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On Anisotropic BV-Spaces

Abstract. This paper is devoted to the study of a new class of functional spaces, the so-called anisotropic BV-spaces with a degenerate weight. We give a precise definition of such spaces and show that they can be viewed as a natural generalization of the standard space of functions with bounded variation.

Key words: Bounded variation, anisotropic space, BV space, generalized weighted gradient

Анотація. Стаття присвячена аналізу одного класу функціональних просторів, а саме анізотропних просторів функцій з обмеженою варіацією та виродженою вагою. Наводиться означення такого класу та вивчаються його основні властивості. Показано, що такі простори є природнім узагальненням класичних BV-просторів.

Ключові слова: обмежена варіація, анізотропний простір, BV-простір, узагальнений ваговий градієнт

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1. Introduction

An $L^1(\Omega)$-function u belongs to $BV(\Omega)$ if and only if its first distributional derivatives are bounded measures. The main fundamental results for BV spaces can be found in [4,5,7,10,22]. The space of BV functions, as the space of functions with bounded variations, appears naturally when one considers variational problems with linear growth. Variational problems with linear growth are currently the subject of intense mathematical research and commonly arise in image processing and denoising, in materials science, and in the other fields of applied mathematics. Some important examples of such minimization problems include the following:

- The minimal surface equation, which corresponds to the minimisation of the area functional [24] $\int_\Omega \sqrt{1+|Du|^2}$, where for the smooth functions $Du = \operatorname{grad} u$, and Du stands for the generalized gradient of u provided $u \in L^1(\Omega)$;
- The minimization of the total variation [5] $\int_\Omega |Du|$;

- The image denoising problem basing on the minimization of the Rudin-Osher-Fatemi functional [34]

$$\int_\Omega |Du| + \frac{\lambda}{2}\int_\Omega (u-f)^2\,dx;$$

- The Mumford-Shah variational problem appearing in the image segmentation [28]

$$\int_{\Omega\setminus K} |\nabla u|^2\,dx + \alpha\int_\Omega (u-f)^2\,dx + \mathcal{H}^{N-1}(K\cap\Omega).$$

A common feature of the above-mentioned variational problems is the fact that the corresponding objective functionals are not lower semi-continuous in the Sobolev space $W^{1,1}(\Omega)$. The main reason is that a bounded sequence in $W^{1,1}(\Omega)$ does not necessarily have a limit in $W^{1,1}(\Omega)$ even in the weak-* topology. Because of this, a larger function space such as $BV(\Omega)$ should be involved.

In this paper, we focus on the study of one class of anisotropic spaces of functions with bounded variation. In principle, anisotropic BV-spaces are not new objects in the literature, and their study is a rich area in functional analysis and the calculus of variations. These spaces can be introduced in a different manner and usually generalize the classical BV spaces, allowing one to control different regularity and variation in different directions. In particular, [36] is arguably one of the early works in which anisotropic BV spaces appear in mechanics and plasticity. In [20] the authors introduce an anisotropic version of perimeters and provide some development and generalization of the classical BV-theory. An interesting application of anisotropic BV-spaces, where they naturally appear as the spaces of solutions for special anisotropic variational problems, is image processing (see, for instance, [3,11,12,23,39]). We also refer to recent publications in this field [32], where the authors study the link of BV spaces to some variational problems and discuss the different classes of diffusion models in the context of BV spaces (see also [25,26,29,31,38]).

As was mentioned above, perhaps one of the most important applications of functions of bounded variation is the field of image processing. Because a function $u\in BV(\Omega)$ with $\Omega\subset\mathbb{R}^N$ may contain discontinuities of dimension $N-1$, it is natural to suppose that the space $BV(\Omega)$ is well suited for modeling real images, which basically consist of rather uniform regions well separated by significant edges. One of the most widely used techniques in image restoration is the total variation regularization, introduced in [34] for the problem of image denoising and after being generalized in [1] to the stable solution of inverse problems. Given a noisy image $f\in L^2(\Omega)$, they consider a minimizer u_α of the so-called ROF-problem

$$I(u) = \frac{1}{2}\|u-f\|^2_{L^2(\Omega)} + \alpha|Du|(\Omega) \tag{1}$$

with some regularisation parameter $\alpha > 0$ as an image which contains the most important features of the original data f while noise, consisting of small oscillations in u, is removed. Moreover, it has been shown that the jump set of u_α is contained in the jump set of f, that is, the restoration of images through the minimization of functional (1) does not create new discontinuities. However, this model fails in the presence of a large number of edges in f. Namely, utilizing the solution of (1), we cannot separate pure noise, i.e., well oscillatory components, from texture, but instead we remove both equally. Also, the ROF model suffers from the so-called blocky effects and it can also develop 'false edges' which can mislead a human or computer into identifying erroneous features not present in the true image. These phenomena were a subject of long and intensive discussions in the literature (see, for example, [6, 13–18], [30, Section 1.3]).

In order to avoid these artifacts, many different variants have recently been proposed for a modification of the standard functional (1), where the isotropic total variation term $|Du|(\Omega)$ would be replaced by an anisotropic space-dependent term (see, for instance, [2,15,17–19,30]). In particular, in [30] the author proposes to define an anisotropic variation of a function $u \in L^1(\Omega)$ as follows

$$\int_\Omega A(Du) := \begin{cases} \displaystyle\int_\Omega A\left(x, \frac{dDu}{d|Du|}(x)\right) d|Du|, & \text{if } \ u \in BV(\Omega), \\ +\infty, & \text{if } \ u \notin BV(\Omega), \end{cases}$$

where the mapping $A : \Omega \times \mathbb{R}^N \to \mathbb{R}_{>0}$ is continuous and satisfy the condition

$$C^{-1}|\xi| \le A(x,\xi) \le C|\xi|, \quad \forall\, (x,\xi) \in \Omega \times \mathbb{R}^N \tag{2}$$

with some constant $C > 1$. In this case, it has been shown that

$$\int\limits_\Omega A(Du) = \sup\left\{ -\int\limits_\Omega u \operatorname{div}\psi\, dx \ : \ \psi \in C_0^1(\Omega;\mathbb{R}^N), A_0(x,\psi(x)) \le 1 \ \forall\, x \in \Omega \right\},$$

$$A_0(x,\xi) := \max\left\{ (\theta,\xi)_{\mathbb{R}^N} \ : \ \theta \in \mathbb{R}^N, A(x,\theta) \le 1 \right\}$$

and the space

$$BV(\Omega;A) := \left\{ u \in L^1(\Omega) \ : \ \int_\Omega A(Du) < +\infty \right\}$$

endowed with the norm $\|u\|_{BV(\Omega;A)} = \|u\|_{L^1(\Omega)} + \int_\Omega A(Du)$ becomes a Banach space such that $BV(\Omega;A) \subset BV(\Omega)$ and the set $C^\infty(\Omega) \cap BV(\Omega)$ is dense in $BV(\Omega;A)$ with respect to the intermediate convergence.

Another variant of anisotropic variation of L^1-functions, has been recently proposed in [19], where the authors define the functions with a bounded anisotropic variation by the rule

$$\sup\left\{ \int_\Omega u \operatorname{div}(A\varphi)\, dx \ : \ \varphi \in C_0^1(\Omega;\mathbb{R}^2), \ |\varphi(x)| \le 1 \ \forall\, x \in \Omega \right\} < +\infty.$$

provided the matrix $A : \Omega \to \mathbb{R}^{2\times 2}$ is such that

$$\begin{aligned} &A(x) = A^t(x), \quad A(\cdot) \in C^\infty(\Omega; \mathbb{R}^{2\times 2}), \\ &\beta^{-1}|\xi|^2 \le (\xi, A(x)\xi) \le \beta|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^2 \end{aligned} \tag{3}$$

for some $\beta > 0$, where A^t stands for the transpose of the matrix A.

A common feature of the two mentioned variants of anisotropy (see (2) and (3)) is the fact that the corresponding anisotropic BV-spaces are subspaces of the standard space $BV(\Omega)$. Moreover, in both cases the anisotropy tensors (see (2)-(3)) are subjected to the coercivity condition which is rather restrictive from practical application point of view.

In contrast to this, the approach that we develop in this article, is to construct an anisotropic space $BV_A(\Omega)$ that is wider than the classical space $BV(\Omega)$, and allow the matrix of anisotropy $A(x)$ to be degenerate in Ω. With that in mind, we introduce a special class of anisotropy tensors (see Section 4) and define the corresponding anisotropic space as the closure of the set $\left\{\psi \in C_0^\infty(\Omega)\ :\ \left|D^A\psi\right|(\Omega) < \infty\right\}$ with respect to the following convergence

$$u_k \to u \quad \text{strongly in} \quad L^1(\Omega) \quad \text{and} \quad \int_\Omega \left|d[D^A u_k]\right| \to \int_\Omega \left|d[D^A u]\right|,$$

and endow it with the norm

$$\|u\|_{BV_A(\Omega)} := \|u\|_{L^1(\Omega)} + \left|D^A u\right|(\Omega),$$

where $\left|D^A u\right|(\Omega)$ stands for

$$\sup\left\{ \int_\Omega u\left(\left[A, \nabla\varphi\right]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A, \varphi)_{\mathbb{R}^N}\right)\, dx \;\middle|\; \begin{array}{c} \varphi \in C_0^\infty(\Omega; \mathbb{R}^N), \\ |\varphi(x)| \le 1\ \forall\ x \in \Omega \end{array} \right\}.$$

Here, the form $\left[A, \nabla\varphi\right]_{\mathbb{R}^{N\times N}}$ will be defined in (11).

We show that in this case the standard BV-space can be viewed as a proper subspace of $BV_A(\Omega)$ and study its main topological properties. In particular, we establish some results concerning a continuous embedding of $BV_A(\Omega)$ into a weighted Lebesgue spaces and establish the format of the anisotropic coarea formula. The paper is organized as follows. In Section 2 we give some preliminaries related to functional spaces and other notions. Section 3 is devoted to the description of the class of feasible tensors of anisotropy. We provide a comparison of the introduced class with the class of A_p weights that was introduced by B. Muckenhoupt and show that the set of feasible tensors of anisotropy may contain tensors which cannot be associated with the Muckenhoupt weights. In Section 4 we give the definition of the A-weighted gradients and specify the class of L^1-functions for which their A-weighted gradients have a finite total variation in Ω. All of these allows us to introduce the space of functions with a bounded anisotropic total variation. We show that the introduced class is a Banach space and study its basic properties in Section 5. The last section is devoted to some natural generalizations of the results of Samson et al. [35].

2. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz boundary $\partial\Omega$. For any subset $E \subset \Omega$ we denote by $|E|$ its N-dimensional Lebesgue measure $\mathcal{L}^N(E)$, and by $\overline{E}$ its closure. For a function u, we denote by $u|_E$ its restriction to the set $E \subseteq \Omega$. Let $C_0^\infty(\Omega)$ be the set of infinitely differentiable functions with compact supports in Ω. To specify the space of compactly supported functions with k-times continuous derivatives in Ω, we use the notation $C_0^k(\Omega)$, whereas $C_0(\Omega)$ stands for the space of continuous functions with compact supports in Ω.

For a Banach space X its dual is X^* and $\langle \cdot, \cdot \rangle_{X^*,X}$ is the duality form on $X^* \times X$. By $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ we denote the weak and weak-* convergence in normed spaces, respectively.

We recall here the most common definitions of some functional spaces that we will use later on. In particular, we will often use the Lebesgue spaces of vector-valued functions. For example, for the L^p-space of vector-valued functions $\boldsymbol{u}(x) = [u_1(x), \dots, u_N(x)]^t$ with $1 \leq p \leq \infty$, we use the notation $L^p(\Omega; \mathbb{R}^N)$, while the notation $L^p(\Omega; \mathbb{R}^{N\times N})$ stands for the space of p-summable functions whose values are square matrices $A(x) = [a_{i,j}(x)]_{i,j=1}^N$. So, $A(\cdot) \in L^p(\Omega; \mathbb{R}^{N\times N})$ if and only if $\|A\|_{L^p(\Omega;\mathbb{R}^{N\times N})} < \infty$, where

$$\|A\|_{L^p(\Omega;\mathbb{R}^{N\times N})} = \left\{ \begin{array}{ll} \left(\displaystyle\int_\Omega |A(x)|^p_{\mathbb{R}^{N\times N}}\, dx \right)^{1/p}, & \text{if } \ 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} |A(x)|_{\mathbb{R}^{N\times N}}, & \text{if } \ p = \infty. \end{array} \right\}$$

Here, $|A(x)|_{\mathbb{R}^{N\times N}}$ is a norm of matrix $A(x) \in \mathbb{R}^{N\times N}$ that can be defined in different ways. For instance, $|A(x)|_{\mathbb{R}^{N\times N}} := \|A(x)\|_2$ is the Euclidean (or the spectral) norm, i.e.,

$$\|A(x)\|_2 = \sup\left\{|A(x)\xi|_{\mathbb{R}^N} \ : \ |\xi|_{\mathbb{R}^N} \leq 1\right\} = \sqrt{\lambda_{max}\left(A^t(x)A(x)\right)}. \tag{4}$$

However, we can use, instead of (4), the so-called Schatten 2-norm $\|A(x)\|_*$, which we define as follows

$$\|A(x)\|_* = \max_{i\in\{1,\dots,N\}} \sqrt{\sum_{j=1}^N a_{i,j}^2(x)}, \quad \forall\, x \in \Omega. \tag{5}$$

In fact, $\|A(x)\|_*$ is the mix of the max norm with the Frobenius norm. Moreover, it is easy to show that in this case the following two-sided estimates

$$\frac{1}{\sqrt{N}}\|A(x)\|_* \leq \|A(x)\|_2 \leq \sqrt{N}\|A(x)\|_*, \quad \forall\, x \in \Omega \tag{6}$$

holds true.

In order to specify what kind of matrix norm we use in definition of the space $L^p(\Omega;\mathbb{R}^{N\times N})$, we make use of the notaions $L^p_2(\Omega;\mathbb{R}^{N\times N})$ or $L^p_*(\Omega;\mathbb{R}^{N\times N})$ for the norms (4) or (5), respectively.

Let $p\in\mathbb{R}$ with $1\le p\le\infty$. By $W^{1,p}(\Omega)$ we denote the standard Sobolev space, i.e.,

$$W^{1,p}(\Omega):=\left\{u\in L^p(\Omega)\;\middle|\;\begin{array}{c}\exists g_1,g_2,\dots,g_N\in L^p(\Omega)\text{ such that}\\ \displaystyle\int_\Omega u\frac{\partial\varphi}{\partial x_i}\,dx=-\int_\Omega g_i\varphi\,dx,\ \forall\, i=1,\dots,N,\\ \forall\,\varphi\in C_0^\infty(\Omega).\end{array}\right\}$$

So, $u\in W^{1,p}(\Omega)$ iff $u\in L^p(\Omega)$ and its gradient in the sense of distributions

$$\nabla u(x)=\begin{bmatrix}\frac{\partial u(x)}{\partial x_1}\\ \dots\\ \frac{\partial u(x)}{\partial x_N}\end{bmatrix}$$

is a function of $L^p(\Omega;\mathbb{R}^N)$.

The space $W^{1,p}(\Omega)$ endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)}=\left(\|u\|^p_{L^p(\Omega)}+\|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^N)}\right)^{1/p}$$

is a Banach space. We also define the Sobolev space $W_0^{1,p}(\Omega)$ with $1\le p\le\infty$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm of $W^{1,p}(\Omega)$. It is clear that $W_0^{1,p}(\Omega)\subset W^{1,p}(\Omega)$ and any function $u\in W_0^{1,p}(\Omega)$ vanish on the boundary $\partial\Omega$ in the sense of the trace.

Among the properties of Sobolev functions (see [8]), let us list these which are the most relevant to the present topic:

(a) Smooth functions with finite Sobolev norm form a dense subset of $W^{1,p}(\Omega)$;

(b) We have the continuous embedding $W^{1,p}(\Omega)\hookrightarrow L^{Np/(N-p)}(\Omega)$ for $1\le p<N$;

(c) For bounded Ω and $p\in[1,N)$, $W^{1,p}(\Omega)\hookrightarrow L^q(\Omega)$ for all $q\in\left[1,\frac{Np}{N-p}\right)$ and this embedding is compact;

(d) We have the Poincaré inequality

$$\|u-u_\Omega\|_{L^{Np/(N-p)}(\Omega)}\le C\left(\int_\Omega|\nabla u|^p\,dx\right)^{1/p}\tag{7}$$

for some constant C depending only on the width of Ω. Here

$$u_\Omega=\frac{1}{|\Omega|}\int_\Omega u(x)\,dx$$

denotes the mean value of u in Ω. We can omit u_Ω in (7) setting $u_\Omega=0$, if $u\in W_0^{1,p}(\Omega)$;

(e) For bounded Ω, there is a bounded and linear trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ with the following property:

$$\int_\Omega u \operatorname{div} \varphi \, dx + \int_\Omega (\varphi, \nabla u)_{\mathbb{R}^N} \, dx = \int_{\partial\Omega} \left(\varphi, \nu^\Omega\right)_{\mathbb{R}^N} Tu \, d\mathcal{H}^{N-1} \tag{8}$$

for all $u \in W^{1,p}(\Omega)$ and $\varphi \in C^1(\mathbb{R}^N; \mathbb{R}^N)$. Here, ν^Ω stands for the unit outward normal to a Lipschitz boundary $\partial\Omega$.

These properties will be utilized in the next sections.

Before proceeding further, we make use of the following observation concerning the anisotropic version of the weak derivative for the Sobolev functions $y \in W^{1,1}(\Omega)$.

Let

$$A(x) = \begin{bmatrix} a_1^t(x) \\ a_2^t(x) \\ \cdots \\ a_N^t(x) \end{bmatrix} \in \mathbb{R}^{N\times N} \quad \text{with} \quad a_i(x) = \begin{bmatrix} a_{i,1}(x) \\ a_{i,2}(x) \\ \cdots \\ a_{i,N}(x) \end{bmatrix}, \quad i = 1, \dots, N, \tag{9}$$

be a given matrix such that $A \in L^\infty(\Omega; \mathbb{R}^{N\times N})$ and $\operatorname{div} A \in L^\infty(\Omega; \mathbb{R}^N)$, where

$$\operatorname{div} A(x) = \begin{bmatrix} \operatorname{div} a_1(x) \\ \operatorname{div} a_2(x) \\ \cdots \\ \operatorname{div} a_N(x) \end{bmatrix} \in \mathbb{R}^N \text{ and } \operatorname{div} a_i(x) = \textstyle\sum_{j=1}^N \frac{\partial a_{i,j}(x)}{\partial x_j},$$

$$i = 1, \dots, N.$$

Let $\varphi \in C_0^1(\Omega; \mathbb{R}^N)$ be a vector-valued test function. Then the following equality

$$\int_\Omega \varphi_i \cdot (\nabla y, a_i)_{\mathbb{R}^N} \, dx = -\int_\Omega \Big[(\nabla \varphi_i, a_i)_{\mathbb{R}^N} + \varphi_i \operatorname{div} a_i \Big] y \, dx$$

holds true for each $i = 1, \dots, N$ provided $y \in W^{1,1}(\Omega)$.

Hence,

$$\int_\Omega \sum_{i=1}^N \varphi_i \cdot (\nabla y, a_i)_{\mathbb{R}^N} \, dx = -\int_\Omega \Big(\sum_{i=1}^N (\nabla \varphi_i, a_i)_{\mathbb{R}^N} + \sum_{i=1}^N \varphi_i \operatorname{div} a_i \Big) y \, dx,$$

where the last equality can be rewritten as follows

$$\int_\Omega (\varphi, A\nabla y)_{\mathbb{R}^N} \, dx = -\int_\Omega \Big(\big[A, \nabla \varphi\big]_{\mathbb{R}^{N\times N}} + (\varphi, \operatorname{div} A)_{\mathbb{R}^N} \Big) y \, dx. \tag{10}$$

Here, by analogy with (9),

$$\nabla \varphi = \begin{bmatrix} \nabla \varphi_1^t \\ \nabla \varphi_2^t \\ \cdot \\ \nabla \varphi_N^t \end{bmatrix} = \begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots & \frac{\partial \varphi_1}{\partial x_N} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \cdots & \frac{\partial \varphi_2}{\partial x_N} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial \varphi_N}{\partial x_1} & \frac{\partial \varphi_N}{\partial x_2} & \cdots & \frac{\partial \varphi_N}{\partial x_N} \end{bmatrix} \in \mathbb{R}^{N\times N}$$

and the bilinear form $\big[\cdot,\cdot\big] : \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times N} \to \mathbb{R}$ is defined as follows

$$\big[A, \nabla\varphi\big]_{\mathbb{R}^{N\times N}} = \sum_{i=1}^{N} \left(\nabla\varphi_i, a_i\right)_{\mathbb{R}^N}. \tag{11}$$

Taking into account that, for an arbitrary matrix $A = \begin{bmatrix} a_1^t \\ \cdot \\ a_N^t \end{bmatrix} \in L^\infty(\Omega;\mathbb{R}^{N\times N})$ and $y \in W^{1,p}(\Omega)$ (with $p \geq 1$), we have the estimate

$$\begin{aligned}
\int_\Omega |A(x)\nabla y(x)|^p\,dx &= \int_\Omega \left(\sum_{i=1}^{N} |\left(a_i(x), \nabla y(x)\right)_{\mathbb{R}^N}|^2\right)^{p/2} dx \\
&\leq \int_\Omega |\nabla y(x)|^p \left(\sum_{i=1}^{N} |a_i(x)|^2\right)^{p/2} dx \\
&\leq N^{p/2} \int_\Omega |\nabla y(x)|^p \left(\max_{1\leq i\leq N} |a_i(x)|^2_{\mathbb{R}^N}\right)^{p/2} dx \\
&= \int_\Omega N^{p/2} |\nabla y(x)|^p |A(x)|^p_* \,dx \\
&\leq N^{p/2} \|A\|^p_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})} \|y\|^p_{W^{1,p}(\Omega)}.
\end{aligned}$$

Hence, $A\nabla y \in L^p(\Omega;\mathbb{R}^N)$, and, therefore, equality (10), which holds true for all test functions $\varphi \in C_0^\infty(\Omega;\mathbb{R}^N)$, can be interpreted as a definition of the weak weighted gradient of y. However, its representation in the form $A\nabla y$ (that is in the sense of the pointwise application of A to ∇y) is valid if only $y \in W^{1,p}(\Omega)$ and $A \in L^\infty(\Omega;\mathbb{R}^{N\times N})$.

3. On Feasible Tensors of Anisotropy

We associate with a bounded open domain $\Omega \in \mathbb{R}^N$ the following space of the matrix-valued functions

$$L^\infty(\Omega, \mathrm{div};\mathbb{R}^{N\times N}) = \left\{A \in L^\infty_*(\Omega;\mathbb{R}^{N\times N}) \ :\ \mathrm{div}\,A \in L^\infty(\Omega;\mathbb{R}^N)\right\}.$$

Hereinafter we assume that $\partial\Omega$ is of class C^1. Setting $Y = L^\infty(\Omega, \mathrm{div};\mathbb{R}^{N\times N})$ and endowing it with the norm

$$\|A\|_Y = \max\left\{N\|A\|_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})}, \|\,\mathrm{div}\,A\|_{L^\infty(\Omega;\mathbb{R}^N)}\right\}, \tag{12}$$

let us show that the following assertion is valid.

Lemma 1. *$\langle Y, \|\cdot\|_Y\rangle$ is a Banach space.*

Proof. To begin with, let us show that the following duality pairing

$$\begin{aligned}\langle A, V\rangle_{Y;X} := &\int_\Omega \left[A, V\right]_{\mathbb{R}^{N\times N}}\,dx\\ = &\int_\Omega \Big(\left[A, V^a\right]_{\mathbb{R}^{N\times N}} - \left[A, \nabla v^b\right]_{\mathbb{R}^{N\times N}}\Big)\,dx\end{aligned}$$

is well defined for every $A \in Y$ and $V \in X$ with the decomposition $V = V^a - \nabla v^b$, where X stands for the renormalized space $L^1(\Omega;\mathbb{R}^{N\times N})$, and the new norm $\|\cdot\|_X$ in $L^1(\Omega;\mathbb{R}^{N\times N})$ is given by the rule (see [9]):

$$\|V\|_X = \inf_{(V^a,v^b)\in\Xi(V)}\left\{\|V^a\|_{L^1_*(\Omega;\mathbb{R}^{N\times N})} + \|v^b\|_{L^1(\Omega;\mathbb{R}^N)}\right\} \tag{13}$$

with

$$\Xi(V) = \left\{\Big(V^a, v^b\Big) \in L^1_*(\Omega;\mathbb{R}^{N\times N}) \times W_0^{1,1}(\Omega;\mathbb{R}^N)\ :\ V = V^a - \nabla v^b\right\}.$$

So, for each $V \in X$, we have

$$V = V^a - \nabla v^b = \left[v_1^a\ v_2^a\ \cdots\ v_N^a\right] - \left[\nabla v_1^b\ \nabla v_2^b\ \cdots\ \nabla v_N^b\right]$$

with

$$v_i^a(x) = \begin{bmatrix} v_{i,1}^a(x)\\ v_{i,2}^a(x)\\ \cdots\\ v_{i,N}^a(x)\end{bmatrix},\quad \nabla v_i^b = \begin{bmatrix}\frac{\partial v_i^b}{\partial x_1}\\ \frac{\partial v_i^b}{\partial x_2}\\ \cdots\\ \frac{\partial v_i^b}{\partial x_N}\end{bmatrix}\quad i = 1,\dots,N,$$

for some $V^a \in L^1_*(\Omega;\mathbb{R}^{N\times N})$ and $v^b \in W_0^{1,1}(\Omega;\mathbb{R}^N)$.

Then, taking into account the fact that

$$-\int_\Omega \left[A, \nabla v^b\right]_{\mathbb{R}^{N\times N}}\,dx \overset{\text{by (8)}}{=} \int_\Omega \Big(\operatorname{div} A, v^b\Big)_{\mathbb{R}^N}\,dx$$

and

$$\int_\Omega \left|\left[A, V^a\right]_{\mathbb{R}^{N\times N}}\right|\,dx \le \int_\Omega \sum_{i=1}^N \left|(v_i^a, a_i)_{\mathbb{R}^N}\right|\,dx$$
$$\le N\|A\|_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})}\|V^a\|_{L^1_*(\Omega;\mathbb{R}^{N\times N})}, \tag{14}$$

$$\int_\Omega \left|\left[A, \nabla v^b\right]_{\mathbb{R}^{N\times N}}\right|\,dx \le \int_\Omega \left|v^b\right|\left|\operatorname{div} A\right|\,dx$$
$$\le \|\operatorname{div} A\|_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})}\|v^b\|_{L^1(\Omega;\mathbb{R}^N)}, \tag{15}$$

we arrive at the estimate $\left|\langle A, V\rangle_{Y;X}\right| \le \|A\|_Y\|V\|_X$. Thus, $\langle\cdot,\cdot\rangle_{Y;X}$ a bilinear continuous form on $Y \times X$. Hence, each element A of Y can be associated with a linear continuous functional

$$F_A(\cdot) = \langle A, \cdot\rangle_{Y;X}\,,\quad F_A \in \mathcal{L}(X,\mathbb{R}).$$

As a result, we deduce:

$$Y \subseteq X^*, \tag{16}$$

where X^* stand for the dual space to X.

In order to establish the reverse inclusion, we fix an arbitrary element $F \in X^*$. Then

$$\|F\|_{X^*} := \sup_{V \in X, V \neq 0} \frac{|F(V)|}{\|V\|_X} < +\infty$$

and there exists a constant $C > 0$ such that

$$\begin{gathered} |F(V)| \le C\|V\|_X = C \inf_{(V^a, v^b) \in \Xi(V)} \left\{ \|V^a\|_{L^1(\Omega;\mathbb{R}^{N\times N})} + \|v^b\|_{L^1(\Omega;\mathbb{R}^N)} \right\}, \\ \forall\, V \in X. \end{gathered} \tag{17}$$

Since $F \in \mathcal{L}(X, \mathbb{R})$ and each element V of X is in $L^1(\Omega;\mathbb{R}^{N\times N})$, it follows from the Riesz representation theorem [8, Theorem 4.14] that there exists a matrix $B_F \in L^\infty(\Omega;\mathbb{R}^{N\times N})$ such that $B_F \in L^\infty_*(\Omega;\mathbb{R}^{N\times N})$ due to (6), and

$$\begin{gathered} F(V) = \int_\Omega \left[B_F, V\right]_{\mathbb{R}^{N\times N}} dx = \int_\Omega \left[B_F, V^a - \nabla v^b\right]_{\mathbb{R}^{N\times N}} dx, \\ \forall\, V \in X \text{ and } \forall\ (V^a, v^b) \in \Xi(V). \end{gathered}$$

Then inequality (17) implies that

$$\left| \int_\Omega \left[B_F, \nabla v^b\right]_{\mathbb{R}^{N\times N}} dx \right| \le C\|v^b\|_{L^1(\Omega;\mathbb{R}^N)}, \quad \forall\, v^b \in W_0^{1,1}(\Omega;\mathbb{R}^N), \tag{18}$$

where the constant $C > 0$ has been defined in (17).

Let $\{\varphi_k\}_{k\in\mathbb{N}} \in C_0^\infty(\Omega)$ be a sequence of test functions such that $\varphi_k \to v^b$ strongly in $W_0^{1,1}(\Omega)$. Then estimate (18) and the Riesz representation theorem imply the existence of a vector-valued function $\operatorname{div} B_F \in L^\infty(\Omega;\mathbb{R}^N)$ such that

$$\int_\Omega \left[B_F, \nabla\varphi_k\right]_{\mathbb{R}^{N\times N}} dx = -\int_\Omega \left(\operatorname{div} B_F, \varphi_k\right)_{\mathbb{R}^N}\, dx \le C\|\varphi_k\|_{L^1(\Omega;\mathbb{R}^N)}, \quad \forall\, k \in \mathbb{N}.$$

Passing to the limit in the last relation as $k \to \infty$, we arrive at the following representation for the functional $F \in X^*$

$$F(V) = \int_\Omega \left(\left[B_F, V^a\right]_{\mathbb{R}^{N\times N}} + \left(\operatorname{div} B_F, v^b\right)_{\mathbb{R}^N} \right) dx, \quad \forall\, V \in X.$$

Moreover, in view of (11) and the definition of matrix norm (5), we see that

$$\begin{aligned}
\|F\|_{X^*} &:= \sup_{V\in X, V\neq 0} \frac{|F(V)|}{\|V\|_X}\\
&= \sup_{V\neq 0} \sup_{\left(V^a,v^b\right)\in\Xi(V)} \frac{\left|\int_\Omega \left[B_F, V\right]_{\mathbb{R}^{N\times N}}\,dx\right|}{\|V^a\|_{L^1_*(\Omega;\mathbb{R}^{N\times N})}+\|v^b\|_{L^1(\Omega;\mathbb{R}^N)}}\\
&= \sup_{V^a-\nabla v^b\neq 0} \frac{\left|\int_\Omega \left(\left[B_F, V^a\right]_{\mathbb{R}^{N\times N}}+\left(\operatorname{div} B_F, v^b\right)_{\mathbb{R}^N}\right)\,dx\right|}{\|V^a\|_{L^1_*(\Omega;\mathbb{R}^{N\times N})}+\|v^b\|_{L^1(\Omega;\mathbb{R}^N)}}\\
&= \sup_{\left(V^a,v^b\right)\neq 0} \frac{\left|\int_\Omega \left(\left[B_F, V^a\right]_{\mathbb{R}^{N\times N}}+\left(\operatorname{div} B_F, v^b\right)_{\mathbb{R}^N}\right)\,dx\right|}{\|V^a\|_{L^1_*(\Omega;\mathbb{R}^{N\times N})}+\|v^b\|_{L^1(\Omega;\mathbb{R}^N)}}\\
&\geq \max\left\{\sup_{\left(0,v^b\right)\neq 0} \frac{\left|\int_\Omega \left(\operatorname{div} B_F, v^b\right)_{\mathbb{R}^N}\,dx\right|}{\|v^b\|_{L^1(\Omega;\mathbb{R}^N)}},\ \sup_{\left(V^a,0\right)\neq 0} \frac{\left|\int_\Omega \left[B_F, V^a\right]_{\mathbb{R}^{N\times N}}\right|}{\|V^a\|_{L^1_*(\Omega;\mathbb{R}^{N\times N})}}\right\}\\
&= \max\left\{N\|B_F\|_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})}, \|\operatorname{div} B_F\|_{L^\infty(\Omega;\mathbb{R}^N)}\right\} = \|B_F\|_Y.
\end{aligned}$$

On the other hand, utilizing estimates (14)–(15), we get

$$\begin{aligned}
\|F\|_{X^*} &:= \sup_{V\in X, V\neq 0} \frac{|F(V)|}{\|V\|_X}\\
&= \sup_{\left(V^a,v^b\right)\neq 0} \frac{\left|\int_\Omega \left(\left[B_F, V^a\right]_{\mathbb{R}^{N\times N}}+\left(\operatorname{div} B_F, v^b\right)_{\mathbb{R}^N}\right)\,dx\right|}{\|V^a\|_{L^1_*(\Omega;\mathbb{R}^{N\times N})}+\|v^b\|_{L^1(\Omega;\mathbb{R}^N)}}\\
&\leq \sup_{\left(V^a,v^b\right)\neq 0} \left[\frac{N\|B_F\|_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})}\|V^a\|_{L^1_*(\Omega;\mathbb{R}^{N\times N})}}{\|V^a\|_{L^1_*(\Omega;\mathbb{R}^{N\times N})}+\|v^b\|_{L^1(\Omega;\mathbb{R}^N)}}\right.\\
&\qquad\left.+\frac{\|\operatorname{div} B_F\|_{L^\infty(\Omega;\mathbb{R}^N)}\|v^b\|_{L^1(\Omega;\mathbb{R}^N)}}{\|V^a\|_{L^1_*(\Omega;\mathbb{R}^{N\times N})}+\|v^b\|_{L^1(\Omega;\mathbb{R}^N)}}\right]\\
&\leq \max\left\{N\|B_F\|_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})}, \|\operatorname{div} B_F\|_{L^\infty(\Omega;\mathbb{R}^N)}\right\} = \|B_F\|_Y.
\end{aligned}$$

Thus, $\|B_F\|_Y = \|F\|_{X^*} < +\infty$ and $B_F \in Y$. Since the functional $F \in X^*$ has been arbitrary chosen, this implies the inclusion $X^* \subseteq Y$. Combining it with (16), we obtain $X^* = Y$.

It remains to notice that the announced property of completeness of the space Y with respect to the norm (12), is a direct consequence of the established equality $X^* = Y$ and the fact that the set of all continuous linear functionals $\mathcal{L}(X,\mathbb{R})$ is always a Banach space.

Remark 1. It is worth to notice that $\langle X, \|\cdot\|_X\rangle$ is a separable Banach space. This conclusion is an immediate consequence of the similar properties of the Lebesgue space $L^1(\Omega;\mathbb{R}^{N\times N})$ and fact that the norm $\|\cdot\|_X$ is weaker then the standard one in $L^1(\Omega;\mathbb{R}^{N\times N})$,

$$(V,0)\in\Xi(V) \Longrightarrow \|V\|_X \overset{\text{by (13)}}{\leq} \|V\|_{L^1(\Omega;\mathbb{R}^{N\times N})}.$$

At the end of this section let us consider the following special subspace of Y:

$$Y_0 = \left\{ A(\cdot) \in L^\infty(\Omega, \mathrm{div}; \mathbb{R}^{N\times N}) \ : \ A(x) = \rho(x) I \right\},$$

where I stands for the unit matrix in $\mathbb{R}^{N\times N}$.

So, if $A = \rho I \in Y_0$, then

$$\rho \in L^\infty(\Omega) \quad \text{and} \quad \nabla \rho \in L^\infty(\Omega; \mathbb{R}^N).$$

As a result, we can identify the subspace Y_0 with the following set

$$Y_0 = \left\{ A(\cdot) = \rho(\cdot) I \in \mathbb{R}^{N\times N} \ : \ \rho \in W^{1,\infty}(\Omega) \right\}.$$

It is clear that in this case we have

$$A \in Y_0 \quad \Longrightarrow \quad \|A\|_Y = \max \left\{ N \|\rho\|_{L^\infty(\Omega)}, \|\nabla \rho\|_{L^\infty(\Omega;\mathbb{R}^N)} \right\} < \infty.$$

The next result is a direct consequence of Theorem 5.8.4 in [21] (see also [22, Theorem 4.2.5]) and specifies the principal properties of the subspace Y_0.

Lemma 2. *A matrix-function $A : \Omega \to \mathbb{R}^{N\times N}$ belongs to Y_0 if and only if $A(x) = \rho(x) I$ for a.e. $x \in \Omega$, where $\rho(\cdot)$ is a bounded and Lipschitz continuous scalar function with a Lipschitz constant depending on Ω.*

Remark 2. We make use of the following observation concerning the class of A_p weights that was introduced by B. Muckenhoupt in [27], where he showed that the A_p weights are precisely those weights ρ for which the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbb{R}^N, \rho\, dx)$ to $L^p(\mathbb{R}^N, \rho\, dx)$ when $1 < p < \infty$ (for more details, we refer to [37]). Here, the corresponding Hardy-Littlewood maximal operator is defined as a mapping which takes $f \in L^p(\mathbb{R}^N, \rho\, dx)$ to the Hardy-Littlerwood maximal function Mf with

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\, dy, \quad B_r(x) = \left\{ y \in \mathbb{R}^N \ : \ |y - x| < r) \right\}.$$

As a result, a weight ρ is said to be an A_p weight, if there exists a positive constant C such that, for every ball $B \subset \mathbb{R}^N$,

$$\left(\frac{1}{|B|} \int_B \rho(x)\, dx \right) \left(\frac{1}{|B|} \int_B \rho^{-1/(p-1)}(x)\, dx \right)^{p-1} \leq C, \quad p > 1.$$

A typical representative of the A_2 class is the weight function $\rho(x) = |x|_{\mathbb{R}^N}$ for $N \geq 2$. It is easy to check that in this case the matrix function $A(x) = |x|_{\mathbb{R}^N} I$ belongs to the class Y_0 with $\Omega = B_1(0)$. However, setting $\rho(x) = |x|^N_{\mathbb{R}^N}$, we see that $|x|^N_{\mathbb{R}^N} \notin A_2$, whereas the matrix $A(x) = |x|^N_{\mathbb{R}^N} I$ is still in Y_0 for any bounded Lipschitz domain Ω. Thus, the set Y_0 of feasible anisotropy tensors contains tensors that cannot be associated with the Muckenhoupt weights.

4. Weak Weighted Gradients and Anisotropic BV Spaces

We begin this section by recalling the following well-known result.

Theorem 1 (Riesz Representation Theorem). *[5, Theorem 2.4.6] Let* $L : C_0(\mathbb{R}^N;\mathbb{R}^N) \to \mathbb{R}$ *be a linear functional that satisfies the condition*

$$\sup\left\{L(f)\ :\ f \in C_0(\mathbb{R}^N;\mathbb{R}^N),\ |f| \le 1,\ \operatorname{supp}(f) \subset K\right\} < \infty$$

for each compact set $K \subset \mathbb{R}^N$*. Then there exists a Radon measure* μ *on* $\mathbb{R}^N$ *and a* μ*-measurable function* $\sigma : \mathbb{R}^N \to \mathbb{R}^N$ *such that*

$$|\sigma(x)| = 1 \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^N$$

and

$$L(f) = \int_{\mathbb{R}^N} (f, \sigma)_{\mathbb{R}^N}\, d\mu$$

for all $f \in C_0(\mathbb{R}^N;\mathbb{R}^N)$*. In this case* μ *is called the variation measure associated with* L*, and in each open set* $V \subset \mathbb{R}^N$ *it holds that*

$$\mu(V) = \sup\left\{L(f)\ :\ f \in C_0(\mathbb{R}^N;\mathbb{R}^N),\ |f| \le 1,\ \operatorname{supp}(f) \subset V\right\}.$$

For the proof of this theorem, we refer to [33, Theorem 6.19].

Having been inspired by this result and by the fact that

$$\left|\int_\Omega y\left(\left[A, \nabla\varphi\right]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A, \varphi)_{\mathbb{R}^N}\right)\, dx\right| \le \int_\Omega |y|\, dx \\ \times \left(N\|A\|_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})}\|\nabla\varphi\|_{C(\Omega;\mathbb{R}^{N\times N})} + \|\operatorname{div} A\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})}\|\varphi\|_{C(\Omega;\mathbb{R}^N)}\right) \quad (19)$$

for any $y \in L^1(\Omega)$, $A \in L^\infty(\Omega, \operatorname{div}; \mathbb{R}^{N\times N})$, and $\varphi \in C_0^\infty(\Omega;\mathbb{R}^N)$, we notice that the map $L_y^A : C_0^\infty(\Omega;\mathbb{R}^N) \to \mathbb{R}$ given by the rule

$$L_y^A(\varphi) := -\int_\Omega y\left(\left[A, \nabla\varphi\right]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A, \varphi)_{\mathbb{R}^N}\right)\, dx$$

is a distribution. Let us show that the functional L_y^A can be extended to the space $C_0(\Omega;\mathbb{R}^N)$ provided the function $y \in L^1(\Omega)$ satisfies condition

$$\sup\left\{\int_\Omega y\Big(\left[A, \nabla\varphi\right]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A, \varphi)_{\mathbb{R}^N}\Big)\, dx\ \middle|\ \begin{array}{c}\varphi \in C_0^\infty(\Omega;\mathbb{R}^N),\\ |\varphi(x)| \le 1\ \forall\ x \in \Omega\end{array}\right\} \le C \quad (20)$$

for some finite positive $C > 0$ (depending on y, A, and Ω).

Indeed, as follows from (20), we have

$$\sup\left\{L_y^A(\varphi)\ :\ \varphi \in C_0^\infty(\Omega;\mathbb{R}^N),\ \|\varphi\|_{C(\Omega;\mathbb{R}^N)} \le 1\right\} \le C$$

and consequently, for all $\varphi \in C_0^\infty(\Omega;\mathbb{R}^N)$

$$|L_y^A(\varphi)| \le C\|\varphi\|_{C(\Omega;\mathbb{R}^N)}. \quad (21)$$

Since, for each $\varphi \in C_0(\Omega;\mathbb{R}^N)$, one can find a sequence $\{\varphi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega;\mathbb{R}^N)$ which converges uniformly to φ, we can define the value $L_y^A(\varphi)$ by the standard rule

$$L_y^A(\varphi) := \lim_{k\to\infty} L_y^A(\varphi_k).$$

In view of estimate (21), we see that this limit exists and does not depend on the choice of the approximating sequence. Therefore, L_y^A can be uniquely extended to a linear functional $L_y^A : C_0(\Omega;\mathbb{R}^N) \to \mathbb{R}^N$ satisfying the condition

$$\sup\Big\{L_y^A(\varphi) \,:\, \varphi \in C_0(\Omega;\mathbb{R}^N),\ \|\varphi\|_{C(\Omega;\mathbb{R}^N)} \le 1\Big\} < \infty$$

provided property (20) holds.

Then the Riesz Representation Theorem implies: there exists a Radon measure μ on $\mathbb{R}^N$ and a μ-measurable function $\sigma : \mathbb{R}^N \to \mathbb{R}^N$ with unit norm such that

$$\begin{gathered} -\int_\Omega y\Big(\big[A,\nabla\varphi\big]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A,\varphi)_{\mathbb{R}^N}\Big)\,dx = \int_\Omega (\varphi,\sigma)_{\mathbb{R}^N}\,d\mu,\\ \forall\,\varphi \in C_0(\Omega;\mathbb{R}^N). \end{gathered} \tag{22}$$

This relation inspires us to introduce the following concept.

Definition 1. Let $y \in L^1(\Omega)$ and $A \in L^\infty(\Omega,\operatorname{div};\mathbb{R}^{N\times N})$ be given functions. We say that a vector-valued Radon measure $D^A y$ is a weak A-weighted gradient of the function y, if

$$\int_\Omega \big(\varphi, d[D^A y]\big) = -\int_\Omega y\left(\big[A,\nabla\varphi\big]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A,\varphi)_{\mathbb{R}^N}\right)dx \tag{23}$$

for all $\varphi \in C_0^\infty(\Omega;\mathbb{R}^N)$.

If a weak A-weighted gradient $D^A y$ of a function $y \in L^1(\Omega)$ admits the representation

$$d[D^A y] = v\,dx \quad \text{for some } \ v \in L^p(\Omega;\mathbb{R}^N) \text{ with } \ p \ge 1,$$

we will associate $D^A y$ with the vector-valued function $v \in L^p(\Omega;\mathbb{R}^N)$.

Remark 3. Returning to representation (22) and comparing it with relation (23), we see that

$$\mu = D^A y \quad \text{and} \quad D^A y = \sigma|D^A y|,$$

provided the total variation of the measure $D^A y$ is bounded over the domain Ω.

In view of the last statement of Theorem 1, it follows immediately from (23) and the fact that the condition

$$\int_\Omega \big(\varphi, d[D^A y]\big) = 0, \quad \forall\,\varphi \in C_0^\infty(\Omega;\mathbb{R}^N)$$

implies: $D^A y(S) = 0$ for all measurable sets $S \subset \Omega$, the weak-weighted gradient $D^A y$ of a given function y is uniquely determined.

It is worth noting that Definition 1 can be specified for the case of weights $A \in Y_0$. Namely, if $A(x) = \rho(x) I$ with $\rho \in W^{1,\infty}(\Omega)$, then $D^A y$ is a weak A-weighted gradient of the function $y \in L^1(\Omega)$ if

$$\int_\Omega \left(\varphi, d[D^A y]\right) = -\int_\Omega y \left(\rho \operatorname{div} \varphi + (\nabla \rho, \varphi)_{\mathbb{R}^N}\right) dx \tag{24}$$

for all $\varphi \in C_0^\infty(\Omega; \mathbb{R}^N)$.

Before proceeding further, we adopt the following notion.

Definition 2. We say that a matrix-valued function $A : \Omega \to \mathbb{R}^{N\times N}$ is an admissible weight (or tensor of anisotropy) if it satisfies the following conditions:

(i) $A \in L^\infty(\Omega, \operatorname{div}; \mathbb{R}^{N\times N}) \cap C(\overline{\Omega}; \mathbb{R}^{N\times N})$;

(ii) $A(x) = A^t(x) \quad \forall\, x \in \Omega$;

(iii) There exists a function $\rho \in W^{1,\infty}(\Omega)$ such that

$$\left.\begin{array}{c} \rho(x) \geq 0 \text{ for each } x \in \Omega, \\ (\xi, A(x)\xi)_{\mathbb{R}^N} \geq \rho(x)|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^N \text{ in } \Omega, \\ \mathcal{L}^N(\Lambda) = 0, \quad \operatorname{closure}_{|\cdot|_{\mathbb{R}^N}}(\Lambda) \subset \Omega, \end{array}\right\} \tag{25}$$

where $\Lambda = \{x \in \Omega \,:\, \rho(x) = 0\}$.

We denote by $\mathfrak{A}(\Omega)$ the set of all admissible weights.

Definition 3. Let $y \in L^1(\Omega)$ and $A \in \mathfrak{A}(\Omega)$ be given functions. We say that a weak A-weighted gradient $D^A y$ of the function y has a finite total variation in Ω, if

$$\left|D^A y\right|(\Omega) = \int_\Omega \left|d[D^A y]\right| < \infty,$$

where $\int_\Omega \left|d[D^A y]\right|$ is given by the rule

$$\int_\Omega \left|d[D^A y]\right| = \sup\left\{ \int_\Omega y \left(\left[A, \nabla\varphi\right]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A, \varphi)_{\mathbb{R}^N}\right) dx \;\middle|\; \begin{array}{c} \varphi \in C_0^\infty(\Omega; \mathbb{R}^N), \\ |\varphi(x)| \leq 1 \;\forall\; x \in \Omega \end{array} \right\}. \tag{26}$$

From now on with each admissible weight $A \in \mathfrak{A}(\Omega)$ we will associate the so-called anisotropic space of functions with bounded variation $BV_A(\Omega)$ which we define as follows:

Definition 4. $BV_A(\Omega)$ is the closure of the set

$$\left\{\psi \in C_0^\infty(\Omega) \ : \ \left|D^A\psi\right|(\Omega) < \infty\right\}$$

with respect to the following convergence

$$u_k \to u \quad \text{strongly in} \quad L^1(\Omega) \quad \text{and} \quad \int_\Omega \left|d[D^A u_k]\right| \to \int_\Omega \left|d[D^A u]\right|, \tag{27}$$

and endow it with the norm

$$\|u\|_{BV_A(\Omega)} := \|u\|_{L^1(\Omega)} + \left|D^A u\right|(\Omega). \tag{28}$$

Remark 4. The norm properties for $\|u\|_{BV_A(\Omega)}$ given by (28) follow easily from the definitions of $\|u\|_{L^1(\Omega)}$ and $\left|D^A u\right|(\Omega)$ and the initial assumptions (25).

Remark 5. As follows from the Riesz Representation Theorem and estimate (19), if u is an element of the $BV_A(\Omega)$ space, then its weak A-weighted gradient $D^A u$ lies in the space of vector-valued Radon measures $\mathcal{M}(\Omega;\mathbb{R}^N)$. In addition, Definition 4 implies one of the fundamental properties of space $BV_A(\Omega)$, namely, the set $C^\infty(\Omega)\cap BV^A(\Omega)$ is dense in $BV_A(\Omega)$ equipped with the intermediate convergence (27).

Remark 6. To emphasize the specificity of Definition 4, we note that, in principle, a A-weighted BV-space could be defined in different ways. That is, we can define $BV^A(\Omega)$ as the set of functions $y \in L^1(\Omega)$ for which the norm $\|y\|_{BV_A(\Omega)}$ if finite. However, it is unknown whether the set of functions $C^\infty(\Omega)\cap BV^A(\Omega)$ will be dense in $BV^A(\Omega)$ with respect to convergence (27). Moreover, in this case we have an obvious inclusion $BV_A(\Omega) \subseteq BV^A(\Omega)$ which may be strict under some special weight matrix $A \in \mathfrak{A}(\Omega)$ (see Remark 8 for more details).

Remark 7. Since $C_0^\infty(\Omega;\mathbb{R}^N)$ is dense in $C_0^1(\Omega;\mathbb{R}^N)$ with respect to the uniform topology, it follows from (26) that for any element $y \in BV_A(\Omega)$ the total variation of $D^A y$ in Ω can be defined by the rule

$$\int_\Omega \left|d[D^A y]\right| = \sup\left\{ \int_\Omega y\left(\left[A, \nabla\varphi\right]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A, \varphi)_{\mathbb{R}^N}\right)\,dx \;\middle|\; \begin{array}{c} \varphi \in C_0^1(\Omega;\mathbb{R}^N), \\ |\varphi(x)| \le 1 \ \forall\ x \in \Omega \end{array} \right\}.$$

One of the most important properties of $BV_A(\Omega)$-functions is demonstrated by the next theorem.

Theorem 2 (Semi-continuity). *Let $\Omega \subset \mathbb{R}^N$ be an open set with a Lipschitz boundary $\partial\Omega$. Let $A \in \mathfrak{A}(\Omega)$ be a given weight function, and let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $BV_A(\Omega)$ that strongly converges in $L^1(\Omega)$ to a function u. Then*

$$\int_\Omega \left|d[D^A u]\right| \le \liminf_{k\to\infty} \int_\Omega \left|d[D^A u_k]\right|. \tag{29}$$

Proof. Let $\varphi \in C_0^1(\Omega;\mathbb{R}^N)$ be such that $|\varphi| \le 1$. Since

$$\left(\left[A, \nabla\varphi\right]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A, \varphi)_{\mathbb{R}^N}\right) \in L^\infty(\Omega),$$

it follows that

$$\begin{aligned}\int_\Omega u\Big(\left[A, \nabla\varphi\right]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A, \varphi)_{\mathbb{R}^N}\Big)\, dx \\ &= -\lim_{k\to\infty}\int_\Omega \left(\varphi, d[D^A u_k]\right) \\ &\le \liminf_{k\to\infty}\int_\Omega \left|d[D^A u_k]\right|\end{aligned}$$

Then (29) follows by taking the supremum over all such test functions φ (see Remark 7).

By analogy to the classical theory of BV-spaces, we define the following type of weak convergence in $BV_A(\Omega)$.

Definition 5. We say that a sequence $\{u_k\}_{k\in\mathbb{N}}$ in $BV_A(\Omega)$ weakly-$*$ converges to some u in $BV_A(\Omega)$, and we write $u_k \overset{*}{\rightharpoonup} u$, iff the two following convergences hold:

$$\begin{aligned} u_k \to u &\quad \text{strongly in } \ L^1(\Omega); \\ D^A u_k \overset{*}{\rightharpoonup} D^A u &\quad \text{weakly-}*\text{ in } \ \mathcal{M}(\Omega;\mathbb{R}^N). \end{aligned}$$

In the proposition below we establish a compactness result related to this convergence.

Proposition 1. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in $BV_A(\Omega)$ strongly converging to some u in $L^1(\Omega)$ and satisfying the condition $\sup_{k\in\mathbb{N}}\int_\Omega \left|d[D^A u_k]\right| < +\infty$. Then

$$u \in BV_A(\Omega) \quad \text{and} \quad u_k \overset{*}{\rightharpoonup} u \ \text{ in } \ BV_A(\Omega). \tag{30}$$

Proof. The first part of the assertion (30) is a direct consequence of the inequality (29). We now establish the second part of (30). Since $u_k \to u$ strongly in $L^1(\Omega)$ and $A \in L^\infty(\Omega, \operatorname{div}; \mathbb{R}^{N\times N})$, it follows from (23) that, for all $\varphi \in C_0^\infty(\Omega;\mathbb{R}^N)$, we have

$$\begin{aligned} -\int_\Omega u_k\left[A, \nabla\varphi\right]_{\mathbb{R}^{N\times N}} dx &\to -\int_\Omega u\left[A, \nabla\varphi\right]_{\mathbb{R}^{N\times N}} dx \\ -\int_\Omega u_k\,(\operatorname{div} A, \varphi)_{\mathbb{R}^N}\, dx &\to -\int_\Omega u\,(\operatorname{div} A, \varphi)_{\mathbb{R}^N}\, dx \end{aligned}$$

as $k \to \infty$.

Then by density of $C_0^\infty(\Omega;\mathbb{R}^N)$ in $C_0(\Omega;\mathbb{R}^N)$ for the uniform norm and the boundedness of $\left\{D^A u_k\right\}_{k\in\mathbb{N}}$, we deduce that

$$\lim_{k\to\infty}\int_\Omega \left(\varphi, d[D^A u_k]\right) = \int_\Omega \left(\varphi, d[D^A u]\right), \quad \forall\varphi \in C_0(\Omega;\mathbb{R}^N),$$

i.e., the sequence of Radon vector-valued measures $\left\{D^A u_k\right\}_{k\in\mathbb{N}}$ weakly-$*$ converges to $D^A u$.

Let us show that, as a consequence of the semi-continuity property (29), $BV_A(\Omega)$ is a complete normed space.

Theorem 3. *$BV_A(\Omega)$ is a Banach space with respect to the norm* (28).

Proof. In view of Remark 4, it only remains to prove completeness. Let $\{u_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in $BV_A(\Omega)$. Then for all $\varepsilon > 0$ there exists N_ε in $\mathbb{N}$ such that

$$\forall\, m,n > N_\varepsilon, \quad \int_\Omega \left|d[D^A\left(u_m - u_n\right)]\right| < \varepsilon.$$

Since $\{u_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega)$, it follows that there exists $u \in L^1(\Omega)$ such that $u_k \to u$ strongly in $L^1(\Omega)$. Hence, $u_m - u_n \to u - u_n$ strongly in $L^1(\Omega)$ as m goes to ∞. Then, the lower semi-continuity property (29) implies that, for $n > N_\varepsilon$,

$$\int_\Omega \left|d[D^A\left(u - u_n\right)]\right| \le \liminf_{m\to\infty} \int_\Omega \left|d[D^A\left(u_m - u_n\right)]\right| \le \varepsilon.$$

Since ε was arbitrary, we obtain: $\int_\Omega \left|d[D^A u]\right| < +\infty$, i.e., $u \in BV^A(\Omega)$ and

$$\lim_{n\to\infty} \int_\Omega \left|d[D^A\left(u - u_n\right)]\right| = 0.$$

Thus, $u_k \to u$ in $BV_A(\Omega)$ as $k \to \infty$.

5. Basic Properties of $BV_A(\Omega)$ Spaces and Embedding Results

In this section, we mainly focus on the embedding theorems and the results concerning the bounds on norms in weighted Lebesgue spaces for $BV_A(\Omega)$-functions in terms of their anisotropic total variation.

We begin with the result showing that the anisotropic space $BV_A(\Omega)$ can be considered as a natural generalization of the classical space of functions with bounded variation.

Theorem 4. *The embedding $BV(\Omega) \hookrightarrow BV_A(\Omega)$ is continuous for each admissible weight $A \in \mathfrak{A}(\Omega)$, where $BV(\Omega)$ stands for the standard space of functions with bounded variations.*

Proof. Let y be an arbitrary $L^1(\Omega)$-function such that

$$\int_\Omega |d[Dy]| := \sup\left\{ \int_\Omega y \operatorname{div}\varphi\, dx \;\middle|\; \begin{array}{c} \varphi \in C_0^\infty(\Omega;\mathbb{R}^N), \\ |\varphi(x)| \le 1 \ \forall\, x \in \Omega \end{array} \right\} < +\infty.$$

i.e. $y \in BV(\Omega)$. Let $A \in \mathfrak{A}(\Omega)$ be a given weight matrix, and let $\varphi \in C_0(\Omega;\mathbb{R}^N)$ be a test function. Let $\{\varphi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega;\mathbb{R}^N)$ be a sequence which converges uniformly to φ. Then

$$\begin{aligned}\int_\Omega \left(\varphi_k, d[D^A y]\right) &= -\int_\Omega y\left(\left[A, \nabla\varphi_k\right]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A, \varphi_k)_{\mathbb{R}^N}\right)\, dx\\ &= -\int_\Omega y \operatorname{div}\left(A\varphi_k\right)\, dx = \int_\Omega \left(A\varphi_k, d[Dy]\right), \quad \forall\, k \in \mathbb{N},\end{aligned}$$

where $\operatorname{div}(A\varphi_k) \in L^\infty(\Omega)$ for each $k \in \mathbb{N}$. This fact follows from the previous line and the definition of the class of admissible matrices. Passing to the limit in this relation as $k \to \infty$, we obtain

$$\int_\Omega \left(\varphi, d[D^A y]\right) = \int_\Omega \left(A\varphi, d[Dy]\right).$$

Then, utilizing the property $A \in C(\overline{\Omega};\mathbb{R}^{N\times N})$ and definition of the total variation of y in Ω, we get

$$\begin{aligned}\int_\Omega \left(\varphi, d[D^A y]\right) &= \int_\Omega \left(A\varphi, d[Dy]\right)\\ &\le \sup_{\substack{\varphi\in C_0(\Omega;\mathbb{R}^N),\\ |\varphi(x)|\le 1\ \forall\, x\in\Omega}} \int_\Omega \left(A\varphi, d[Dy]\right)\\ &\overset{\text{by (6)}}{\le} \sqrt{N}\|A\|_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})} \sup_{\substack{\varphi\in C_0(\Omega;\mathbb{R}^N),\\ |\varphi(x)|\le 1\ \forall\, x\in\Omega}} \int_\Omega |\varphi(x)|\, |d[Dy]|\\ &= \sqrt{N}\|A\|_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})} \int_\Omega |d[Dy]|. \end{aligned} \tag{31}$$

As a result, the inequality $\left|D^A y\right|(\Omega) \le \sqrt{N}\|A\|_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})} \left|Dy\right|(\Omega)$ follows by taking the supremum in the left hand side of (31) over all test functions $\varphi \in C_0(\Omega;\mathbb{R}^N)$ such that $|\varphi(x)| \le 1$ in Ω. To conclude the proof, it remains to notice that $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$, and therefore,

$$\|u\|_{BV_A(\Omega)} \le \max\left\{1, \sqrt{N}\|A\|_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})}\right\} \|u\|_{BV(\Omega)}, \quad \forall\, u \in BV(\Omega). \tag{32}$$

Theorem 5. *Let $A \in \mathfrak{A}(\Omega)$ be a given weight matrix. If this matrix is positive definite in Ω, that is, there exists a constant $\alpha > 0$ such that*

$$(\xi, A(x)\xi)_{\mathbb{R}^N} \ge \rho(x)|\xi|^2 \ge \alpha|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^N \ \textit{ and for all } x \in \Omega, \tag{33}$$

then the norms $\|\cdot\|_{BV_A(\Omega)}$ and $\|\cdot\|_{BV(\Omega)}$ are equivalent in $BV_A(\Omega)$, that is, $BV(\Omega)$ and $BV_A(\Omega)$ are identical as topological spaces.

Proof. Let $\varphi \in C_0(\Omega;\mathbb{R}^N)$ be an arbitrary test function, and let y be an $L^1(\Omega)$-function such that $\|y\|_{BV_A(\Omega)} < \infty$. Since A is a symmetric matrix, it follows from (33) that

$$\left(\xi, A^{-1}(x)\xi\right)_{\mathbb{R}^N} \le \frac{1}{\alpha}|\xi|^2, \tag{34}$$

for all $\xi \in \mathbb{R}^N$ and $x \in \Omega$. Hence,

$$|A^{-1}\xi|^2 \overset{\text{by (33)}}{\le} \frac{1}{\alpha}\left(\xi, A^{-1}(x)\xi\right)_{\mathbb{R}^N} \overset{\text{by (34)}}{\le} \frac{1}{\alpha^2}|\xi|^2.$$

Therefore, setting $\psi(x) = A^{-1}(x)\varphi(x)\ \forall\, x \in \Omega$, we see that $\psi \in C_0(\Omega;\mathbb{R}^N)$ and

$$|\psi(x)| = |A^{-1}(x)\varphi(x)| \le \alpha^{-1}|\varphi(x)|, \quad \forall\, x \in \Omega.$$

Thus, $|\psi(x)| \le 1$ in Ω provided $\|\varphi\|_{C(\Omega;\mathbb{R}^N)} \le \alpha$.

So, letting $\|\varphi\|_{C(\Omega;\mathbb{R}^N)} \le \alpha$ and $\widehat{\varphi} = \alpha^{-1}\varphi$, we have

$$\begin{aligned} -\int_\Omega y \operatorname{div}(\widehat{\varphi})\,dx &= -\frac{1}{\alpha}\int_\Omega y \operatorname{div}(A\psi)\,dx \\ &\le \frac{1}{\alpha}\sup\left\{ -\int_\Omega y \operatorname{div}(A\psi) \;\middle|\; \begin{array}{c} \psi \in C_0(\Omega;\mathbb{R}^N), \\ |\psi(x)| \le 1\ \forall\ x \in \Omega \end{array} \right\} \\ &= \frac{1}{\alpha}\int_\Omega \left|d[D^A y]\right| < +\infty. \end{aligned} \tag{35}$$

Taking the supremum in the left hand side of (35) over all test functions $\widehat{\varphi} \in C_0(\Omega;\mathbb{R}^N)$ such that $|\widehat{\varphi}(x)| \le 1$ in Ω, we arrive at the inequality

$$\int_\Omega |d[Dy]| \le \frac{1}{\alpha}\int_\Omega \left|d[D^A y]\right|.$$

Combining this estimate with (32), we finally get

$$\begin{aligned} \min\left\{1, \frac{1}{\sqrt{N}}\|A\|^{-1}_{L^\infty_*(\Omega;\mathbb{R}^{N\times N})}\right\}\|y\|_{BV_A(\Omega)} &\le \|y\|_{BV(\Omega)} \\ &\le \max\left\{1, \frac{1}{\alpha}\right\}\|y\|_{BV_A(\Omega)}, \quad \forall\, y \in BV_A(\Omega). \end{aligned} \tag{36}$$

Remark 8. It is easy to show that, in general, the inclusion $BV(\Omega) \subseteq BV_A(\Omega)$ can be strict. Indeed, let us consider a classical example of a function with two variables that is not of bounded variation,

$$u(x_1,x_2) = \sin\left(|x|^{-2}_{\mathbb{R}^2}\right), \quad \text{for } (x_1,x_2) \ne (0,0), \text{ and } \ u(0,0) = 0.$$

Let Ω be an open ball $B_1(0)$ with radius 1 and centered at the origin. The problem occurs near the singularity at $(0,0)$. As $(x_1,x_2) \to (0,0)$, the function

u begin to oscillate infinitely many times. The gradient $\nabla u(x_1,x_2)$ in $\Omega\setminus(0,0)$ takes the form

$$\nabla u(x_1,x_2)=\left[\frac{2x_1\cos|x|^{-2}_{\mathbb{R}^2}}{|x|^4_{\mathbb{R}^2}},\frac{2x_2\cos|x|^{-2}_{\mathbb{R}^2}}{|x|^4_{\mathbb{R}^2}}\right]^t$$

and tends to explode as $(x_1,x_2)\to(0,0)$. As a result, the total variation of ∇u is unbounded in any neighbourhood around $(0,0)$ and, therefore, $u\notin BV(\Omega)$. However, setting $A(x)=\rho(x)I$ with $\rho(x)=|x|^4_{\mathbb{R}^2}$, we see that this weight function satisfies all conditions of Definition 2, i.e., $A\in\mathfrak{A}(\Omega)$, and for each $\varphi\in C_0^\infty(\Omega;\mathbb{R}^2)$,

$$\begin{aligned}\int_\Omega\left(\varphi,d[D^Au]\right)&\overset{\text{by }(24)}{=}-\int_\Omega u\left(\rho\operatorname{div}\varphi+(\nabla\rho,\varphi)_{\mathbb{R}^2}\right)dx\\&=-\int_\Omega u\operatorname{div}(\rho\varphi)\,dx=\int_\Omega(\varphi,\rho\nabla u)_{\mathbb{R}^2}\,dx\\&=2\int_\Omega(\varphi(x),x)_{\mathbb{R}^2}\cos|x|^{-2}_{\mathbb{R}^2}\,dx\\&\le 2\pi\|\varphi\|_{C(\overline{\Omega};\mathbb{R}^2)}<+\infty\end{aligned}\tag{37}$$

Hence, passing in (37) to the supremum over all test functions $\varphi\in C_0(\Omega;\mathbb{R}^2)$ such that $|\varphi(x)|\le 1$ in Ω, we infer

$$\left|D^Au\right|(\Omega)=\int_\Omega\left|d[D^Au]\right|<+\infty,$$

i.e., $u\in BV_A(\Omega)$ provided $A(x)=|x|^4_{\mathbb{R}^2}I$.

We make use of the following observation concerning the pointwise properties of the weak A-weighted gradients D^Ay

Proposition 2. Let $u\in BV(\Omega)$ be a given function and let $A=\rho I\in\mathfrak{A}(\Omega)$ be an admissible weight. Then the representation for Radon measure Du

$$Du(x)=\frac{1}{\rho(x)}D^Au(x)\tag{38}$$

holds true almost everywhere in Ω.

Proof. Since $A=\rho I$ in Ω and $A\in\mathfrak{A}(\Omega)$, it follows that $A\in Y_0$ and $\rho\in W^{1,\infty}(\Omega)$. Then Lemma 2 and properties (25) imply that $\rho(\cdot)$ is a bounded and Lipschitz continuous scalar function such that $\mathcal{L}^N(\Lambda_\rho)=0$, where $\Lambda_\rho:=\{x\in\Omega\,:\,\rho(x)=0\}$. Since $\Omega_\rho=\Omega\setminus\operatorname{closure}\Lambda_\rho$ is an open set, it follows that, for each $\varphi\in C_0^\infty(\Omega_\rho;\mathbb{R}^N)$ there exists $\delta>0$ such that $\rho(x)\ge\delta$ for each $x\in\operatorname{supp}\varphi$. Hence $\widehat{\varphi}=\rho^{-1}\varphi$, as an element of $W^{1,\infty}(\Omega;\mathbb{R}^N)$ with a compact support in Ω_ρ, can be used as a test function in (24) with $y = u$, where $u \in BV(\Omega)$ is a given function. This yields

$$-\int_{\Omega_\rho} u \operatorname{div} \varphi \, dx = -\int_{\Omega_\rho} u \operatorname{div} (\rho\widehat{\varphi}) \, dx = -\int_{\Omega_\rho} u \left(\rho \operatorname{div} \widehat{\varphi} + (\nabla \rho, \widehat{\varphi})_{\mathbb{R}^N}\right) dx$$

$$= \int_{\Omega_\rho} \left(\widehat{\varphi}, d[D^A u]\right) = \int_{\Omega_\rho} \left(\varphi, d\left[\frac{1}{\rho} D^A u\right]\right), \quad \forall\, \varphi \in C_0^\infty(\Omega_\rho; \mathbb{R}^N).$$

Thus, we arrive at the representation (38) for the weak gradient Du in Ω_ρ. Since $\mathcal{L}^N(\Omega \setminus \Omega_\rho) = 0$, it follows that this representation holds for almost all $x \in \Omega$.

Let us show that Definition 4 allows to extend the compactness embedding results which are well known for the class $BV(\Omega)$ (see [5, Section 10.1]) to the space $BV_A(\Omega)$. Keeping that in mind, we make use of some preliminaries.

Let $A \in \mathfrak{A}(\Omega)$ be a given weight matrix, and let ρ be a $W^{1,\infty}(\Omega)$-function such that (see (25))

$$(\xi, A(x)\xi)_{\mathbb{R}^N} \geq \rho(x)|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^N \text{ and } \forall\, x \in \Omega.$$

Let Λ be the zero-level set of the function ρ (see (25)). Since ρ is Lipschitz continuous, it follows that Λ has a finite Hausdorff measure, $\mathcal{H}^{N-1}(\Lambda) < \infty$.

Definition 6. We say that a function $u \in L^1(\Omega)$ possesses the (β)-property if there exists $\varepsilon > 0$ and a finite collection of points $\left\{x_j^*\right\}_{j\in J}$ such that

(a) $x_j^* \in \Lambda, \quad \forall\, j \in J$;

(b) $|u(y)| \leq \dfrac{1}{\varepsilon |y - x_j^*|_{\mathbb{R}^N}^N}$ for a.e. $y \in \overline{B_\varepsilon}(x_j^*) \cap \Omega$, where

$$\overline{B_\varepsilon}(x_j^*) = \left\{y \in \mathbb{R}^N \,:\, |y - x_j^*|_{\mathbb{R}^N} \leq \varepsilon\right\} \quad \forall\, j \in J;$$

(c) $\Lambda \subset \Lambda_\varepsilon$, where

$$\Lambda_\varepsilon = \Omega \cap \bigcup_{j\in J} \overline{B_\varepsilon}(x_j^*). \tag{39}$$

So, when we talk about an L^1-function with the (β)-property, we admit that this function can be unbounded in some neighborhoods of separated points on Λ. For further convenience, we denote the set of functions $u \in BV_A(\Omega)$ possessing the (β)-property by $BV_A^{(\beta)}(\Omega)$.

Theorem 6. *Let $A \in \mathfrak{A}(\Omega)$ be a given weight matrix. Then for any exponent p, $1 \leq p \leq \frac{N}{N-1}$, the following inclusion holds*

$$BV_A^{(\beta)}(\Omega) \subset L^p(\Omega, \rho^{N(p-1)}\, dx).$$

Moreover, for any function $u \in BV_A^{(\beta)}(\Omega)$ *there exists a constant* C *such that*

$$\left(\int_\Omega |u|^p \rho^{N(p-1)}\,dx\right)^{\frac{1}{p}} \le C \max\left\{\|u\|_{BV_A(\Omega)}, \|u\|^{\frac{1}{p}}_{BV_A(\Omega)}\right\} \\ \forall\, p \in \left[1, \tfrac{N}{N-1}\right]. \tag{40}$$

Proof. Let $A \in \mathfrak{A}(\Omega)$, and $p \in [1, \frac{N}{N-1}]$ be given. We fix an arbitrary element $u \in BV_A^{(\beta)}(\Omega)$. Then there exists $\varepsilon > 0$ such that $\Omega' := \Omega \setminus \Lambda_\varepsilon$ is a nonempty open set. Without loss of generality, we can suppose that Ω' has a Lipschitz boundary. Then we deduce from Theorem 5 that there exists a constant $\alpha = \alpha(\varepsilon) > 0$ such that

$$(\xi, A(x)\xi)_{\mathbb{R}^N} \ge \rho(x)|\xi|^2 \ge \alpha(\varepsilon)|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^N \text{ and for all } x \in \Omega'$$

and

$$\|u\|_{BV(\Omega')} \le \max\left\{1, \frac{1}{\alpha(\varepsilon)}\right\} \|u\|_{BV_A(\Omega')}.$$

Since the embedding $BV(\Omega') \hookrightarrow L^p(\Omega')$ is continuous for $1 \le p \le \frac{N}{N-1}$ [5, Theorem 10.1.3], it follows that there exists a constant C, which depends only on Ω', p, and N such that

$$\left(\int_{\Omega'} |u|^p\,dx\right)^{\frac{1}{p}} \le C\left(\|u\|_{L^1(\Omega')} + \int_{\Omega'} |d[Du]|\right) < \infty. \tag{41}$$

Hence,

$$\begin{aligned}\int_{\Omega'} |u|^p \rho^{N(p-1)}\,dx &\le \|\rho\|^{N(p-1)}_{W^{1,\infty}(\Omega)} \int_{\Omega'} |u|^p\,dx \\ &\overset{\text{by (41)}}{\le} C\|\rho\|^{N(p-1)}_{W^{1,\infty}(\Omega)} \left(\|u\|_{L^1(\Omega')} + \int_{\Omega'} |d[Du]|\right)^p \\ &\overset{\text{by (36)}}{\le} \widetilde{C}\left(\|u\|_{L^1(\Omega)} + \int_\Omega \left|d[D^A u]\right|\right)^p \end{aligned} \tag{42}$$

with $\widetilde{C} = C\|\rho\|^{N(p-1)}_{W^{1,\infty}(\Omega)} \max\left\{1, \frac{1}{\alpha^p(\varepsilon)}\right\}$. Since

$$\int_\Omega |u|^p \rho^{N(p-1)}\,dx = \int_{\Omega'} |u|^p \rho^{N(p-1)}\,dx + \int_{\Lambda_\varepsilon} |u|^p \rho^{N(p-1)}\,dx, \tag{43}$$

we dwell at the evaluation of the second term in (43). With that in mind, we consider the following collection of sets

$$\begin{aligned}
S(x_1^*) &:= B_\varepsilon(x_1^*),\\
S(x_2^*) &:= B_\varepsilon(x_2^*) \setminus \overline{B_\varepsilon(x_1^*)},\\
S(x_3^*) &:= B_\varepsilon(x_3^*) \setminus \bigcup_{j=1}^{2} \overline{B_\varepsilon(x_j^*)},\\
&\dots\\
S(x_M^*) &:= B_\varepsilon(x_M^*) \setminus \bigcup_{j=1}^{M-1} \overline{B_\varepsilon(x_j^*)},
\end{aligned}$$

where $M = \sharp J$. Then the representation (39) implies that

$$\begin{aligned}
&\Lambda_\varepsilon = \bigcup_{j\in J} \left(\Omega \cap \overline{S(x_j^*)}\right) \quad \text{and}\\
&|u(y)| \le \frac{1}{\varepsilon |y - x_j^*|_{\mathbb{R}^N}^N} \quad \text{a.e. in } \Omega \cap \overline{S(x_j^*)} \quad \forall\, j \in J.
\end{aligned} \tag{44}$$

Since $S(x_i^*) \cap S(x_j^*) = \emptyset$ for $i \ne j$, it follows from (44) that

$$\begin{aligned}
\int_{\Lambda_\varepsilon} |u|^p \rho^{N(p-1)}\, dx &= \sum_{j\in J} \int_{\Omega\cap\overline{S(x_j^*)}} |u|^p \rho^{N(p-1)}\, dx\\
&\le \frac{1}{\varepsilon^{p-1}} \sum_{j\in J} \int_{\Omega\cap\overline{S(x_j^*)}} |u| \rho^{N(p-1)} \frac{1}{|x - x_j^*|_{\mathbb{R}^N}^{N(p-1)}}\, dx.
\end{aligned} \tag{45}$$

Taking into account that the function ρ is Lipschitz continuous in Ω, we deduce the existence of a constant $L > 0$ such that

$$|\rho(x) - \rho(z)| \le L|x - z|_{\mathbb{R}^N}, \quad \forall\, z, x \in \Omega.$$

Hence, setting $z = x_j^*$, $x \in \Omega \cap \overline{S(x_j^*)}$, and noticing that $\rho(x_l^*) = 0$, we see that

$$\rho(x) = \left|\rho(x) - \rho(x_j^*)\right| \le L|x - x_j^*|_{\mathbb{R}^N}, \quad \forall\, x \in \Omega \cap \overline{S(x_j^*)}.$$

Then (45) leads us to the estimate

$$\begin{aligned}
\int_{\Lambda_\varepsilon} |u|^p \rho^{N(p-1)}\, dx &\le \frac{L^{N(p-1)}}{\varepsilon^{p-1}} \sum_{j\in J} \int_{\Omega\cap\overline{S(x_j^*)}} |u|\, dx\\
&\le \frac{L^{N(p-1)}}{\varepsilon^{p-1}} \|u\|_{L^1(\Omega)} \le \frac{L^{N(p-1)}}{\varepsilon^{p-1}} \|u\|_{BV_A(\Omega)}.
\end{aligned}$$

Combining this result with (42) and (43), we finally get

$$\|u\|^p_{L^p(\Omega,\rho^{N(p-1)}\, dx)} := \int_\Omega |u|^p \rho^{N(p-1)}\, dx \le \widetilde{C} \|u\|^p_{BV_A(\Omega)} + \widehat{C} \|u\|_{BV_A(\Omega)}, \tag{46}$$

where

$$\widetilde{C} = C\|\rho\|^{N(p-1)}_{W^{1,\infty}(\Omega)} \max\left\{1, \frac{1}{\alpha^p(\varepsilon)}\right\} \quad \text{and} \quad \widetilde{C} = \frac{L^{N(p-1)}}{\varepsilon^{p-1}}.$$

As a result, the announced estimate (40) is a direct consequence of inequality (46).

Due to this theorem, we can specify the main property of $BV_A(\Omega)$, given by Definition 4, as follows:

Proposition 3. Let $A \in \mathfrak{A}(\Omega)$ be a given weight function and let $u \in BV_A^{(\beta)}(\Omega)$. Then there exists a sequence of functions $\{u_k\}_{\mathbb{N}}$ in $C_0^\infty(\Omega) \cap BV_A(\Omega)$ such that

(i) $u_k \to u$ strongly in $L^p(\Omega, \rho^{N(p-1)}\, dx)$ for all $p \in \left[1, \frac{N}{N-1}\right)$;

(ii) $\left|D^A u_k\right|(\Omega) \to \left|D^A u\right|(\Omega)$ as $k \to \infty$.

Proof. The existence of a sequence $\{u_k\}_{\mathbb{N}}$ in $C_0^\infty(\Omega) \cap BV_A(\Omega)$ which converges to u with respect to intermediate convergence (27) is guaranteed by Definition 4. So, property (ii) is immediate. Therefore, we dwell in property (i). As follows from Theorem 6, the sequence is $\{u_k\}_{\mathbb{N}}$ is bounded in $L^{\frac{N}{N-1}}(\Omega, \rho^{\frac{N}{N-1}}\, dx)$. Since $u_k \to u$ strongly in $L^1(\Omega)$, without loss of generality, we can suppose (passing to a subsequence if necessary) $u_k(x) \to u(x)$ a.e. in Ω. Hence, $u_k(x)\rho(x) \to u(x)\rho(x)$ a.e. in Ω. Taking into account that boundedness of any sequence $\{v_k\}_{\mathbb{N}}$ in $L^p(\Omega)$ implies equiintegrability of $\{|v_k|^r\}_{\mathbb{N}}$ for all $1 \le r < p$, the strong convergence $u_k \to u$ in $L^p(\Omega, \rho^{N(p-1)}\, dx)$ for all $p \in \left[1, \frac{N}{N-1}\right)$ is a direct consequence of Vitali's convergence theorem [7].

6. Other Properties of Anisotropic BV Spaces

We begin this section with the following useful observation (for comparison, we refer to Proposition 1.13 in [24] where the similar result is stated for the classical BV–spaces).

Proposition 4. Let $A \in \mathfrak{A}(\Omega)$ be a given weight matrix, and let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in $BV_A(\Omega)$ such that

$$u_k \to u \quad \text{strongly in} \quad L^1(\Omega) \quad \text{and} \quad \lim_{k\to\infty} \int_\Omega \left|d[D^A u_k]\right| = \int_\Omega \left|d[D^A u]\right|. \tag{47}$$

Then for every open set $Q \subseteq \Omega$

$$\int_{\overline{Q}\cap\Omega} \left|d[D^A u]\right| \ge \limsup_{k\to\infty} \int_{\overline{Q}\cap\Omega} \left|d[D^A u_k]\right|, \tag{48}$$

and, in addition, if $\int_{\partial Q\cap\Omega} \left|d[D^A u]\right| = 0$, then

$$\int_Q \left|d[D^A u]\right| = \lim_{k\to\infty} \int_Q \left|d[D^A u_k]\right|, \tag{49}$$

Proof. Let $Q \subseteq \Omega$ be an open set. Since $S = \Omega \setminus \overline{Q}$ is the open set as well, it follows from Theorem 2 that

$$\begin{aligned}\int_Q \left|d[D^A u]\right| &\le \liminf_{k\to\infty} \int_Q \left|d[D^A u_k]\right|,\\ \int_S \left|d[D^A u]\right| &\le \liminf_{k\to\infty} \int_S \left|d[D^A u_k]\right|. \end{aligned} \tag{50}$$

Taking into account that $\Omega = S \cup \left(\overline{Q} \cap \Omega\right)$ and $S \cap \left(\overline{Q} \cap \Omega\right) = \emptyset$, we get

$$\begin{aligned}\int_{\overline{Q}\cap\Omega} \left|d[D^A u]\right| + \int_S \left|d[D^A u]\right| &= \int_\Omega \left|d[D^A u]\right|\\ &\overset{\text{by (47)}}{=} \lim_{k\to\infty} \int_\Omega \left|d[D^A u_k]\right|\\ &\ge \limsup_{k\to\infty} \int_{\overline{Q}\cap\Omega} \left|d[D^A u_k]\right| + \liminf_{k\to\infty} \int_S \left|d[D^A u_k]\right|\\ &\overset{\text{by (50)}}{\ge} \limsup_{k\to\infty} \int_{\overline{Q}\cap\Omega} \left|d[D^A u_k]\right| + \int_S \left|d[D^A u]\right|. \end{aligned}$$

Then the announced inequality (48) is immediate. As for the equality (49), it follows from (48). Indeed, since

$$\begin{aligned}\int_{\overline{Q}\cap\Omega} \left|d[D^A u]\right| &= \int_{\partial Q\cap\Omega} \left|d[D^A u]\right|\\ &\quad + \int_Q \left|d[D^A u]\right| \overset{\text{by (48)}}{\ge} \limsup_{k\to\infty} \int_{\overline{Q}\cap\Omega} \left|d[D^A u_k]\right|, \end{aligned} \tag{51}$$

and $\int_{\partial Q\cap\Omega} \left|d[D^A u]\right| = 0$, we see that

$$\begin{aligned}\liminf_{k\to\infty} \int_Q \left|d[D^A u_k]\right| &\overset{\text{by (50)}}{\ge} \int_Q \left|d[D^A u]\right|\\ &\overset{\text{by (51)}}{\ge} \limsup_{k\to\infty} \int_{\overline{Q}\cap\Omega} \left|d[D^A u_k]\right| \ge \limsup_{k\to\infty} \int_Q \left|d[D^A u_k]\right|. \end{aligned}$$

This concludes the proof.

Before proceeding further with the next results, we remind here the well-known facts concerning the mollified functions (see, for instance, [24, Chapter 1.14]). Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be a positive symmetric mollifier, i.e., $\eta(x)$ is zero outside a compact set $B_1 = \left\{x \in \mathbb{R}^2 : |x| \le 1\right\}$,

$$\int_{B_1} \eta(x)\,dx = 1, \quad \eta(x) \ge 0, \text{ and } \nu(x) = \mu(|x|) \text{ for some function } \mu : \mathbb{R}^+ \to \mathbb{R}.$$

Given such an η and a function $f \in L^1(\Omega)$ (where f is extended to be 0 outside of Ω if necessary), we set for each $\varepsilon > 0$

$$f_\varepsilon = \eta_\varepsilon * f \quad \text{with} \quad \eta_\varepsilon(x) = \varepsilon^{-N}\eta\left(\frac{x}{\varepsilon}\right),$$

that is,

$$\begin{aligned} f_\varepsilon(x) &= \varepsilon^{-N}\int_\Omega \eta\left(\frac{x-z}{\varepsilon}\right) f(z)\,dz = (-1)^N\varepsilon^{-N}\int_{\mathbb{R}^N}\eta\left(\frac{z}{\varepsilon}\right) f(x-z)\,dz \\ &= \int_{\mathbb{R}^N}\eta(w) f(x+\varepsilon w)\,dw. \end{aligned}$$

Then, using the standard properties of mollifiers, it can be shown that

(i) $f_\varepsilon \in C^\infty(\mathbb{R}^N)$, $f_\varepsilon \to f$ in $L^1(\Omega)$ as $\varepsilon \to 0$;

(ii) If $C_1 \le f \le C_2$ in Ω, then $C_1 \le f_\varepsilon(x) \le C_2$ for all $x \in \Omega$;

(iii) If $f, g \in L^1(\Omega)$, then $\int_\Omega f_\varepsilon g\,dx = \int_\Omega f g_\varepsilon\,dx$;

(iv) If $f \in C^1(\mathbb{R}^N)$, then $\dfrac{\partial f_\varepsilon}{\partial x_i} = \left[\dfrac{\partial f}{\partial x_i}\right]_\varepsilon$;

(v) If $\operatorname{supp} f \subseteq E$, then $\operatorname{supp} f_\varepsilon \subseteq E_\varepsilon = \big\{x \in \mathbb{R}^N : \operatorname{dist}(x, E) \le \varepsilon\big\}$.

The following property can be considered as a natural generalization of Proposition 11.15 in [24].

Proposition 5. Let $A \in \mathfrak{A}(\Omega)$ and $f \in BV_A(\Omega)$ be given functions. It is assumed that f is extended by zero outside Ω, whereas the matrix-function A can be extended to Ω_1 such that $A \in \mathfrak{A}(\Omega_1)$, where $\Omega_1 = \big\{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < 1\big\}$. Let E be an open subset of Ω such that

$$\overline{E} \subset \Omega \quad \text{and} \quad \int_{\partial E} \left|d[D^A f]\right| = 0. \tag{52}$$

Then

$$\int_E \left|d[D^A f]\right| = \lim_{\varepsilon\to 0}\int_E \left|d[D^A f_\varepsilon]\right|, \tag{53}$$

where f_ε is a mollified function described above.

Proof. Assuming that the small parameter ε varies within strictly decreasing sequences of positive numbers in $(0,1)$ that converge to 0 and taking into account that $f_\varepsilon \to f$ strongly in $L^1(\Omega)$, we see that

$$\int_E \left|d[D^A f]\right| \le \liminf_{\varepsilon\to 0}\int_E \left|d[D^A f_\varepsilon]\right| \quad \text{(by Theorem 2)}. \tag{54}$$

So, to establish equality (53), it remains to prove the reverse inequality to (54).

With that in mind, we fix an arbitrary test vector-valued function $\varphi \in C^1_0(E;\mathbb{R}^N)$ such that $|\varphi(x)| \le 1$ in E. Then by properties (ii)–(v) of mollified functions, we have

$$\begin{aligned}\int_\Omega f_\varepsilon\Big(\big[A,\nabla\varphi\big]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A,\varphi)_{\mathbb{R}^N}\Big)\,dx &= \int_\Omega f_\varepsilon \operatorname{div}(A\varphi)\;dx\\ &= \int_{\Omega_1} f\,[\operatorname{div}(A\varphi)]_\varepsilon\;dx = \int_{\Omega_1} f \operatorname{div}[A\varphi]_\varepsilon\;dx. \qquad (55)\end{aligned}$$

Taking into account that

$$|[A\varphi]_\varepsilon| \le |A\varphi| \le \sqrt{N}\|A\|_{L^\infty_*(E;\mathbb{R}^{N\times N})}|\varphi|$$

and

$$\operatorname{supp} A\varphi \subset E \implies \operatorname{supp}\,[A\varphi]_\varepsilon \subset E_\varepsilon = \big\{x\in\mathbb{R}^N : \operatorname{dist}(x,E)\le\varepsilon\big\},$$

we can estimate the right-hand side of (55) as follows

$$\begin{aligned}\int_{\Omega_1} f \operatorname{div}[A\varphi]_\varepsilon\;dx &= \int_{E_\varepsilon} f \operatorname{div}[A\varphi]_\varepsilon\;dx\\ &\le \sup\left\{\int_{E_\varepsilon} f\operatorname{div}(A\varphi)\;dx \;:\; \varphi\in C^1_0(E_\varepsilon;\mathbb{R}^N),\ |\varphi(x)|\le 1\ \forall\, x\in E_\varepsilon\right\} + O(\varepsilon)\\ &= \int_{E_\varepsilon}\big|d[D^Af]\big| + O(\varepsilon).\end{aligned}$$

Passing to the supremum in the left-hand side of this estimate over all $\varphi \in C^1_0(E;\mathbb{R}^N)$ such that $|\varphi(x)| \le 1$ in E, we obtain

$$\int_E \big|d[D^Af_\varepsilon]\big| \le \int_{E_\varepsilon}\big|d[D^Af]\big| + O(\varepsilon), \quad \forall\varepsilon>0.$$

Hence,

$$\limsup_{\varepsilon\to0}\int_E\big|d[D^Af_\varepsilon]\big| \le \lim_{\varepsilon\to0}\left(\int_{E_\varepsilon}\big|d[D^Af]\big| + O(\varepsilon)\right) = \int_{\overline{E}}\big|d[D^Af]\big|.$$

Since

$$\int_{\overline{E}}\big|d[D^Af]\big| = \int_{\partial E}\big|d[D^Af]\big| + \int_E\big|d[D^Af]\big| \overset{\text{by (52)}}{=} \int_E\big|d[D^Af]\big|,$$

it finally follows that

$$\limsup_{\varepsilon\to0}\int_E\big|d[D^Af_\varepsilon]\big| \le \int_E\big|d[D^Af]\big| \overset{\text{by (54)}}{\le} \liminf_{\varepsilon\to0}\int_E\big|d[D^Af_\varepsilon]\big|.$$

Then the announced equality (53) is immediate.

As an obvious consequence of this result, we have the following property, which can be viewed as a natural generalization of the results of Samson et al. [35].

Corollary 1. *Let $A \in \mathfrak{A}(\Omega)$ be a given weight matrix and let E be an open set such that $\overline{E} \subset \Omega$. Suppose that E has a finite A-perimeter* $\mathrm{Per}(E; A; \Omega)$*, that is,*

$$\mathrm{Per}(E; A; \Omega) := \int_\Omega \left| d[D^A \chi_E] \right| < +\infty, \tag{56}$$

where $\chi_E(x)$ stands for the characteristic function of the set E. Let $[\chi_E]_\varepsilon$ be a mollified characteristic function. Then

$$\lim_{\varepsilon \to 0} \left| d[D^A (\chi_E)_\varepsilon] \right| = \left| d[D^A \chi_E] \right|.$$

In fact, in (56) we used an anisotropic version of perimeter, saying that an $\mathcal{L}^N$-measurable subset $E \subset \Omega$ has a finite A-perimeter if $\chi_E \in BV_A(\Omega)$. In this case, we write down

$$\begin{aligned} &\mathrm{Per}(E; A; \Omega) = (\text{by } (56)) \\ &= \sup \left\{ \int_E \left(\left[A, \nabla\varphi\right]_{\mathbb{R}^{N\times N}} + (\operatorname{div} A, \varphi)_{\mathbb{R}^N} \right) dx \;\middle|\; \begin{array}{c} \varphi \in C_0^\infty(\Omega; \mathbb{R}^N), \\ |\varphi(x)| \le 1 \ \forall\ x \in \Omega. \end{array} \right\}. \end{aligned}$$

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