u_2)|^p ds.$$

Passing in the resulting estimate to the limit as $\varepsilon \to 0$, we obtain $|||x^{\uparrow}|||_p^p \geq \int_m^M |\varphi_{\lambda,r}^{\alpha,\beta}(s) - \varphi_{\lambda,r}^{\alpha,\beta}(u)|^p ds \geq |||(\varphi_{\lambda,r}^{\alpha,\beta})^{\uparrow}|||_p^p$. This immediately implies (2.5).

Let us now prove

$$L\left(x_{\pm}^{(k)}\right)_q \leq L\left((\varphi_{\lambda,r-k}^{\alpha,\beta})_{\pm}\right)_q. \tag{2.16}$$

Let $q = 1$ first. We fix an arbitrary interval (a, b) such that $|x^{(k)}(t)| > 0$ for $t \in (a, b)$ and the constant sign interval (A, B) of the function $\varphi_{\lambda,r-k}^{\alpha,\beta}$ between its adjacent zeros A and B. Then from (1.2), in view of (2.3) and (2.4), we obtain the inequalities

$$\|x_{\pm}^{(k)}\|_{\infty} \leq \|(\varphi_{r-k}^{\alpha,\beta})_{\pm}\|_{\infty}, \quad k = 1, 2, \ldots, r-1. \tag{2.17}$$

That's why $|||x^{(k-1)}|||_{\infty} \leq |||\varphi_{\lambda,r-(k-1)}^{\alpha,\beta}|||_{\infty}$ and

$$\left|\int_a^b x^{(k)}(t)dt\right| = \left|x^{(k-1)}(b) - x^{(k-1)}(a)\right| \leq 2|||x^{(k-1)}|||_{\infty} \leq 2|||\varphi_{\lambda,r-(k-1)}^{\alpha,\beta}|||_{\infty} =$$

$$= \left|\varphi_{\lambda,r-(k-1)}^{\alpha,\beta}(B) - \varphi_{\lambda,r-(k-1)}^{\alpha,\beta}(A)\right| = \left|\int_A^B \varphi_{\lambda,r-k}^{\alpha,\beta}(t)dt\right| = L((\varphi_{\lambda,r-k}^{\alpha,\beta})_{\pm})_1.$$

(The last equality follows from the fact that the spline $\varphi_{\lambda,r-k}^{\alpha,\beta}$ is equal to zero on the period on average). This immediately implies (2.6) for $q = 1$.

Let now $q > 1$. Note that if the function $f \in L_{\infty}^s(\mathbf{R})$, $s \in \mathbf{N}$, satisfies the condition $L(f)_p < \infty$ for some $p > 0$ and $|f(t)| > 0$, $t \in (a, b)$, with $a = -\infty$ or $b = +\infty$, then $f(t) \to 0$, for $t \to -\infty$ or $t \to +\infty$. In this case, we will set $f(-\infty) = 0$ and $f(+\infty) = 0$.

Since inequality (2.16), proved for $q = 1$, implies the estimate $L\left(x_{\pm}^{(k)}\right)_1 < \infty$, then if $|x_{\pm}^{(k)}(t)| > 0$ on the infinite interval (a, b), we can assume that $x_{\pm}^{(k)}(-\infty) = 0$ (if $a = -\infty$) and $x_{\pm}^{(k)}(+\infty) = 0$ (if $b = +\infty$).

Proceeding to the proof of (2.16) for $q > 1$, we fix the segment $[a, b]$ (finite or infinite) for which $|x_{\pm}^{(k)}(t)| > 0$ for $t \in (a, b)$ and denote by $x_{\sigma}^{(k)}$ the restriction of the function $x_{\pm}^{(k)}$ to $[a, b]$. Since the derivative $x^{(k)}$ is continuous and, in view of the above, it is sufficient to take the least upper bound in the definition of $L(x^{(k)})_p$ over segments $[a, b]$ such that

$$x^{(k)}(a) = x^{(k)}(b) = 0. \tag{2.18}$$

Let further the interval (A, B) of constant sign of the spline $\varphi_{\lambda,r-k}^{\alpha,\beta}$ between its adjacent zeros A and B be such that the sign of the spline on (A, B) is the same as the sign of the derivative $x^{(k)}(t)$ on (a, b). We denote by $(\varphi_{\lambda,r-k}^{\alpha,\beta})_{\sigma}$ the restriction of the spline $\varphi_{\lambda,r-k}^{\alpha,\beta}$ to the segment $[A, B]$.

Let's prove the inequality

$$\int\limits_0^{\xi} r(x_{\sigma}^{(k)}, t)dt \leq \int\limits_0^{\xi} r((\varphi_{\lambda,r-k}^{\alpha,\beta})_{\sigma}, t)dt, \quad \xi > 0. \tag{2.19}$$

First of all, we note that since the permutation preserves the uniform norm, then (2.17) implies the inequalities $r(x_{\sigma}^{(k)}, 0) \leq r((\varphi_{\lambda,r-k}^{\alpha,\beta})_{\sigma}, 0)$. Let us show further that the difference $\Delta(t) := r(x_{\sigma}^{(k)}, t) - r((\varphi_{\lambda,r-k}^{\alpha,\beta})_{\sigma}, t)$ changes sign (from – to +) at most once. To do this, note that, in view of (2.17) and (2.18), for any $y \in (0, \|x_{\sigma}^{(k)}\|_{\infty})$ there are points $t_i \in (a, b)$, $i = 1, ..., m$, $m \geq 2$, and two points $y_j \in (A, B)$ such that $y = |x_{\sigma}^{(k)}(t_i)| = (\varphi_{\lambda,r-k}^{\alpha,\beta})_{\sigma}(y_j)$. According to Hörmander's comparison theorem [18] applied to the function $x^{(k)}$ (its conditions are satisfied due to (2.3) and (2.17)), the inequality $|x_{\sigma}^{(k+1)}(t_i)| \leq |(\varphi_{\lambda,r-k-1}^{\alpha,\beta})_{\sigma}(y_j)|$ holds. Therefore, by the permutation derivative theorem (see, for example, [20, Proposition 1.3.2]), if the points θ_1 and θ_2 are chosen so that

$$y = r(x_{\sigma}^{(k)}, \theta_1) = r((\varphi_{\lambda,r-k}^{\alpha,\beta})_{\sigma}, \theta_2), \tag{2.20}$$

then

$$|r'(x_{\sigma}^{(k)}, \theta_1)| = \left[\sum_{i=1}^{m} |x_{\sigma}^{(k+1)}(t_i)|^{-1}\right]^{-1} \leq$$

$$\leq \left[\sum_{j=1}^{2} |(\varphi_{\lambda,r-k-1}^{\alpha,\beta})_{\sigma}(y_j)|^{-1}\right]^{-1} = |r'((\varphi_{\lambda,r-k}^{\alpha,\beta})_{\sigma}, \theta_2)|. \tag{2.21}$$

It follows from this that the difference Δ does not change sign (from – to +) more than once. Consider the integral $I(\xi) := \int\limits_0^{\xi} \left[r(x_{\sigma}^{(k)}, t) - r((\varphi_{\lambda,r-k}^{\alpha,\beta})_{\sigma}, t)\right] dt$.

It is clear that $I(0) = 0$. Further, in view of (2.16) for $q = 1$

$$\int\limits_0^{b-a} r(x_\sigma^{(k)}, t)dt = \int\limits_a^b |x_\sigma^{(k)}(t)|dt \le L(x_\sigma^{(k)})_1 \le$$

$$\le L((\varphi_{\lambda,r-k}^{\alpha,\beta})_\sigma)_1 = \int\limits_0^{B-A} r((\varphi_{\lambda,r-k}^{\alpha,\beta})_\sigma, t)dt.$$

Therefore, if we put $\mathcal{M} := \max\{b-a, B-A\}$, then $I(\mathcal{M}) \le 0$. In addition, $I'(t) = \Delta(t)$ changes sign (from − to +) no more than once. Thus, $I(\xi) \le 0$ for all $\xi \ge 0$ i.e., inequality (2.19) is proved. From (2.19), in virtue of the Hardy–Littlewood–Polya theorem (see, for example, [20, Statement 1.3.11]) inequality (2.16) follows.

Let us finally prove (2.1) and (2.2). Using (2.16), (2.5), the obvious equalities

$$L((\varphi_{\lambda,r}^{\alpha,\beta})_\pm)_q = \lambda^{-\frac{rq+1}{q}} L((\varphi_r^{\alpha,\beta})_\pm)_q,\ |||(\varphi_{\lambda,r}^{\alpha,\beta})^{\uparrow\downarrow}|||_p = \lambda^{-\frac{rp+1}{p}} |||(\varphi_r^{\alpha,\beta})^{\uparrow\downarrow}|||_p, \quad (2.22)$$

and the definition of δ, we have

$$L\left(x_\pm^{(k)}\right)_q \le L\left((\varphi_{\lambda,r-k}^{\alpha,\beta})_\pm\right)_q M\left(\frac{|||x|||_p}{|||\varphi_{\lambda,r}^{\alpha,\beta}|||_p}\right)^\delta =$$

$$= \lambda^{-(r-k)-1/q} \cdot L((\varphi_{r-k}^{\alpha,\beta})_\pm)_q \left(\lambda^{r+1/p} \cdot M\left(\frac{|||x|||_p}{|||\varphi_r^{\alpha,\beta}|||_p}\right)\right)^\delta =$$

$$= L((\varphi_{r-k}^{\alpha,\beta})_\pm)_q \cdot M\left(\frac{|||x|||_p}{|||\varphi_r^{\alpha,\beta}|||_p}\right)^\delta.$$

Now (2.1) follows in view of (2.3). Similarly, from (2.3) — (2.5) inequality (2.2) can be deduced. Theorem 1 is proved.

For a function f continuous on a segment $[a, b]$, we put $\mu(f) := \mu\{t \in [a, b] : f(t) > 0\}$. In particular, $\mu(\varphi_{\lambda,r}^{\alpha,\beta})_\pm := \mu\{t \in [0, 2\pi/\lambda] : (\varphi_{\lambda,r}^{\alpha,\beta})_\pm(t) > 0\}$.

Theorem 2. *Let $k, r \in \mathbf{N}$, $k < r$, $q \ge 1, p > 0$, $\alpha, \beta > 0$. For any function $x \in L_\infty^r(\mathbf{R})$ satisfying condition $|||x|||_p < \infty$, and an arbitrary segment $[a, b] \subset \mathbf{R}$ such that*

$$x^{(k)}(a) = x^{(k)}(b) = 0, \quad (2.23)$$

the inequality

$$\left\|\frac{x_\pm^{(k)}}{\mu(x_\pm^{(k)})}\right\|_{L_q[a,b]} \le \left\|\frac{\left(\varphi_{r-k}^{\alpha,\beta}\right)_\pm}{\mu(\varphi_{r-k}^{\alpha,\beta})_\pm}\right\|_q M\left(\frac{|||x|||_p}{|||\varphi_r^{\alpha,\beta}|||_p}\right)^\delta \|x^{(r)}\|_{\infty,\alpha^{-1},\beta^{-1}}^{1-\delta} \quad (2.24)$$

holds, where $\delta = (r-k)/(r+1/p)$.

Proof. In view of the homogeneity of inequality (2.24), we can assume that

$$\|x^{(r)}\|_{\infty,\alpha^{-1},\beta^{-1}} = 1. \tag{2.25}$$

Let's choose λ from the condition

$$M\left(\frac{|||x|||_p}{|||\varphi^{\alpha,\beta}_{\lambda,r}|||_p}\right) = 1. \tag{2.26}$$

Then inequality (2.1) implies that

$$L\left(x^{(k)}_{\pm}\right)_q \le L\left((\varphi^{\alpha,\beta}_{\lambda,r-k})_{\pm}\right)_q. \tag{2.27}$$

Let us first prove (2.24) in the case when

$$x^{(k)}(a) = x^{(k)}(b) = 0, \qquad |x^{(k)}(t)| > 0, \quad t \in (a,b). \tag{2.28}$$

We denote by (A,B) an interval of constant sign of the spline $\varphi^{\alpha,\beta}_{\lambda,r-k}$ between its adjacent zeros A and B such that the sign of the spline on (A,B) is the same as the sign of the derivative $x^{(k)}(t)$ on (a,b). We denote by $(\varphi^{\alpha,\beta}_{\lambda,r-k})_\sigma$ the restriction of the spline $\varphi^{\alpha,\beta}_{\lambda,r-k}$ to the segment $[A,B]$, and by $x^{(k)}_\sigma$ the restriction of the derivative $x^{(k)}$ to the segment $[a,b]$. We put $l := b-a$, $L := B-A$. Then from inequality (2.27) we have $\int_0^l r^q(x^{(k)}_\sigma, t)dt \le \int_0^L r^q((\varphi^{\alpha,\beta}_{\lambda,r-k})_\sigma, t)dt$. The last inequality implies the existence of a point $y \in [0,L]$ for which

$$\int\limits_0^l r^q(x^{(k)}_\sigma, t)dt = \int\limits_y^L r^q((\varphi^{\alpha,\beta}_{\lambda,r-k})_\sigma, t)dt. \tag{2.29}$$

Note that in the proof of Theorem 1 it was established that if the points θ_1 and θ_2 are chosen so that (2.20) holds i.e., $r(x^{(k)}_\sigma, \theta_1) = r((\varphi^{\alpha,\beta}_{\lambda,r-k})_\sigma, \theta_2)$, then inequality (2.21) holds i.e., $|r'(x^{(k)}_\sigma, \theta_1)| \le |r'((\varphi^{\alpha,\beta}_{\lambda,r-k})_\sigma, \theta_2)|$. Hence, in view of equality (2.29), the estimate $l \ge L - y$ follows. Moreover, it is easy to see that the function $\frac{1}{L-y}\int\limits_0^{L-y} r^q((\varphi^{\alpha,\beta}_{\lambda,r-k})_\sigma, t)dt$ decreases on $[0,L]$. That's why $\frac{1}{l}\int\limits_0^l r^q(x^{(k)}_\sigma, t)dt \le \frac{1}{L-y}\int\limits_0^{L-y} r^q((\varphi^{\alpha,\beta}_{\lambda,r-k})_\sigma, t)dt \le \frac{1}{L}\int\limits_0^L r^q((\varphi^{\alpha,\beta}_{\lambda,r-k})_\sigma, t)dt$. Hence, in view of (2.25) and (2.26), the following inequality holds:

$$\int\limits_a^b \frac{|x^{(k)}_\sigma(t)|^q}{\mu(x^{(k)}_\sigma)}dt \le$$

$$\le \int\limits_A^B \frac{|(\varphi^{\alpha,\beta}_{\lambda,r-k})_\sigma(t)|^q}{\mu(\varphi^{\alpha,\beta}_{\lambda,r-k})_\sigma}dt M\left(\frac{|||x|||_p}{|||\varphi^{\alpha,\beta}_{\lambda,r}|||_p}\right)^{\delta q}\|x^{(r)}\|^{(1-\delta)q}_{\infty,\alpha^{-1},\beta^{-1}}. \tag{2.30}$$

Using equalities (2.22) and $\mu_\pm(\varphi_{\lambda,r}^{\alpha,\beta}) = \lambda^{-1}\mu_\pm(\varphi_r^{\alpha,\beta})$, we see that the right-hand side of this inequality does not depend on λ. Thus (2.24) in case (2.28) is proved.

Let us now prove (2.24) in the general case. Consider the collections $I_\pm$ of all segments $[a_j^\pm, b_j^\pm] \subset [a,b]$ such that $x^{(k)}(a_j^\pm) = x^{(k)}(b_j^\pm) = 0$, $x_\pm^{(k)}(t) > 0$, $t \in (a_j^\pm, b_j^\pm)$. It's clear that

$$\sum_{j\in I_\pm}(b_j^\pm - a_j^\pm) = \mu(x_\pm^{(k)}), \qquad \left\|x_\pm^{(k)}\right\|_{L_q[a,b]}^q = \sum_{j\in I_\pm}\int\limits_{a_j^\pm}^{b_j^\pm}\left|x_\pm^{(k)}(t)\right|^q dt. \tag{2.31}$$

Let us estimate the integrals $\int_{a_j^\pm}^{b_j^\pm}\left|x_\pm^{(k)}(t)\right|^q dt$ in (2.31) using inequality (2.30). Setting for brevity $S_\pm := \left\|\frac{\left(\varphi_{r-k}^{\alpha,\beta}\right)_\pm}{\mu(\varphi_{r-k}^{\alpha,\beta})_\pm}\right\|_q^q M\left(\frac{|||x|||_p}{|||\varphi_r^{\alpha,\beta}|||_p}\right)^{\delta q}\|x^{(r)}\|_{\infty,\alpha^{-1},\beta^{-1}}^{(1-\delta)q}$, from (2.31) we deduce the estimate $\left\|x_\pm^{(k)}\right\|_{L_q[a,b]}^q \le \sum\limits_{j\in I_\pm}(b_j^\pm - a_j^\pm)S_\pm = \mu(x_\pm^{(k)})S_\pm$, which is equivalent to (2.24). Theorem 2 is proved.

Corollary 1. *If under the conditions of Theorem 2 the number r is even or $p=\infty$, then*

$$\left\|\frac{x_\pm^{(k)}}{\mu(x_\pm^{(k)})}\right\|_{L_q[a,b]} \le \left\|\frac{\left(\varphi_{r-k}^{\alpha,\beta}\right)_\pm}{\mu(\varphi_{r-k}^{\alpha,\beta})_\pm}\right\|_q \left(\frac{|||x|||_p}{|||\varphi_r^{\alpha,\beta}|||_p}\right)^{\delta}\|x^{(r)}\|_{\infty,\alpha^{-1},\beta^{-1}}^{1-\delta},$$

where $\delta = (r-k)/(r+1/p)$.

Corollary 2. *If under the conditions of Theorem 2 the number $r-k$ is odd, then*

$$\left\|x_\pm^{(k)}\right\|_{L_q[a,b]} \le \left(\frac{\mu(x_\pm^{(k)})}{2\pi}\right)^{\frac{1}{q}}\left\|\varphi_{r-k}^{\alpha,\beta}\right\|_q M\left(\frac{|||x|||_p}{|||\varphi_r^{\alpha,\beta}|||_p}\right)^{\delta}\|x^{(r)}\|_{\infty,\alpha^{-1},\beta^{-1}}^{1-\delta}$$

and

$$\left\|x^{(k)}\right\|_{L_q[a,b]} \le \left(\frac{b-a}{2\pi}\right)^{\frac{1}{q}}\left\|\varphi_{r-k}^{\alpha,\beta}\right\|_q M\left(\frac{|||x|||_p}{|||\varphi_r^{\alpha,\beta}|||_p}\right)^{\delta}\|x^{(r)}\|_{\infty,\alpha^{-1},\beta^{-1}}^{1-\delta}$$

with the same exponent δ.

Corollary 3 [21]. *Under the conditions of Theorem 2*

$$\left\|x_\pm^{(k)}\right\|_{L_q[a,b]} \le \left(\frac{\mu_\pm(x^{(k)})}{2\pi}\right)^{\frac{1}{q}}\|\varphi_{r-k}\|_q\left(\frac{|||x|||_p}{|||\varphi_r|||_p}\right)^{\delta}\cdot\|x^{(r)}\|_\infty^{1-\delta}$$

and

$$\left\|x^{(k)}\right\|_{L_q[a,b]} \le \left(\frac{b-a}{2\pi}\right)^{\frac{1}{q}} \|\varphi_{r-k}\|_q \left(\frac{|||x|||_p}{|||\varphi_r|||_p}\right)^{\delta} \cdot \|x^{(r)}\|_\infty^{1-\delta},$$

where $\delta = (r-k)/(r+1/p)$.

Remark 1. Theorems 1, 2 and Corollaries 1 — 3 are analogues of the statements obtained in [11], in which the local norms $L(x)_p$ were used instead of the local norms $|||x|||_p$. Note that the use of $|||x|||_p$ is preferable, because $|||x|||_p^p < L(x_+)_p^p + L(x_-)_p^p$, but it is easy to give examples of infinitely differentiable functions x for which the fraction $\frac{|||x|||_p}{L(x)_p}$ is arbitrarily small.

It is easy to check that in all the inequalities obtained above, the exponent at $|||x|||_p$ is the maximum possible. Ligun [22] found that if the number of sign changes of the derivative is taken into account in the Kolmogorov-type inequality, then the exponent at the norm of the function can be increased. The following theorem shows that this Ligun effect is preserved in inequalities with local norms.

For a function f continuous on a segment $[a,b]$, denote by $F_{[a,b]}$ the set of its zeros on this segment, and by the symbol $\nu_{[a,b]}(f)$ the number of constituent intervals of an open set $(a,b) \setminus F_{[a,b]}$.

Theorem 3. *Let* $k, r \in \mathbf{N}$, $k < r$, $r-k$ *be odd,* $q \ge 1, p > 0$, $\alpha, \beta > 0$. *For any function* $x \in L_\infty^r(\mathbf{R})$ *satisfying* $|||x|||_p < \infty$, *and an arbitrary segment* $[a,b] \subset \mathbf{R}$ *such that* $x^{(k)}(a) = x^{(k)}(b) = 0$, *the following inequality*

$$\left\|x^{(k)}\right\|_{L_q[a,b]} \le$$

$$\left(\frac{\nu_{[a,b]}(x^{(k)})}{2}\right)^{\frac{1}{q}} \left\|\varphi_{r-k}^{\alpha,\beta}\right\|_q M \left(\frac{|||x|||_p}{|||\varphi_r^{\alpha,\beta}|||_p}\right)^{\gamma} \|x^{(r)}\|_{\infty,\alpha^{-1},\beta^{-1}}^{1-\gamma} \tag{2.32}$$

holds, where $\gamma = (r-k+1/q)/(r+1/p)$.

Proof. Following the proof of Theorem 2, we assume that condition (2.25) is satisfied, and we choose λ from condition (2.26). Then inequality (2.1) implies (2.27) i.e., $L\left(x_\pm^{(k)}\right)_q \le L\left((\varphi_{\lambda,r-k}^{\alpha,\beta})_\pm\right)_q$. Let us represent the segment $[a,b]$ as $\cup_{i=1}^{\nu}[a_i, a_{i+1}]$, where a_i are the zeros of the derivative $x^{(k)}$, $a_1 = a$, $a_{\nu+1} = b$, $\nu = \nu_{[a,b]}(x^{(k)})$. Since $r-k$ is odd, we have $L\left((\varphi_{\lambda,r-k}^{\alpha,\beta})_+\right)_q = L\left((\varphi_{\lambda,r-k}^{\alpha,\beta})_-\right)_q$. Therefore, applying (2.27), we obtain $\|x^{(k)}\|_{L_q[a,b]} = \sum_1^{\nu_{[a,b]}(x^{(k)})} \|x^{(k)}\|_{L_q[a_i,a_{i+1}]} \le \nu_{[a,b]}(x^{(k)}) L\left((\varphi_{\lambda,r-k}^{\alpha,\beta})\right)_q = \frac{\nu_{[a,b]}(x^{(k)})}{2} \left\|\varphi_{\lambda,r-k}^{\alpha,\beta}\right\|_q$. From here, in virtue of (2.25) and (2.26), the inequality $\|x^{(k)}\|_{L_q[a,b]} \le \left(\frac{\nu_{[a,b]}(x^{(k)})}{2}\right)^{\frac{1}{q}} \left\|\varphi_{\lambda,r-k}^{\alpha,\beta}\right\|_q M \left(\frac{|||x|||_p}{|||\varphi_{\lambda,r}^{\alpha,\beta}|||_p}\right)^{\gamma} \|x^{(r)}\|_{\infty,\alpha^{-1},\beta^{-1}}^{1-\gamma}$, follows, whose right-hand side, in view of equalities (2.22) and the definition of γ, does not depend on λ. Theorem 3 is proved.

Remark 2. For $\alpha = \beta$ inequalities (2.1), (2.2), (2.24), and (2.32) were obtained in [21].

The exponent γ in (2.32) is larger than the corresponding exponent δ in (2.24). This happens because (2.32) takes into account the characteristic $\nu_{[a,b]}(x^{(k)})$, which is (under the conditions of Theorem 3) equal to the number of zeros of the derivative $x^{(k)}$ on the interval $[a, b)$.

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