Res. Math. **32**(3), 2024, p. 12–25
doi:10.15421/242430
ISSN (Print) 2664-4991
ISSN (Online) 2664-5009

UDK 517.98+515.1

D. Y. Dolishniak, A. V. Zagorodnyuk
Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk 76018
E-mails: daryna.dolishniak@pnu.edu.ua, andriy.zagorodnyuk@pnu.edu.ua

Metric Semigroups and Groups of Multisets [1]

Abstract. We investigate the algebraic and topological properties of sets of complex multisets associated with Banach spaces having symmetric bases. We consider algebraic structures on the sets of multisets and compare some natural metrics on the (semi)groups of multisets. Also, we construct nonlinear analogs of the weighted backward shift operator on metric spaces of multisets, establish conditions of topological transitivity, and prove an analog of the Topological Transitivity Criterion for metric semigroups.
Key words: symmetric functions on Banach spaces, metric semigroups of multisets, topological transitivity, hypercyclicity

Анотація. Досліджено алгебраїчні та топологічні властивості множин комплексних мультимножин, асоційованих із банаховими просторами із симетричними базисами. Розглянуто алгебраїчні структури на множинах мультимножин і зроблено порівняння деяких природних метрик на (напів)групах мультимножин. Також побудовано нелінійні аналоги оператора зваженого лівого зсуву на метричних просторах мультимножин, встановлено умови топологічної транзитивності та доведено аналог критерію топологічної транзитивності для метричних напівгруп.
Ключові слова: симетричні функції на банахових просторах, метричні напівгрупи мультимножин, топологічна транзитивність, гіперциклічність

MSC2020: Pri 46B70, Sec 47A16, 22A05

1. Introduction and Preliminaries

Let S be a semigroup of isometric operators on a Banach space X. Then the quotient space X/S, consisting of orbits with respect to actions of S on X, can be considered as a natural domain of S-symmetric mappings. The set X/S may have nontrivial algebraic and topological structures. The case when $X=\ell_1$ and S is the group of permutations of standard basis vectors was considered in [6, 15] and in [9] for a more general situation. In particular, it

[1]This research was supported by the National Research Foundation of Ukraine, grant No. 2023.03/0198

was investigated semiring and ring structures related to X/S, topologizations of X/S, and applications to symmetric functions on Banach spaces. In this paper, we continue the investigation of X/S and its supersymmetric version if X is a Banach space with a symmetric basis (e_i) and S is the group of permutations of vectors e_i, $i \in \mathbb{N}$. Note that in this case, X/S may be identified with a set of multisets of numbers because any element of X/S is a family of unordered numbers with possible repetitions, that is, a multiset. We compare different metrizable topologies on these sets and investigate the dynamics of some analogs of the backward shift operator on corresponding metric spaces of multisets. Possible applications of polynomial dynamics on a ring of multisets in economics were proposed in [13]. Some applications of the semiring of integer multisets in cryptography were considered in [10], and applications in quantum physics in [8]. Many authors studied symmetric functions with respect to various groups and semigroups of operators S. In [1, 7, 14] were considered algebras of symmetric polynomials and analytic functions on Banach spaces ℓ_p, $1 \le p < \infty$ with respect to the group of permutations of basis vectors, their algebraic bases and spectra, in [4, 16, 22] were investigated so-called block symmetric polynomials on ℓ_p and L_p. Algebras of symmetric analytic functions with respect to abstract groups of operators were studied in [2, 3, 11].

Let X be a Banach space with a symmetric normalised monotone Schauder basis (e_n), $n \in \mathbb{N}$ and a symmetric norm $\|\cdot\|$ over the field $\mathbb{C}$ of complex numbers. In other words, (e_n) is equivalent to $(e_{\sigma(n)})$ for every permutation σ on the set positive integers $\mathbb{N}$, $\|e_n\| = 1$ and the basis constant of (e_n) is equal to 1 (see [17] for details). In particular, for a given $x \in X$,

$$x = (x_1, \ldots, x_n, \ldots) = \sum_{n=1}^{\infty} x_n e_n,$$

$\|x\| \ge \|x\|_{c_0} = \max_n |x_n|$. The *support* of x is defined as $\operatorname{supp}(x) = \{k \in \mathbb{N} \colon x_k \ne 0\}$. We define an equivalence relation "$\sim$" so that $x \sim z$ if and only if there is a bijection $\sigma \colon \operatorname{supp}(z) \to \operatorname{supp}(x)$ such that

$$\sum_{n \in \operatorname{supp}(x)} x_n e_n = \sum_{n \in \operatorname{supp}(z)} z_n e_{\sigma(n)}.$$

The class of equivalence containing x will be denoted by $[x]$. The quotient set $\mathcal{M}_X^+ = X/\sim$ consists of sets of (possibly infinite) multisets of nonzero numbers $[x] = \{x_k \colon k \in \operatorname{supp}(x), x \in X\}$, and the class $[0] = \{(0, 0, \ldots)\}$. Note that $(x_1, \ldots, x_n, \ldots) \sim (0, x_1, \ldots, x_n, \ldots)$ and since the series $\sum_{n=1}^{\infty} x_n e_n$ converges in X, every nonzero coordinate x_k has a finite multiplicity in $[x]$.

Let us define a semigroup associative operation on $\mathcal{M}_X^+$ by $[x] + [z] = [x] \cup [z]$. Then $\left(\mathcal{M}_X^+, +\right)$ is a commutative semigroup. Note that if $[x]+[z] = [x]+[u]$, then $[z] = [u]$, that is, we have the cancelation law. If we denote by $x \bullet z$ the vector $(x_1, z_1, x_2, z_2, \ldots,)$, then $[x] + [z] = [x \bullet z]$.

It well-known that any commutative semigroup with the cancelation law can be embedded (as a semigroup) to a commutative group using so-called Grothendieck's extension which is unique up to an isomorphism. The Grothendieck extension of $\big(\mathcal{M}_X^+, +\big)$ can constructed by the following way. Let

$$\mathcal{M}_X^+ \times \mathcal{M}_X^+ = \{([y], [x])\colon [x], [y] \in \mathcal{M}_X^+\}$$

be the Cartesian square of $\mathcal{M}_X^+$. Consider the following relation of equivalence on $\mathcal{M}_X^+ \times \mathcal{M}_X^+$: $([y], [x]) \approx ([v], [u])$ if and only if $[y] \setminus [x] = [v] \setminus [u]$ and $[x] \setminus [y] = [u] \setminus [v]$. In particular, we have that $([y] + [a], [x] + [a]) \approx ([y], [x])$ for every $[a] \in \mathcal{M}_X^+$. We will use notation $[(y|x)] = \big[[y] \,|\, [x]\big] = [(\ldots, y_2, y_1|x_1, x_2, \ldots)]$ for the class containing $([y], [x])$, and $\mathcal{M}_X$ for the quotient set $\mathcal{M}_X^+ \times \mathcal{M}_X^+ / \approx$. The semigroup operation can be naturally extended to $\mathcal{M}_X$ by

$$\big[[y] \,|\, [x]\big] + \big[[v] \,|\, [u]\big] = \big[[y] + [v] \,|\, [x] + [u]\big].$$

Since $[(x|x)] = [(0|0)] = 0$ in $\mathcal{M}_X$, for the inverse element we have $-[(y|x)] = [(x|y)]$. Thus $\big(\mathcal{M}_X, +\big)$ is a commutative group. Using the "symmetric translation" operation on X, $x \bullet z$ we can write

$$\big[[y] \,|\, [x]\big] + \big[[v] \,|\, [u]\big] = [(y|x)] + [(v|u)] = [(y \bullet v|x \bullet u)].$$

Throughout the paper we assume that $\mathcal{M}_X^+$ is a subset of $\mathcal{M}_X$ with respect to the embedding $[x] \mapsto [(0|x)]$.

A representative $(y'|x')$ of $[(y|x)]$ is *irreducible* if for every pair of indexes i and j, $x'_i \neq y'_j$. It is known [9, 15] that for every $[(y|x)] \in \mathcal{M}_X$ there exists an irreducible representative and it is unique up to permutations (separately for coordinates of x and of y).

Let M be a metric space and T be a continuous mapping $T\colon M \to M$. We say that the pair (T, M) is a *(discrete) dynamical system* on M considering the sequence of maps $\{T^n\}$, $n \in \mathbb{N}$ [12, p. 4].

Definition 1. A dynamical system (T, M) is called topologically transitive if for any pair U, V of nonempty open subsets of M there exists some integer $k \geq 0$ such that $T^k(U) \cap V \neq \emptyset$.

It is well known [21] that the weighted backward shift

$$B_\lambda\colon (x_1, \ldots, x_n, \ldots) \mapsto \lambda(x_2, \ldots, x_n, \ldots),$$

$\lambda \in \mathbb{C}$, $|\lambda| > 1$, is a topologically transitive linear operator if $X = \ell_p$, $1 \leq p < \infty$ or if $X = c_0$. In [18, 20, 24], there were considered some generalizations of the weighted backward shift for nonseparable Banach spaces. A continuous mapping $T\colon M \to M$ is said to be *hypercyclic* if there is an element $a \in M$ such that the orbit $\{T^n(a)\colon n \in \mathbb{N}\}$ is dense in M. It is well-known that if M is a complete separable space, then T is hypercyclic if and only if it is topologically transitive [12, p. 10].

In Section 2, we discuss different ways of introducing metrizable topologies on $\mathcal{M}_X^+$ and $\mathcal{M}_X$. In Section 3, we prove topological transitivity of some nonlinear symmetric modifications of the weighted backward shift operator.

For general information about the theory of topologically transitive mappings we refer the reader to [5, 12].

2. Metrics on Semigroups and Groups of Multisets

Let us define a norm $\big\|[x]\big\| := \|x\|$ on $\mathcal{M}_X^+$. Since the original norm on X supposed to be symmetric, the defined norm on $\mathcal{M}_X^+$ does not depend on the representative. Clearly, (c.f. [15]) that $\big\|[x]+[z]\big\| \le \big\|[x]\big\| + \big\|[z]\big\|$ and $\big\|[\lambda x]\big\| = |\lambda|\big\|[x]\big\|$, for all $[x],[z] \in \mathcal{M}_X^+$ and $\lambda \in \mathbb{C}$.

The function $\big\|[x]\big\|$ generates a metric d on $\mathcal{M}_X^+$:

$$d\big([x],[z]\big) = \big\|[x]\triangle[z]\big\|,$$

where $[x]\triangle[z] = \big([x]\setminus[z]\big) \cup \big([z]\setminus[x]\big)$ is the symmetric difference. Here we understand the symmetric difference by taking into account the multiplicities of elements. For example,

$$[(1,1,2,2,2,3)]\triangle[(1,2,2,2,2,2,4)] = [(1,2,2,3,4)].$$

In [9, 15] it was introduced a norm on $\mathcal{M}_X$ by the following formula:

$$\big\|\big[[y]\,|\,[x]\big]\big\| := \sup\big\{\big\|[y']\big\| + \big\|[x']\big\|\colon\ \big[[y]\,|\,[x]\big] \approx \big[[y']\,|\,[x']\big]\big\}.$$

This norm generate a metric on $\mathcal{M}_X$ by $d\big(\big[[y]\,|\,[x]\big],\big[[v]\,|\,[u]\big]\big) = \big\|\big[[y]\,|\,[x]\big] - \big[[v]\,|\,[u]\big]\big\|$. This metric is an extension of the metric of $\mathcal{M}_X^+$ introduced above, because

$$d\big([x],[z]\big) = \big\|[x]\triangle[z]\big\| = \big\|\big[[0]\,|\,[x]\big] - \big[[0]\,|\,[z]\big]\big\| = d\big(\big[[0]\,|\,[x]\big],\big[[0]\,|[z]\,\big]\big).$$

Note that the metric space $\big(\mathcal{M}_X, d\big)$ endowed with more algebraic operations, for the case $X = \ell_1$ was introduced in [15] and further investigated in [8, 10]. The general case of X was considered in [9]. In particular, in [9] it was proved that $\big(\mathcal{M}_X, d\big)$ is compete and so $\big(\mathcal{M}_X^+, d\big)$ as a closed subspace.

Let us denote by $\big(\mathcal{M}_{X,0}, d\big)$ the subset of $\big(\mathcal{M}_X, d\big)$ consisting of elements $[(y|x)] = \big[[y]\,|\,[x]\big]$ such that there are representatives $y' \in [y]$ and $x' \in [x]$ with finite supports. Clearly, $\big(\mathcal{M}_{X,0}, d\big)$ is a subgroup of $\big(\mathcal{M}_X, d\big)$. Also, we denote $\big(\mathcal{M}_{X,0}^+, d\big) = \big(\mathcal{M}_X^+, d\big) \cap \big(\mathcal{M}_{X,0}^+, d\big)$. Elements of $\mathcal{M}_X$ admit the multiplication by constants $\lambda \in \mathbb{C}$,

$$\mathbb{C} \times \mathcal{M}_X \ni \big(\lambda, [(y|x)]\big) \longmapsto \lambda[(y|x)] := [(\lambda y|\lambda x)] \in \mathcal{M}_X.$$

The most important results that we may obtain using [9, 15] about $\big(\mathcal{M}_X, d\big)$, can be gathered in the following theorem.

Theorem 1. (c.f. [9]).

(i) $\big(\mathcal{M}_X, d\big)$ *is a complete metric space.*

(ii) *The operation of addition is continuous in the metric* d*.*

(iii) *The multiplication by a constant is discontinuous but* $\lambda[(y|x)] \to 0$ *if* $\lambda \to 0$, *for every fixed* $[(y|x)]$.

(iv) *The metric space* $\big(\mathcal{M}_X, d\big)$ *is nonseparable.*

(v) $\big(\mathcal{M}^+_{X,0}, d\big)$ *is a dense subspace in* $\big(\mathcal{M}_X, d\big)$.

Proof. Items *(i)* and *(ii)* are proved in [9]. To prove *(iii)* and *(iv)* we observe that for every $\varepsilon > 0$ and $[(y|x)] \neq 0$, $d\big(\lambda[(y|x)], (\lambda+\varepsilon)[(y|x)]\big) = 2\lambda + \varepsilon\lambda \nrightarrow 0$ as $\varepsilon \to 0$ if $\lambda \neq 0$. Thus, the operation of multiplication by constants is discontinuous and the interval $\lambda[(y|x)]$, $0 < \lambda_1 < \lambda < \lambda_2$ is an uncountable nowhere dense set. On the other hand, $\big\|\lambda[(y|x)]\big\| = |\lambda|\big\|[(y|x)]\big\| \to 0$ as $\lambda \to 0$.

To prove *(v)* we notice that for every representative $x \in [x]$ and $y \in [y]$, and natural numbers n and m,

$$
\begin{aligned}
&\big\|[(y|x)] - [(\ldots, 0, y_m, \ldots, y_1|x_1, \ldots, x_n, 0, \ldots)]\big\| \\
&= \big\|[(\ldots, y_{m+k}, \ldots, y_{m+1}|x_{n+1}, \ldots, x_{n+k}, \ldots)]\big\| \\
&\le \Big\|\sum_{k=m+1}^{\infty} y_k e_k\Big\| + \Big\|\sum_{k=n+1}^{\infty} x_k e_k\Big\| \to 0
\end{aligned}
$$

as $\min\{n, m\} \to \infty$.

Note that the multiplication by a fixed constant λ, $[(y|x)] \mapsto \lambda[(y|x)]$ is obviously continuous with respect to $[(y|x)]$.

Every function g on $\mathcal{M}_X$ (or on $\mathcal{M}^+_X$) can be extended to a function $\breve{g}$ on $X \times X$ (on X) by $\breve{g}(y|x) = g\big([(y|x)]\big)$ (or $\breve{g}(x) = g\big([x]\big)$ respectively). We say that a function f on $X \times X$ (resp. on X) is supersymmetric (resp. symmetric) if there is a function g on $\mathcal{M}_X$ (resp. on $\mathcal{M}^+_X$) such that $f(y|x) = g\big([(y|x)]\big)$ (resp. $f(x) = g\big([x]\big)$).

Proposition 1. Let $[x^{(m)}]$ be a sequence in $\mathcal{M}^+_X$ that converges to an element $[x^{(0)}]$. Then $[x^{(m)}]$ is of the form

$$[x^{(m)}] = [(x_1^{(m)}, z_1^{(m)}, x_2^{(m)}, z_2^{(m)}, \ldots)],$$

where $[z^{(m)}] = [(z_1^{(m)}, z_2^{(m)}, \ldots)] \to [0]$ as $m \to \infty$.

Proof. Clearly, $[x^{(m)}] - [x^{(0)}] =: [z^{(m)}] \to [0]$ as $m \to \infty$, and so $[x^{(m)}]$ is as required.

Theorem 2. *The quotient map* $x \mapsto [x]$ *is open but it is discontinuous at any point of* X *excepting* 0.

Proof. An open ball of radius r in X centered at a point x_0 contains, in particular, all point of the form x_0+z such that $\operatorname{supp}(x_0)\cap\operatorname{supp}(z)=\varnothing$ and $\|z\|<r$. But for this case, $[x_0+z]=[x_0]+[z]$ and the set $\{[x_0]+[z]\colon \|[z]\|<r\}$ is exactly the open ball of radius r centered at $[x_0]\in\mathcal{M}_X^+$. Thus, the range of any open ball in X under the quotient map contains an open ball of $\mathcal{M}_X^+$ and so must be open.

Let $x\in X$ and $x\neq 0$. Let us choose a sequence $x^{(n)}=\big(x_1^{(n)},x_2^{(n)},\dots\big)$ so that $\big|x_k^{(n)}-x_k\big|<1/2^{n+k}$ and $x_k^{(n)}\neq x_j$ for any pair of indexes k and j. Then

$$\|x-x^{(n)}\|\le\frac{1}{2^n}\sum_{k=1}^{\infty}\frac{1}{2^k}=\frac{1}{2^n}\to 0\quad\text{as}\quad n\to\infty.$$

On the other hand

$$d\big(x,x^{(n)}\big)=\Big\|\big[\big(x^{(n)}|x\big)\big]\Big\|=\|x\|+\|x^{(n)}\|>\|x\|>0.$$

Thus, the quotient map is discontinuous at x.

According to the definition of d, a sequence $x^{(m)}\in X$ tends to 0 if and only if $\|[x^{(m)}]\|=\|x^{(m)}\|$ tends to 0, that is, $[x^{(m)}]\to[0]$ in $\mathcal{M}_X^+$. Thus, $x\mapsto[x]$ is continuous at 0.

Corollary 1. *If a symmetric function f on X is continuous, then $\widehat{f}\colon[x]\to f(x)$ is continuous on $\mathcal{M}_X^+$.*

A symmetric function f on X is continuous at zero if and only if $\widehat{f}$ is continuous at $[0]$.

Example 1. Let $X=\ell_1$ and χ be the indicator function of the open unit disc $D\subset\mathbb{C}$, that is, $\chi(t)=1$ if $|t|<1$ and $\chi(t)=0$ if $|t|\ge 1$. Set

$$g(x)=\sum_{n=1}^{\infty}x_n\chi(x_n),\quad x\in\ell_1.$$

Evidently, g is symmetric and discontinuous. But $\widehat{g}$ is continuous on $\mathcal{M}_X^+$. Indeed, let $[x^{(m)}]\to[x]$ as $m\to\infty$. It is enough to check the case when $g\big(x^{(m)}\big)\not\to g(x)$. If for all coordinates $x_i^{(m)}$ of vectors $x^{(m)}$ we have $\big|x^{(m)}\big|<1$ and there is a coordinate x_j of x with $|x_j|\ge 1$, then $d\big([x^{(m)}],[x]\big)\ge|x_j|\ge 1$. So, for this case, $[x^{(m)}]\not\to[x]$. Clearly that for other cases, we have convergence. For example, if absolute values of all coordinates of x are less than 1, then $\widehat{g}\big(x^{(m)}\big)=x^{(m)}\to x=\widehat{g}(x)$ as $m\to\infty$. Thus $\widehat{g}$ is continuous while g is not.

The following example shows that the convergence of $x^{(m)}$ in X does not imply the convergence of $[x^{(m)}]$ in $\mathcal{M}_X^+$.

Example 2. The set of complex numbers can be naturally included into $\mathcal{M}_X^+$ by

$$\mathbb{C}\ni\lambda\rightsquigarrow[\lambda]=\lambda[e_1]\in\mathcal{M}_X^+.$$

But

$$d(\lambda_1, \lambda_2) = \begin{cases} |\lambda_1| + |\lambda_2| & \text{if } \lambda_1 \neq \lambda_2, \\ 0 & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

In other words, the restriction of the metric d to the range of any closed subset of $\mathbb{C}$ that does not contain zero, generates the discrete topology. In particular, if λ_n converges to $\lambda \neq 0$ in $\mathbb{C}$ and $\lambda_n \neq \lambda$ for all n, then $\lambda_n e_1$ converges to λe_1 in X, but $[\lambda_n]$ does not converges to $[\lambda]$ in $\mathcal{M}_X^+$.

We can see that the topology generated by the metric d looks too strong. Let us introduce another metric on $\mathcal{M}_X$ generating a weaker topology. A mapping f from a metric space (X, ρ_1) to a metric space (Y, ρ_2) is *Lipschitz* if the *Lipschitz constant*

$$L(f) = \sup_{x,z \in X} \frac{\rho_2(f(x), f(z))}{\rho_1(x, z)}$$

is finite. If $L(f) \leq 1$, then f is called *nonexpansive*. According to [23, pp. 10-13], the following theorem holds

Theorem 3. *Let (X, ρ_0) be a complete metric space and "$\sim$" be a relation of equivalence on X. Then the following function on $(X/\sim) \times (X/\sim)$,*

$$\rho\big([x], [z]\big) = \inf\{\rho_0(x', q_1) + \rho_0(q_1', q_2) + \cdots + \rho_0(q_{n-1}', z') \colon x \sim x', z \sim z', q_j \sim q_j'\}$$

is a pseudometric and

$$\rho\big([x], [z]\big) = \sup |f(x) - f(z)|,$$

where f goes over the set of real valued nonexpansive functions on X such that $f(x) = f(y)$ if $x \sim y$. Moreover, if there is a sequence of nonexpansive functions $f_k \colon X \to \mathbb{R}$ such that $x \sim z$ if and only if $f_k(x) = f_k(z)$, $k \in \mathbb{N}$, then ρ is a metric.

Corollary 2. *The function $\rho(\,\cdot\,,\,\cdot\,)$, defined as*

$$\rho\big([u], [v]\big) = \inf\{\|u' - q_1\| + \|q_1' - q_2\| + \cdots + \|q_{n-1}' - v'\| \colon u \approx u', v \approx v', q_j \approx q_j'\},$$

is a pseudometric on $\mathcal{M}_X$, and it is a metric if $X = \ell_p$, $1 \leq p < \infty$, where $\|u\| = \|x\| + \|y\|$, $u = (y|x)$ is the standard norm in $X \times X$.

Proof. By Theorem 3 applying to $\rho_0(u, v) = \|u - v\|$, $u, v \in X \times X$ and to the equivalence "$\approx$", the function $\rho(\,\cdot\,,\,\cdot\,)$ is a pseudometric on $\mathcal{M}_X$. Let $X = \ell_p$ for some $1 \leq p < \infty$. We claim that polynomials

$$T_k(u) = T_k(y|x) = \sum_{j=1}^{\infty} x_j^k - \sum_{j=1}^{\infty} y_j^k$$

are such that $u \approx v$ if and only if $T_k(u) = T_k(v)$, $k \in \mathbb{N}$, $k \geq \lceil p \rceil$, where $\lceil p \rceil$ is the ceil of p. Indeed, in [1] it is proved that $x \sim y$ in ℓ_p, $1 \leq p < \infty$ if and only if

$$\sum_{j=1}^{\infty} x_j^k = \sum_{j=1}^{\infty} y_j^k, \quad \text{that is,} \quad T_k(y|x) = 0$$

for all $k \geq m \geq \lceil p \rceil$. Hence, $[(y|x)] = [0]$ if and only if $x \sim y$ if and only if $T_k(y|x) = 0$ for all $k \geq m \geq \lceil p \rceil$. Let now $u = (y|x)$, $v = (d|b)$, and $T_k(u) = T_k(v)$ for all $k \geq m \geq \lceil p \rceil$. Then

$$\sum_{j=1}^{\infty} x_j^k - \sum_{j=1}^{\infty} y_j^k = \sum_{j=1}^{\infty} b_j^k - \sum_{j=1}^{\infty} d_j^k$$

and so $T_k(y \bullet b | x \bullet d) = 0$. Thus $\big[[y] - [b] \,|\, [x] - [d]\big] = [(y|x)] - [(d|b)] = [0]$, that is, $u \approx v$.

Note that for $k \geq \lceil p \rceil$ and $u = (y|x)$ with $\|u\| \leq 1$ we have

$$|T_k(u)|^{1/k} \leq \Big(\sum_{j=1}^{\infty} |x_j|^k + \sum_{j=1}^{\infty} |y_j|^k \Big)^{1/k} \leq \big(\|x\|^p + \|y\|^p \big)^{1/p} \leq \|u\|.$$

Let us define $f_k(u) := |T_k(u)|^{1/k}$. Since the function $u \mapsto \|u\|$ is nonexpansive and $|T_k(u)|^{1/k} \leq \|u\|$, the sequence f_k is as required in Theorem 3. Thus ρ is a metric.

For the general case of X the amount of symmetric polynomials may be not enough. For example c_0 does not admit any nonconstant symmetric polynomial. However, there are other nonexpansive symmetric functions on X. Consider the following linear order "$\prec$" on the set of complex numbers $\mathbb{C}$. Let $a = |a|(\cos\theta_a + i\sin\theta_a)$ and $b = |b|(\cos\theta_b + i\sin\theta_b)$ be complex numbers. Here we assume that θ_a and θ_b are in the interval $[0, 2\pi)$. If $|a| \neq |b|$ we say that $a \prec b$ if $|a| < |b|$. If $|a| = |b|$, then $a \prec b$ if $\theta_a < \theta_b$. Clearly, that "$\prec$" is a linear order and any finite set of complex numbers has a maximal and a minimal element with respect to this order. For a given subset K of complex numbers we denote by $\mathrm{Max}(K)$ the maximal element in K (if exists) with respect to "$\prec$". Let $[u] = [(y|x)] \in \mathcal{M}_X$ and suppose that $(y|x)$ is the irreducible representation of $[u]$. We define $\mathfrak{m}_1^+([u]) = \mathrm{Max}_n(x_n)$ and $\mathfrak{m}_1^-([u]) = \mathrm{Max}_n(y_n)$. If $\mathfrak{m}_k^+([u])$ and $\mathfrak{m}_k^-([u])$ are defined, then

$$\begin{aligned}\mathfrak{m}_{k+1}^+([u]) &= \mathfrak{m}_1^+\big([u] - \big[0|\mathfrak{m}_1^+([u]), \mathfrak{m}_2^+([u]), \ldots, \mathfrak{m}_k^+([u])\big]\big) \\ &= \mathrm{Max}\{[x] \setminus \big\{\mathfrak{m}_1^+([u]), \mathfrak{m}_2^+([u]), \ldots, \mathfrak{m}_k^+([u])\big\},\end{aligned}$$

and

$$\begin{aligned}\mathfrak{m}_{k+1}^-([u]) &= \mathfrak{m}_1^-\big([u] - \big[\mathfrak{m}_1^-([u]), \mathfrak{m}_2^-([u]), \ldots, \mathfrak{m}_k^-([u])|0\big]\big) \\ &= \mathrm{Max}\{[y] \setminus \big\{\mathfrak{m}_1^-([u]), \mathfrak{m}_2^-([u]), \ldots, \mathfrak{m}_k^-([u])\big\}.\end{aligned}$$

In other words, the two-sides sequence

$$\big(\ldots, \mathfrak{m}_k^-([u]), \ldots, \mathfrak{m}_2^-([u]), \mathfrak{m}_1^-([u]) \big| \mathfrak{m}_1^+([u]), \mathfrak{m}_2^+([u]), \ldots, \mathfrak{m}_k^+([u])\big)$$

is a reordering of the set

$$(\ldots, y_k, \ldots, y_2, y_1 | x_1, x_2, \ldots, x_k, \ldots)$$

such that $\mathfrak{m}_1^+([u]) \succ \mathfrak{m}_2^+([u]) \succ \cdots$ and $\mathfrak{m}_1^-([u]) \succ \mathfrak{m}_2^-([u]) \succ \cdots$. Thus, $[u] = [v]$ if and only if $\mathfrak{m}_k^\pm([u]) = \mathfrak{m}_k^\pm([v])$ for every $k \in \mathbb{N}$. Since X has a symmetric basis and a symmetric norm, the reordering is in $X \times X$ and preserves the norm. Also, it is easy to check that $\|\mathfrak{m}_k\| = 1$. However, the function $u \mapsto \mathfrak{m}_k^\pm([u])$ are not nonexpansive and even not continuous. Thus, we can not apply Theorem 3 and we do not know if ρ is a metric in the general case of $\mathcal{M}_X$. Let $\mathcal{M}_X^\pm$ be the following subset of $\mathcal{M}_X$:

$$\mathcal{M}_X^\pm = \{[u] = [(y|x)] \in \mathcal{M}_X : x_i \geq 0, \quad y_j \leq 0, \quad i, j \in \mathbb{N}\}.$$

Clearly, $\mathcal{M}_X^\pm$ is a semigroup but not a group. In [19] it was proved that the restriction of functions $u \mapsto \mathfrak{m}_k^+[u]$ and $u \mapsto \mathfrak{m}_k^-[u]$ to the metric subspace

$$\{(y, x) \in X \times X : x_i \geq 0, \quad y_j \leq 0, \quad i, j \in \mathbb{N}\}$$

is a Lipschitz function with a Lipschitz constant equals 1. Thus, we have the following corollary.

Corollary 3. *The restriction of the pseudometric ρ to $\mathcal{M}_X^\pm$ is a metric.*

Proof. It is enough to take $f_{2k} = \mathfrak{m}_k^+$ and $f_{2k-1} = \mathfrak{m}_k^-$, and apply Theorem 3.

Proposition 2. Functions $\mathfrak{m}_k^+[u]$ and $\mathfrak{m}_k^-[u]$ are continuous on the metric space $\big(\mathcal{M}_X, d\big)$.

Proof. Let $[u] = [(y|x)]$. Then $\mathfrak{m}_k^+[u] \leq \|x\| \leq \|[u]\|$ and $\mathfrak{m}_k^-[u] \leq \|y\| \leq \|[u]\|$. Hence, if $\|[u]\| \to 0$, then both $\mathfrak{m}_k^+[u]$ and $\mathfrak{m}_k^-[u]$ tend to 0 for every k.

Proposition 3. The quotient map $u \mapsto [u]$ is continuous as a mapping from $X \times X$ to $\big(\mathcal{M}_X, \rho\big)$.

Proof. By definition of ρ, $\rho\big([u], [v]\big) \leq \|u - v\|$. Thus, if $u_n \to v$ in $X \times X$ as $n \to \infty$, then $[u_n] \to [v]$ in $\big(\mathcal{M}_X, \rho\big)$ as $n \to \infty$ and so, the quotient map is continuous.

Proposition 4. The operation of addition $\big([u], [v]\big) \mapsto [u] + [v]$ is continuous in $\big(\mathcal{M}_X, \rho\big)$.

Proof. Let $[z]$, $[w]$ in $\mathcal{M}_X$ and $\mu \in \mathbb{C}$ be such that $\rho\big([u], [z]\big) < \varepsilon/2$, and $\rho\big([v], [w]\big) < \varepsilon/2$. Then, for some $n, m \in \mathbb{N}$ there are $q_1, \ldots, q_{n-1}$ and $s_1, \ldots, s_{m-1}$ in $\mathcal{M}_X$ such that

$$\|u' - q_1\| + \|q_1' - q_2\| + \cdots + \|q_{n-1}' - z'\| < \varepsilon/2,$$

where $u \approx u', z \approx z'$ and $q_j \approx q_j'$, and

$$\|v' - s_1\| + \|s_1' - s_2\| + \cdots + \|s_{m-1}' - w'\| < \varepsilon/2,$$

where $v \approx v', w \approx w'$ and $s_j \approx s_j'$. Thus,

$$\begin{aligned}\rho\big([u]+[v],[z]+[w]\big) &\le \|u'+v'-q_1-s_1\|+\|q_1'+s_1'-q_2-s_2\|+\cdots\\ &+\|q_{n-1}'+s_{n-1'}-z'-w'\| \le \|u'-q_1\|+\|q_1'-q_2\|+\cdots\\ &+\|q_{n-1}'-z'\|+\|v'-s_1\|+\|s_1'-s_2\|+\cdots\\ &+\|s_{m-1}'-w'\|<\varepsilon.\end{aligned}$$

Hence, if $[z]$ approaches $[u]$ and $[w]$ approaches $[v]$, then $[z]+[w]$ approaches $[u]+[v]$. That is, the addition is continuous.

Lemma 1. *For any $[u]$ and $[v]$ in $\mathcal{M}_X$ there are representatives $\widetilde{u} \approx u$ and $\widetilde{v} \approx v$ such that $d\big([u],[v]\big) = \|\widetilde{u}-\widetilde{v}\|$.*

Proof. Let $u = (y|x) = (\ldots, y_2, y_1|x_1, x_2, \ldots)$ and $v = (d|b) = (\ldots, d_2, d_1|b_1, b_2, \ldots)$. without loss of generality we may assume that representatives $(y|x)$ and $(d|b)$ are irreducible. We set

$$\widetilde{u} = (\ldots, y_3, 0, y_2, 0, y_1|x_1, 0, x_2, 0, x_3, \ldots)$$

and $\widetilde{v} = (\ldots, \widetilde{d}_2, \widetilde{d}_1|\widetilde{b}_1, \widetilde{b}_2, \ldots)$, such that $b_k = \widetilde{b}_{\sigma(k)}$, and $d_j = \widetilde{d}_{\sigma(j)}$, where μ and σ are injections from $\mathbb{N}$ to itself, defined by the following way. If $b_1 = x_k$ then $\sigma(1) = k$, if $b_1 \neq x_k$ for any $k \in \mathbb{N}$, then $\sigma(1) = 2$; if $d_1 = y_j$ then $\mu(1) = j$, if $d_1 \neq x_j$ for any $j \in \mathbb{N}$, then $\sigma(1) = 2$, Suppose that $\sigma(n)$ and $\mu(n)$ are defined. If $b_{n+1} = x_k$ for some $k \in \mathbb{N} \setminus \{\sigma(1), \cdots, \sigma(n)\}$, then $\sigma(n+1) = 2k-1$, otherwise, $\sigma(n+1) = 2n$; if $d_{n+1} = y_j$ for some $j \in \mathbb{N} \setminus \{\mu(1), \cdots, \mu(n)\}$, then $\mu(n+1) = 2j-1$, otherwise, $\mu(n+1) = 2n$. By this way we have constructed the injective (but not necessary surjective) maps σ and μ. Also, we define $\widetilde{b}_k = 0$ and $\widetilde{b}_j = 0$ for all k and j that are not in the range of σ and μ respectively. For example, let $u = (\ldots, 0, -1, 3, 2|1, 1, 4, -5, 0, \ldots)$ and $v = (\ldots, 0, 2, 3|1, 6, 0, \ldots)$, then $\widetilde{u} = (\ldots, 0, -1, 0, 3, 0, 2|1, 0, 1, 0, 4, 0, -5, \ldots)$ and $\widetilde{v} = (\ldots, 0, 3, 0, 2|1, 6, 0, \ldots)$.

We can see that for such representatives $\widetilde{u}$ and $\widetilde{v}$, $d\big([u],[v]\big) = \|\widetilde{u}-\widetilde{v}\|$.

Corollary 4. *The topology generated by ρ on $\mathcal{M}_X$ is weaker than the topology generated by d.*

Proof. By the definition of ρ and Lemma 1,

$$\rho\big([u],[v]\big) \le \|\widetilde{u}-\widetilde{v}\| = d\big([u],[v]\big).$$

Hence, ρ is continuous in $\big(\mathcal{M}_X, d\big)$ and so the topology generated by ρ is weaker than the topology generated by d.

It is easy to check that for the case if ρ is a metric, $\big(\mathcal{M}_X, \rho\big)$ is a separable metric space. We do not know if the quotient map is open in the norm-to-ρ topology, but from Theorem 3 it follows that for every Lipschitz supersymmetric function f on $X \times X$ the function $[(y|x)] \mapsto f(y,x)$ is continuous on $\big(\mathcal{M}_X, \rho\big)$. Also, we do not know if $\big(\mathcal{M}_X, \rho\big)$ is complete.

3. Topological Transitivity of a Nonlinear Backward Shift

Let us denote by ℓ_1^+ the subset of $x \in \ell_1$ such that all coordinates x_k of x are real nonnegative numbers. Also, we denote by $\mathcal{R}_+^X$ the following subset in $\mathcal{M}_X^+$:

$$\mathcal{R}_+^X = \{[x] \in \mathcal{M}_X^+ \colon x \in \ell_1^+\}.$$

Clearly, $\big(\mathcal{R}_+^X, d\big)$ is a metric semigroup.

For every $x = (x_1, \ldots, x_n, \ldots) \in \ell_1^+$ we denote by $M(x) = [(\max_i x_i, 0, 0, \ldots)] = [(\mathfrak{m}_1([x]), 0, 0, \ldots)]$. For a given $\lambda > 1$ we define the following map $\mathfrak{T}_\lambda \colon \mathcal{R}_+^X \to \mathcal{R}_+^X$,

$$\mathfrak{T}_\lambda([x]) = \lambda\big([x] - M(x)\big).$$

In other words, $\mathfrak{T}_\lambda$ cancels the maximal coordinate of x and multiply the result by λ. It is easy to check that $\mathfrak{T}_\lambda$ is continuous in $\big(\mathcal{R}_+^X, d\big)$. Note that $\mathfrak{T}_\lambda$ is not additive.

Example 3.

$$\mathfrak{T}_\lambda\big([(1,2,2)] + [(1,2,3)]\big) = \lambda[(1,1,2,2,2)]$$

$$\neq \lambda[(1,1,2,2)] = \mathfrak{T}_\lambda\big([(1,2,2)]\big) + \mathfrak{T}_\lambda\big([(1,2,3)]\big).$$

Theorem 4. *Operator $\mathfrak{T}_\lambda$ is topologically transitive on $\big(\mathcal{R}_+^X, d\big)$ for every $\lambda > 1$.*

Proof. Let $\mathcal{R}_0$ be the subset of $\mathcal{R}_+$ consisting of all elements $[x]$ such that only a finite number of coordinates of x is not equal to zero. Note that $\mathcal{R}_0$ is a dense subset in $\big(\mathcal{R}_+^X, d\big)$. Let us define a mapping $S \colon \mathcal{R}_+^X \to \mathcal{R}_+^X$ by

$$S([y]) = \frac{1}{\lambda}\big(M(y) + [y]\big).$$

Note that

$$\|S^n([y])\| \leq \frac{(n+1)\|y\|}{\lambda^n} \to 0 \quad \text{as} \quad n \to \infty$$

for every $[y] \in \mathcal{R}_+^X$.

Let U and V be open subsets in $\big(\mathcal{R}_+^X, d\big)$, $[x] \in U \cap \mathcal{R}_0$ and $[y] \in V$. Suppose that $x = (x_1, \ldots, x_m, 0, 0, \ldots)$, $x_j > 0$, $j = 1, \ldots, m$. Let us choose an integer k such that

1. $S^k([y]) + [x] \in U$;
2. $\mathfrak{m}_1\big(S^k([y])\big) < x_j$, $j = 1, \ldots, m$.

Let
$$[z] = \frac{S^k([y])}{\lambda^m} + [x].$$
Since $S^k([y])+[x] \in U$, it follows that $[z] \in U$ as well. Let us compute $\mathfrak{T}_\lambda^{k+m}([z])$.
$$\mathfrak{T}_\lambda^{k+m}([z]) = \mathfrak{T}_\lambda^{k+m}\Big[\Big(x_1, \dots, x_m, \underbrace{\frac{\mathfrak{m}_1([y])}{\lambda^{m+k}}, \dots, \frac{\mathfrak{m}_1([y])}{\lambda^{m+k}}}_{k}, \frac{y_1}{\lambda^{m+k}}, \frac{y_2}{\lambda^{m+k}}, \dots\Big)\Big]$$
$$= \mathfrak{T}_\lambda^{k}\Big[\Big(\underbrace{\frac{\mathfrak{m}_1([y])}{\lambda^{k}}, \dots, \frac{\mathfrak{m}_1([y])}{\lambda^{k}}}_{k}, \frac{y_1}{\lambda^{k}}, \frac{y_2}{\lambda^{k}}, \dots\Big)\Big] = [y].$$
Thus, we found $[z] \in U$ such that $\mathfrak{T}_\lambda^{k+m}([z]) = [y] \in V$. So $\mathfrak{T}_\lambda$ is topologically transitive.

Note that the proof of Theorem 4 is still true if we replace the topology of $\big(\mathcal{R}_+^X, d\big)$ by any metrizable semigroup topology such that $\mathfrak{T}_\lambda$ is continuous, $\mathcal{R}_0$ is dense in $\mathcal{R}_+^X$ and $S^n([y]) \to 0$ as $n \to \infty$.

Corollary 5. *$\mathfrak{T}_\lambda$ is topologically transitive (and so hypercyclic) on the metric space $\big(\mathcal{R}_+^X, \rho\big)$.*

The following theorem can be considered as an analog of the Topological Transitivity Criterion (c.f. [5, pp. 4,5]) for metric semigroups.

Theorem 5. *Let $\mathcal{Q}$ be a metric semigroup, and T be a mapping from $\mathcal{Q}$ to itself. Suppose that there is a dense subsets $\Omega \subset \mathcal{Q}$ and $\Xi \in \mathcal{Q}$, and for every $u \in \Omega$ there is a number $m \in \mathbb{N}$ and a sequence of maps $S_{u,k} \colon \Xi \to \mathcal{Q}$, $k \in \mathbb{N}$ such that*

(i) $S_{u,k}(v) \to 0$ for every $v \in \Xi$ as $k \to \infty$;

(ii) for each $u \in \Omega$, $T^{k+m}[S_{u,k}(v) + u] \to v$ for every $u \in \Omega$ as $k \to \infty$.

Then T is topologically transitive.

Proof. Let U and V be open and nonempty subsets of $\mathcal{Q}$. Pick $u \in U \cap \Omega$, $v \in V \cap \Xi$. Then, there is k' such that $S_{u,k}(v) + u \in U$ for $k > k'$. According to *(ii)*, there is k'' such that $T^{k+m}[S_{u,k}(v) + u] \in V$ for $k > k''$. Thus, for $k > \max(k', k'')$ we have that T^{k+m} maps the point $S_{u,k}(v) + u \in U$ to V. So, T is topologically transitive.

Let us extend the operator $\mathfrak{T}_\lambda$ to $\mathcal{M}_X^\pm$ by the following way: first of all, we define
$$M([u]) = \begin{cases} [(0|\max_i(x_i))] & \text{if } \max_i(x_i) \geq -\min_j(y_j); \\ [(\min_j(y_j)|0)] & \text{if } \max_i(x_i) < -\min_j(y_j), \end{cases}$$
where $(y|x)$ is an irreducible representation of $[u]$. Then
$$\mathfrak{T}_\lambda([u]) = \lambda\big([u] - M([u])\big).$$

Theorem 6. $\mathfrak{T}_\lambda$ *is topologically transitive in* $\big(\mathcal{M}_X^\pm, d\big)$ *and hypercyclic in* $\big(\mathcal{M}_X^\pm, \rho\big)$ *whenever* $|\lambda| > 1$.

Proof. Let

$$S_{[u],k}([v]) = \frac{\overbrace{M([v]) + \cdots + M([v])}^{k} + [v]}{\lambda^{m([u])}},$$

where $m(u) = |\mathrm{supp}\,(x)| + |\mathrm{supp}\,(y)|$ is the sum of cardinalities of $\mathrm{supp}\,(x)$ and $\mathrm{supp}\,(y)$. Also, the dense subset $\Omega = \Xi = \mathcal{M}_0^\pm$ is the subset of elements in $\mathcal{M}_X^\pm$ such that their irreducible representatives have finite support. Let us check we can apply Theorem 5. Indeed, since $|\lambda| > 1$, $\|S_{u,k}([v])\| \le k\|v\|/\lambda^k \to 0$ as $k \to 0$. Let $[u] = [(y|x)]$ and $[v] = [(d|b)]$ be in $\mathcal{M}_0^\pm$, and we suppose that $\max_i(b_i) \ge -\min_j(d_j)$. If k is big enough, $m_1\big(S_{[u],k}([v])\big) < \min_{i,j}\{|x_i|, |y_j|\}$, $i \in \mathrm{supp}\,(x)$, $j \in \mathrm{supp}\,(y)$. Then

$$\begin{aligned}
\mathfrak{T}_\lambda^{k+m}\big(S_{[u],k}([v]) + [u]\big) &= \mathfrak{T}_\lambda^{k+m}\Big[\Big(y \bullet d \Big| x \bullet \underbrace{\frac{\mathfrak{m}_1([b])}{\lambda^{m+k}}, \ldots, \frac{\mathfrak{m}_1([b])}{\lambda^{m+k}}}_{k}, \frac{b_1}{\lambda^{m+k}}, \frac{b_2}{\lambda^{m+k}}, \ldots\Big)\Big] \\
&= \mathfrak{T}_\lambda^{k} \circ \mathfrak{T}_\lambda^{m}\Big[\Big(y \bullet d \Big| x \bullet \underbrace{\frac{\mathfrak{m}_1([b])}{\lambda^{m+k}}, \ldots, \frac{\mathfrak{m}_1([b])}{\lambda^{m+k}}}_{k}, \frac{b_1}{\lambda^{m+k}}, \frac{b_2}{\lambda^{m+k}}, \ldots\Big)\Big] \\
&= \mathfrak{T}_\lambda^{k}\Big[\Big(\ldots, \frac{d_2}{\lambda^k}, \frac{d_1}{\lambda^k} \Big| \underbrace{\frac{\mathfrak{m}_1([b])}{\lambda^{k}}, \ldots, \frac{\mathfrak{m}_1([b])}{\lambda^{k}}}_{k}, \frac{b_1}{\lambda^{k}}, \frac{b_2}{\lambda^{k}}, \ldots\Big)\Big] = [v],
\end{aligned}$$

where $m = m(u)$. By Theorem 5, $\mathfrak{T}_\lambda$ is topologically transitive in $\big(\mathcal{M}_X^\pm, d\big)$ and it is hypercyclic in $\big(\mathcal{M}_X^\pm, \rho\big)$ because $\big(\mathcal{M}_X^\pm, \rho\big)$ is separable.

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Received: 09.11.2024. *Accepted:* 13.12.2024