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Solving intuitionistic fuzzy fractional differential equations with $\psi-$Caputo fractional derivative

Abstract. In this research paper, we have attempted to establish the definition of the fractional derivative $\psi-$Caputo, the $\psi-$fractional integral, and the $\psi-$Laplace transform in the fuzzy intuitionistic sense, along with their properties. Furthermore, our objective is to explore the existence and uniqueness of solutions for certain intuitionistic fuzzy fractional differential equations (IFFDE) under the $\psi-$Caputo derivative of order $q \in (0,1)$. Lastly, we present an application example at the end to demonstrate how these findings can be applied in practice.
Key words: intuitionistic fuzzy set, intuitionistic fuzzy $\psi-$fractional integral, intuitionistic fuzzy $\psi-$Caputo fractional derivative, intuitionistic fuzzy fractional differential equations

Анотація. У цій дослідницькій статті ми спробували встановити визначення дробової похідної $\psi-$Капуто, $\psi-$дробного інтеграла та $\psi-$перетворення Лапласа в нечіткому інтуїтивістському сенсі, а також їх властивості. Крім того, нашою метою є дослідити існування та унікальність розв'язків для певних інтуїціоністських нечітких дробових диференціальних рівнянь (IFFDE) за похідною $\psi-$Капуто порядку $q \in (0,1)$. Нарешті, ми наводимо приклад застосування в кінці, щоб продемонструвати, як ці висновки можна застосувати на практиці.
Ключові слова: інтуїціоністська нечітка множина, інтуїціоністський нечіткий $\psi-$дробовий інтеграл, інтуїціоністська нечітка $\psi-$дробова похідна Капуто, інтуїціоністські нечіткі дробові диференціальні рівняння

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1. Introduction

After the creation of the theory of Fuzzy Subsets (FS) with Lotfi Zadeh in 1965 (see [16]), as a kind of generalization of the theory of classical subsets, Atanassov in 1986 (see [3]) follows the same approach as Zadeh and introduces

the concept of the Theory of Intuitionistic Fuzzy Subsets (IFS). This theory is a generalization of the definition of fuzzy subsets by adding the notion of membership and non-membership grades.
An intuitionistic fuzzy subset $\mathcal{A}$ of $\mathbb{R}$ is defined by:

$$\mathcal{A} = \{a \in \mathbb{R} \ ; \ \mu_{\mathcal{A}}(a), \nu_{\mathcal{A}}(a) \in [0,1], \ \mu_{\mathcal{A}}(a) + \nu_{\mathcal{A}}(a) \leq 1\},$$

where $\mu_{\mathcal{A}} : \mathbb{R} \to [0,1]$ and $\nu_{\mathcal{A}} : \mathbb{R} \to [0,1]$ represent, respectively, the degree of membership and the degree of non-membership of the element $a \in \mathbb{R}$ to the intuitionistic fuzzy subset $\mathcal{A}$.

After the notion mentioned, the resolution of Intuitionistic Fuzzy Fractional Differential Equations (IFFDE) has become a focal point for mathematicians. In references [11, 12], we encounter the definition of metric space and the distinction of Intuitionistic Fuzzy Hukuhara. Subsequently, numerous articles have been written on resolving Intuitionistic Fuzzy Fractional Differential Equations, such as studying the Cauchy Problem in [5], the Evolution Problem in [8], and the Dirichlet Problem in [10]. For further information, please refer to sources [9, 13, 14].
The idea of our manuscript is to create the definition of the intuitionistic fuzzy fractional derivative $\psi-$Caputo and to demonstrate its properties, as well as to give the definition of the $\psi-$fractional fuzzy intuitionistic integral and its properties. Additionally, we aim to introduce the transform $\psi-$Laplace. Once these new fuzzy intuitionistic concepts are established, we will proceed to study the existence and uniqueness of solutions for two related problems:

$$\begin{cases} {}^{C}D^{\gamma,\psi}_{0^+,gH}\varphi(t) = \mathcal{F}(t,\varphi(t)) \quad , \quad t \in I = [0,T] \\ \varphi(0) = \varphi_0 \in \mathbb{F}, \end{cases}$$

and,

$$\begin{cases} {}^{C}D^{\gamma,\psi}_{0^+,gH}\varphi(t) = \mathcal{A}(t)\varphi(t) \quad , \quad t \in I = [0,T] \\ \varphi(0) = \varphi_0 \in \mathbb{F}, \end{cases}$$

with, ${}^{C}D^{\gamma,\psi}_{0^+,gH}$ is the intuitionistic fuzzy fractional derivative $\psi-$Caputo of order $\gamma \in (0,1)$, $\mathcal{F} : I \times \mathbb{F} \longrightarrow \mathbb{F}$ is a function with an intuitionistic fuzzy value and $\mathcal{A}(t)$ is a bounded linear operator.
The structure of our paper is as follows:
Section 2 discusses the basic notions and necessary results of the intuitionistic fuzzy theory. In Section 3, we provide the definition of the fractional derivative $\psi-$Caputo, the $\psi-$fractional integral, and the $\psi-$Laplace transform in the fuzzy intuitionistic sense along with their properties. Moving on to Section 4, it is divided into two parts: first for resolving the nonlinear fractional differential equation 4.1 and second to examine the existence and uniqueness of solutions for the linear evolution problem 4.3. To further illustrate these concepts, Section 5 presents an example of application for each of these two problems.

2. Preliminaires

In this part we will try to present all the basic concepts necessary in our study.

Definition 1. [11] The set of intuitionistic fuzzy numbers is defined by:

$$\mathbb{F} = \mathbb{F}(\mathbb{R}) = \{\langle \mu, \nu \rangle : \mathbb{R} \longrightarrow [0,1]^2 \ , \ 0 \leq \mu + \nu \leq 1\},$$

and it checks the following properties:

1- For all $\langle \mu, \nu \rangle \in \mathbb{F}$ is normal, i.e :
There exists $a, b \in \mathbb{R}$ such that : $\mu(a) = 1$ and $\nu(b) = 1$.

2- For all $\langle \mu, \nu \rangle \in \mathbb{F}$ is intuitionistic convex, that's to say :
μ is fuzzy covex : $\mu(\lambda a + (1-\lambda)b) \geq \min\{\mu(a), \mu(b)\}$, $\forall a, b \in \mathbb{R}$, $\forall \lambda \in [0,1]$.
ν is fuzzy concave : $\nu(\lambda a + (1-\lambda)b) \geq \max\{\nu(a), \nu(b)\}$, $\forall a, b \in \mathbb{R}$, $\forall \lambda \in [0,1]$.

3- For all $\langle \mu, \nu \rangle \in \mathbb{F}$, μ is lower continuous and ν is appear continuous .

4- $supp\langle \mu, \nu \rangle = \overline{\{a \in \mathbb{R} \ , \ \nu(a) < 1\}}$ is bounded.

And we define zero intuitionistic fuzzy by:

$$\tilde{0}(a) = \begin{cases} (1,0) \ ; \ a = 0, \\ (0,1) \ ; \ a \neq 0 \end{cases}$$

Definition 2. [11] For $\alpha \in [0,1]$, we define the appear and lower $\alpha-$cut as following :

$$[\langle \mu, \nu \rangle]_\alpha = \{a \in \mathbb{R} \ , \ \mu(a) \geq \alpha\}.$$

$$[\langle \mu, \nu \rangle]^\alpha = \{a \in \mathbb{R} \ , \ \nu(a) \leq 1 - \alpha\}.$$

And we can write:

$$[\langle \mu, \nu \rangle]_\alpha = \left[[\langle \mu, \nu \rangle]_l^+ (\alpha), [\langle \mu, \nu \rangle]_r^+ (\alpha) \right]$$

and

$$[\langle \mu, \nu \rangle]^\alpha = \left[[\langle \mu, \nu \rangle]_l^- (\alpha), [\langle \mu, \nu \rangle]_r^- (\alpha) \right] ,$$

where:

$[\langle \mu, \nu \rangle]_l^+ (\alpha)$ and $[\langle \mu, \nu \rangle]_r^+ (\alpha)$ denote the left and right endpoints, respectively, of the upper α-cut interval, $[\langle \mu, \nu \rangle]_l^- (\alpha)$ and $[\langle \mu, \nu \rangle]_r^- (\alpha)$ denote the left and right endpoints of the lower α-cut interval.

Proposition 1. [11] Let $\langle \mu_1, \nu_1 \rangle, \langle \mu_2, \nu_2 \rangle \in \mathbb{F}$, we have :

1) $\langle \mu_1, \nu_1 \rangle = \langle \mu_2, \nu_2 \rangle \Leftrightarrow [\langle \mu_1, \nu_1 \rangle]_\alpha = [\langle \mu_2, \nu_2 \rangle]_\alpha$, $[\langle \mu_1, \nu_1 \rangle]^\alpha = [\langle \mu_2, \nu_2 \rangle]^\alpha$, $\forall \alpha \in [0,1]$.

2) Let $\langle\mu_1,\nu_1\rangle$ and $\langle\mu_2,\nu_2\rangle$ be two intuitionistic fuzzy numbers. Then, the sum (denoted by $\oplus$) of the two is defined (in the sense of Zadeh's extension principle) by:

$$[\langle\mu_1,\nu_1\rangle \oplus \langle\mu_2,\nu_2\rangle]_\alpha = [\langle\mu_1,\nu_1\rangle]_\alpha + [\langle\mu_2,\nu_2\rangle]_\alpha ,$$

$$[\langle\mu_1,\nu_1\rangle \oplus \langle\mu_2,\nu_2\rangle]^\alpha = [\langle\mu_1,\nu_1\rangle]^\alpha + [\langle\mu_2,\nu_2\rangle]^\alpha .$$

3) $\lambda \odot \langle\mu_1,\nu_1\rangle = \langle\lambda\mu_1,\lambda\nu_1\rangle \ , \ \forall\lambda \in \mathbb{R}$,
and according to the extension of Zadeh, we have:

$$[\lambda \odot \langle\mu_1,\nu_1\rangle]_\alpha = \lambda\,[\langle\mu_1,\nu_1\rangle]_\alpha ,$$

$$[\lambda \odot \langle\mu_1,\nu_1\rangle]^\alpha = \lambda\,[\langle\mu_1,\nu_1\rangle]^\alpha .$$

If $\lambda = 0$ then: $\lambda\langle\mu_1,\nu_1\rangle = \tilde{0}$.

Theorem 1. *[11] Let $\mathfrak{M} = \{\mathcal{M}_\alpha, \mathcal{M}^\alpha \ , \ \alpha \in [0,1]\}$ be the family of subsets of $\mathbb{R}$, verifies the following properties:*

1) $\alpha \le \beta \Rightarrow \mathcal{M}_\beta \subset \mathcal{M}_\alpha$ and $\mathcal{M}^\beta \subset \mathcal{M}^\alpha$ for all $\alpha,\beta \in [0,1]$.

2) $\mathcal{M}_\alpha$ and $\mathcal{M}^\alpha$ are two non-empty compact convex subsets in $\mathbb{R}$ for all $\alpha \in [0,1]$.

3) For all nondecreansing sequence $\alpha_i \to \alpha$ on $[0,1]$, we have:

$$\mathcal{M}_\alpha = \cap_i \mathcal{M}_{\alpha_i} \ , \ \mathcal{M}^\alpha = \cap_i \mathcal{M}^{\alpha_i}.$$

Then, we define μ and ν by:

$$\mu(a) = \begin{cases} 0 \ , \ a \notin \mathcal{M}_0, \\ \sup_{\alpha\in[0,1]} \mathcal{M}_\alpha \ , \ a \in \mathcal{M}_0 \end{cases}$$

$$\nu(a) = \begin{cases} 1 \ , \ a \notin \mathcal{M}^0, \\ 1 - \sup_{\alpha\in[0,1]} \mathcal{M}^\alpha \ , \ a \in \mathcal{M}^0 \end{cases}$$

Therefore, $\langle\mu,\nu\rangle \in \mathbb{F}$, $\mathcal{M}_\alpha = [\langle\mu,\nu\rangle]_\alpha$ and $\mathcal{M}^\alpha = [\langle\mu,\nu\rangle]^\alpha$.

Remark 1. i- The family $\{[\langle\mu,\nu\rangle]_\alpha, [\langle\mu,\nu\rangle]^\alpha \ , \ \alpha \in [0,1]\}$ satisfies the previous properties of theorem 1.

ii- For all $\alpha \in [0,1]$ we have:

$$[\langle\mu,\nu\rangle]_\alpha \subset [\langle\mu,\nu\rangle]^\alpha .$$

Definition 3. [11] Let $\langle\mu_1,\nu_1\rangle, \langle\mu_2,\nu_2\rangle \in \mathbb{F}$. We define the following two distances on $\mathbb{F}$:

$$\begin{aligned} d_\infty\big(\langle\mu_1,\nu_1\rangle, \langle\mu_2,\nu_2\rangle\big) = & \frac{1}{4}\sup_{\alpha\in(0,1]} \big|[\langle\mu_1,\nu_1\rangle]_r^+(\alpha) - [\langle\mu_2,\nu_2\rangle]_r^+(\alpha)\big| \\ & + \frac{1}{4}\sup_{\alpha\in(0,1]} \big|[\langle\mu_1,\nu_1\rangle]_l^+(\alpha) - [\langle\mu_2,\nu_2\rangle]_l^+(\alpha)\big| \\ & + \frac{1}{4}\sup_{\alpha\in(0,1]} \big|[\langle\mu_1,\nu_1\rangle]_r^-(\alpha) - [\langle\mu_2,\nu_2\rangle]_r^-(\alpha)\big| \\ & + \frac{1}{4}\sup_{\alpha\in(0,1]} \big|[\langle\mu_1,\nu_1\rangle]_l^-(\alpha) - [\langle\mu_2,\nu_2\rangle]_l^-(\alpha)\big|, \end{aligned}$$

and

$$\begin{aligned} d_p\big(\langle\mu_1,\nu_1\rangle, \langle\mu_2,\nu_2\rangle\big) = & \bigg(\frac{1}{4}\int_0^1 \big|[\langle\mu_1,\nu_1\rangle]_r^+(\alpha) - [\langle\mu_2,\nu_2\rangle]_r^+(\alpha)\big|^p d\alpha \\ & + \frac{1}{4}\int_0^1 \big|[\langle\mu_1,\nu_1\rangle]_l^+(\alpha) - [\langle\mu_2,\nu_2\rangle]_l^+(\alpha)\big|^p d\alpha \\ & + \frac{1}{4}\int_0^1 \big|[\langle\mu_1,\nu_1\rangle]_r^-(\alpha) - [\langle\mu_2,\nu_2\rangle]_r^-(\alpha)\big|^p d\alpha \\ & + \frac{1}{4}\int_0^1 \big|[\langle\mu_1,\nu_1\rangle]_l^-(\alpha) - [\langle\mu_2,\nu_2\rangle]_l^-(\alpha)\big|^p d\alpha\bigg)^{\frac{1}{p}}. \end{aligned}$$

then, $(\mathbb{F}, d_p)$ is a complete metric space.

Now, we move on to define the hukuhara difference between two intuitionistic fuzzy numbers.

Definition 4. [12] Let $\langle\mu_1,\nu_1\rangle, \langle\mu_2,\nu_2\rangle \in \mathbb{F}$. The generalized Hukuhara difference between these two elements is defined by:

$$\langle\mu_1,\nu_1\rangle \ominus_{gH} \langle\mu_2,\nu_2\rangle = \langle\mu,\nu\rangle \quad \Leftrightarrow \quad \langle\mu_1,\nu_1\rangle = \langle\mu_2,\nu_2\rangle \oplus \langle\mu,\nu\rangle.$$

Let $\mathfrak{h} : [0,T] \to \mathbb{F}$ be a function whose values are intuitionistic fuzzy numbers. For each fixed $t \in [0,T]$, the function value $\mathfrak{h}(t)$ is an intuitionistic fuzzy number, represented by a pair of functions $\mu_{\mathfrak{h}(t)}$ and $\nu_{\mathfrak{h}(t)}$, satisfying:

$$\mu_{\mathfrak{h}(t)} : \mathbb{R} \to [0,1], \quad \nu_{\mathfrak{h}(t)} : \mathbb{R} \to [0,1], \quad \text{with} \quad \mu_{\mathfrak{h}(t)}(x) + \nu_{\mathfrak{h}(t)}(x) \le 1.$$

Then, for a fixed level $\alpha \in [0,1]$, the α-level representation of the intuitionistic fuzzy-valued function $\mathfrak{h}$ is given by two interval-valued functions:

$$[\mathfrak{h}(t)]_\alpha = [\mathfrak{h}_{\alpha,l}(t),\ \mathfrak{h}_{\alpha,r}(t)],$$

$$[\mathfrak{h}(t)]^\alpha = \left[\mathfrak{h}^{\alpha,l}(t),\ \mathfrak{h}^{\alpha,r}(t)\right],$$

where:

$$\mathfrak{h}_{\alpha,l}(t) = \inf\left\{x \in \mathbb{R} : \mu_{\mathfrak{h}(t)}(x) \geq \alpha\right\},$$
$$\mathfrak{h}_{\alpha,r}(t) = \sup\left\{x \in \mathbb{R} : \mu_{\mathfrak{h}(t)}(x) \geq \alpha\right\},$$
$$\mathfrak{h}^{\alpha,l}(t) = \inf\left\{x \in \mathbb{R} : \nu_{\mathfrak{h}(t)}(x) \leq 1-\alpha\right\},$$
$$\mathfrak{h}^{\alpha,r}(t) = \sup\left\{x \in \mathbb{R} : \nu_{\mathfrak{h}(t)}(x) \leq 1-\alpha\right\}.$$

Definition 5. [12] Let $\mathfrak{h} : [0,T] \longrightarrow \mathbb{F}$, then the generalized Hukuhara derivative of $\mathfrak{h}$ at t_0 is defined by:

$$\mathfrak{h}'_{gH}(t_0) = \lim_{t\to t_0} \frac{\mathfrak{h}(t) \ominus_{gH} \mathfrak{h}(t_0)}{t-t_0},$$

if $\mathfrak{h}'_{gH}(t_0) \in \mathbb{F}$, and we say that $\mathfrak{h}$ is generalized Hukuhara differentiable ($gH-$differentiable) at t_0.

Therefore, we can separate two types of $gH-$differentiable for a function with value in $\mathbb{F}$.
We say that $\mathfrak{h}$ is $[(i)-gH]-$differentaible et t_0 if :

$$\left[\mathfrak{h}'_{gH}\right]_\alpha = \left[(\mathfrak{h}_{\alpha,l})', (\mathfrak{h}_{\alpha,r})'\right],$$

$$\left[\mathfrak{h}'_{gH}\right]^\alpha = \left[\left(\mathfrak{h}^{\alpha,l}\right)', (\mathfrak{h}^{\alpha,r})'\right].$$

We say that $\mathfrak{h}$ is $[(ii)-gH]-$differentaible et t_0 if :

$$\left[\mathfrak{h}'_{gH}\right]_\alpha = \left[(\mathfrak{h}_{\alpha,r})', (\mathfrak{h}_{\alpha,l})'\right],$$

$$\left[\mathfrak{h}'_{gH}\right]^\alpha = \left[(\mathfrak{h}^{\alpha,r})', \left(\mathfrak{h}^{\alpha,l}\right)'\right].$$

Remark 2. We can define the generalized derivative of higher order by:

$$\begin{cases} \mathfrak{h}^0 = \mathfrak{h}, \\ \mathfrak{h}^{(n)}_{gH} = (\mathfrak{h}^{(n-1)}_{gH})'_{gH}. \end{cases}$$

Definition 6. [12] Let $\mathfrak{h} : [0,T] \longrightarrow \mathbb{F}$, we say that $\mathfrak{h}$ is of class $\mathcal{C}^m$, $m \in \mathbb{N}$ if $\mathfrak{h}^{(m)}_{gH}$ exists and continues, by respect to metric d_∞.

If $\mathfrak{h}_{\alpha,l}, \mathfrak{h}_{\alpha,r}, \mathfrak{h}^{\alpha,l}$and $\mathfrak{h}^{\alpha,r}$ are Riemann integrable on $[0,T]$.
Then,

$$\int_{[0,T]} \mathfrak{h} = \left\{\left[\int_{[0,T]} \mathfrak{h}_{\alpha,l}, \int_{[0,T]} \mathfrak{h}_{\alpha,r}\right], \left[\int_{[0,T]} \mathfrak{h}^{\alpha,l}, \int_{[0,T]} \mathfrak{h}^{\alpha,r}\right]\right\}.$$

Definition 7. [12] Let $\mathfrak{h} : [0,T] \longrightarrow \mathbb{F}$, we say that $\mathfrak{h}$ is integrable on $[0,T]$, if $\mathfrak{h}_{\alpha,l}, \mathfrak{h}_{\alpha,r}, \mathfrak{h}^{\alpha,l}$and $\mathfrak{h}^{\alpha,r}$ are integrable on $[0,T]$.

Now, we remind you of the $\psi-$fractional calculation in the fuzzy case.
Let us first clarify the notations used in the following definition:

- $L([0,T],\mathbb{E}^1)$ denotes the space of all strongly measurable and Lebesgue integrable functions from the interval $[0,T]$ into $\mathbb{E}^1$, where $\mathbb{E}^1$ is the set of all normal, convex, upper semi-continuous, and compactly supported fuzzy numbers on $\mathbb{R}$.
- $\psi : [0,T] \to \mathbb{R}$ is a given increasing and continuously differentiable function. It serves as a kernel or a change-of-variable function in the definition of the generalized fractional integral.
- $\Gamma(q)$ is the Gamma function, defined for all $q > 0$ by:

$$\Gamma(q) = \int_0^\infty x^{q-1} e^{-x}\, dx,$$

which generalizes the factorial function, i.e., $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}^*$.

Definition 8. [15] Let $\mathfrak{h} \in L([0,T],\mathbb{E}^1)$, The fuzzy $\psi-$fractional integral at order $q \in (0,1)$ of h is defined by:

$$I^{q,\psi}_{0^+,gH}\mathfrak{h}(t) = \frac{1}{\Gamma(q)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{q-1} \odot \mathfrak{h}(s)ds.$$

Definition 9. [15] Let $\mathfrak{h} \in C([0,T],\mathbb{E}^1) \cap L([0,T],\mathbb{E}^1)$, the fuzzy $\psi-$Caputo fractional derivative of order $q \in (0,1)$ of $\mathfrak{h}$ is defined by:

$${}^C D^{q,\psi}_{0^+,gH}\mathfrak{h}(t) = \frac{1}{\Gamma(q)} \int_0^t (\psi(t) - \psi(s))^{q-1} \odot \mathfrak{h}^{'}_{gH}(s)ds.$$

3. The intuitionistic fuzzy extension of $\psi-$fractional derivative and of $\psi-$fractional integral

In this section, we create new concepts concerning the fuzzy fractional $\psi-$Caputo derivative and the $\psi-$fractional integral in the intuitionistic case. Before we commence, it is imperative that we acquire the following notations: Let the function $\psi \in C^1([0,T],\mathbb{R}^+)$, such that $\psi^{'}(t) > 0$, $\forall t \in [0,T]$.

Here, $\mathbb{R}^+$ denotes the set of strictly positive real numbers, i.e., $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.
Let $\mathbb{F}$ denote the space of intuitionistic fuzzy numbers, i.e., ordered pairs $\langle \mu, \nu \rangle$ of functions $\mu, \nu : \mathbb{R} \to [0,1]$ satisfying:

$$0 \le \mu(x) + \nu(x) \le 1, \quad \forall x \in \mathbb{R},$$

where $\mu(x)$ represents the degree of membership and $\nu(x)$ the degree of non-membership. The topology on $\mathbb{F}$ is induced by a metric d_∞ (the supremum metric) defined via the level-set approach. In some settings, a general d_p metric may be considered, where $p \in [1, \infty]$, but in this work, we assume the uniform topology associated with d_∞.
And we note in the following,
$C^{\mathbb{F}} = C([0,T], \mathbb{F})$: the space of continuous intuitionnistic fuzzy-valued function (IFVF) of $[0,T]$.
$L^{\mathbb{F}} = L([0,T], \mathbb{F})$: the space of Lebesgue integrable intuitionnistic fuzzy-valued function (IFVF) of $[0,T]$.

Definition 10. Let $\mathfrak{f} \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$ be an intuitionnistic fuzzy-valued function, then the intuitionistic fuzzy $\psi-$fractional integral of order $q \in (0,1)$ of $\mathfrak{f}$ is defined by:

$$\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}\right)(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1} \odot \mathfrak{f}(s)ds \ , \ \forall t > 0.$$

We pose the $\alpha-$level representation of the function f with an intuitionistic fuzzy value:

$$[\mathfrak{f}(t)]_\alpha = [\mathfrak{f}_{\alpha,l}(t), \mathfrak{f}_{\alpha,r}(t)] \, ,$$

$$[\mathfrak{f}(t)]^\alpha = \left[\mathfrak{f}^{\alpha,l}(t), \mathfrak{f}^{\alpha,r}(t)\right].$$

Theorem 2. *Let $\mathfrak{f} \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$ be an intuitionnistic fuzzy-valued function, then the intuitionistic fuzzy $\psi-$fractional integral of $\mathfrak{f}$ can be expressed by:*

$$\left[I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}\right]_\alpha(t) = \left[\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,l}\right)(t), \left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,r}\right)(t)\right],$$

$$\left[I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}\right]^\alpha(t) = \left[\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,l}\right)(t), \left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,r}\right)(t)\right].$$

With,

$$\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,l}\right)(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1}\mathfrak{f}_{\alpha,l}(s)ds \ , \ \forall t > 0,$$

$$\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,r}\right)(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1}\mathfrak{f}_{\alpha,r}(s)ds \ , \ \forall t > 0,$$

$$\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,l}\right)(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1}\mathfrak{f}^{\alpha,l}(s)ds \ , \ \forall t > 0,$$

$$\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,r}\right)(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1}\mathfrak{f}^{\alpha,r}(s)ds \ , \ \forall t > 0.$$

Proof. Let $\mathfrak{f} \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$ be a continuous and integrable intuitionistic fuzzy-valued function on $[0, T]$.

We aim to show that the generalized intuitionistic fuzzy ψ-fractional integral of order $\gamma \in (0, 1)$ of $\mathfrak{f}$ can be expressed via its α-level sets as follows:

$$\left[I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}\right]_\alpha (t) = \left[\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,l}\right)(t), \left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,r}\right)(t)\right], \quad \forall t \in [0, T].$$

By definition, the generalized Hukuhara fractional integral of $\mathfrak{f}$ is given by:

$$I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\gamma-1} \odot \mathfrak{f}(s)\, ds,$$

where $\odot$ denotes the scalar multiplication between a positive real number and an intuitionistic fuzzy number.

For each $\alpha \in [0, 1]$, the fuzzy function $\mathfrak{f}(s)$ is characterized by its α-cut:

$$[\mathfrak{f}(s)]_\alpha = [\mathfrak{f}_{\alpha,l}(s), \mathfrak{f}_{\alpha,r}(s)],$$

where $\mathfrak{f}_{\alpha,l}(s)$ and $\mathfrak{f}_{\alpha,r}(s)$ are the lower and upper bounds of the fuzzy interval at level α.

The scalar multiplication $\lambda \odot \mathfrak{f}(s)$, with

$$\lambda = \frac{1}{\Gamma(\gamma)}\psi'(s)(\psi(t) - \psi(s))^{\gamma-1} > 0,$$

acts on the α-cut bounds as:

$$[\lambda \odot \mathfrak{f}(s)]_\alpha = [\lambda\mathfrak{f}_{\alpha,l}(s),\ \lambda\mathfrak{f}_{\alpha,r}(s)].$$

The integral of a fuzzy function can be interpreted as the integral of the α-cut intervals (see fuzzy number theory), provided the kernel $g(s) \geq 0$ holds. Here,

$$g(s) := \frac{1}{\Gamma(\gamma)}\psi'(s)(\psi(t) - \psi(s))^{\gamma-1} > 0.$$

Hence,

$$\left[\int_0^t g(s) \odot \mathfrak{f}(s) ds\right]_\alpha = \left[\int_0^t g(s)\mathfrak{f}_{\alpha,l}(s) ds,\ \int_0^t g(s)\mathfrak{f}_{\alpha,r}(s) ds\right].$$

Therefore,

$$\left[I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}\right]_\alpha (t) = \left[\int_0^t g(s)\mathfrak{f}_{\alpha,l}(s) ds,\ \int_0^t g(s)\mathfrak{f}_{\alpha,r}(s) ds\right].$$

By the classical definition of fractional integrals on real-valued functions, we define:

$$\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,l}\right)(t) := \int_0^t g(s)\mathfrak{f}_{\alpha,l}(s) ds,$$

$$\left(I_{0^+,gH}^{\gamma,\psi}\mathfrak{f}_{\alpha,r}\right)(t) := \int_0^t g(s)\mathfrak{f}_{\alpha,r}(s)ds.$$

Hence,

$$\left[I_{0^+,gH}^{\gamma,\psi}\mathfrak{f}\right]_\alpha(t) = \left[\left(I_{0^+,gH}^{\gamma,\psi}\mathfrak{f}_{\alpha,l}\right)(t),\ \left(I_{0^+,gH}^{\gamma,\psi}\mathfrak{f}_{\alpha,r}\right)(t)\right].$$

The same reasoning applies for the non-membership part, $\left[I_{0^+,gH}^{\gamma,\psi}\mathfrak{f}\right]^\alpha(t)$.

Example 1. We consider the function $\psi(t) = t^2$ and $\gamma = \dfrac{1}{2}$.
For the function $\mathfrak{f}$ we consider the following example: $[\mathfrak{f}(t)]_\alpha = [0, t]$ and $[\mathfrak{f}(t)]^\alpha = [-3, 0]$.
Then,the intuitionistic fuzzy $\psi-$fractional integral of $\mathfrak{f}$ using Theorem 2 is:

$$\left(I_{0^+,gH}^{\frac{1}{2},t^2}\mathfrak{f}_{\alpha,l}\right)(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (s^2)'(t^2 - s^2)^{\frac{1}{2}-1}0\ ds = 0,$$

$$\left(I_{0^+,gH}^{\frac{1}{2},t^2}\mathfrak{f}_{\alpha,r}\right)(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (s^2)'(t^2 - s^2)^{\frac{1}{2}-1}s\ ds = \frac{\pi t^2}{2\Gamma\left(\frac{1}{2}\right)} = \frac{1}{2\sqrt{\pi}}t^2,$$

$$\left(I_{0^+,gH}^{\frac{1}{2},t^2}\mathfrak{f}^{\alpha,l}\right)(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (s^2)'(t^2 - s^2)^{\frac{1}{2}-1}(-3)\ ds = \frac{-6t}{\Gamma\left(\frac{1}{2}\right)} = \frac{-6}{\sqrt{\pi}}t,$$

$$\left(I_{0^+,gH}^{\frac{1}{2},t^2}\mathfrak{f}^{\alpha,r}\right)(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (s^2)'(t^2 - s^2)^{\frac{1}{2}-1}0\ ds = 0.$$

Therefore, the intuitionistic fuzzy $\psi-$fractional integral of f is given by :

$$\left(I_{0^+,gH}^{\frac{1}{2},t^2}\mathfrak{f}\right)(t) = \left[0\ ,\ \frac{t^2}{2\sqrt{\pi}}\ ;\ \frac{-6t}{\sqrt{\pi}}\ ,\ 0\right].$$

Proposition 2. Let $\mathfrak{f}, \mathfrak{h} \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$ be an intuitionnistic fuzzy-valued function and $\lambda \in \mathbb{F}$, then the intuitionistic fuzzy $\psi-$fractional integral check the following properties:

1) $\left(I_{0^+,gH}^{\gamma,\psi}(\lambda\mathfrak{f})\right)(t) = \lambda\left(I_{0^+,gH}^{\gamma,\psi}\mathfrak{f}\right)(t).$

2) $\left(I_{0^+,gH}^{\gamma,\psi}(\mathfrak{f} + \mathfrak{h})\right)(t) = \left(I_{0^+,gH}^{\gamma,\psi}\mathfrak{f}\right)(t) + \left(I_{0^+,gH}^{\gamma,\psi}\mathfrak{h}\right)(t).$

3) $\left(I_{0^+,gH}^{\gamma,\psi}I_{0^+,gH}^{\delta,\psi}\mathfrak{f}\right)(t) = \left(I_{0^+,gH}^{\gamma+\delta,\psi}\mathfrak{f}\right)(t)\ ,\ \forall\gamma, \delta \in (0, 1).$

Proof. Let $\mathfrak{f},\mathfrak{h}\in C^{\mathbb{F}}\cap L^{\mathbb{F}}$ with:

$$[\mathfrak{f}(t)]_\alpha=[\mathfrak{f}_{\alpha,l}(t),\mathfrak{f}_{\alpha,r}(t)]\ ,\ [\mathfrak{f}(t)]^\alpha=\left[\mathfrak{f}^{\alpha,l}(t),\mathfrak{f}^{\alpha,r}(t)\right],$$

and,

$$[\mathfrak{h}(t)]_\alpha=[\mathfrak{h}_{\alpha,l}(t),\mathfrak{h}_{\alpha,r}(t)]\ ,\ [\mathfrak{h}(t)]^\alpha=\left[\mathfrak{h}^{\alpha,l}(t),\mathfrak{h}^{\alpha,r}(t)\right],$$

and let $\lambda\in\mathbb{F}$, then:

1) We have:

$$\begin{aligned}\left[I^{\gamma,\psi}_{0^+,gH}(\lambda\mathfrak{f})\right]_\alpha(t)&=\left[\left(I^{\gamma,\psi}_{0^+,gH}(\lambda\mathfrak{f})_{\alpha,l}\right)(t),\left(I^{\gamma,\psi}_{0^+,gH}(\lambda\mathfrak{f})_{\alpha,r}\right)(t)\right]\\&=\left[\lambda\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,l}\right)(t),\left(\lambda I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,r}\right)(t)\right]\\&=\lambda\left[\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,l}\right)(t),\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,r}\right)(t)\right]=\lambda\left[I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}\right]_\alpha(t),\end{aligned}$$

and,

$$\begin{aligned}\left[I^{\gamma,\psi}_{0^+,gH}(\lambda\mathfrak{f})\right]^\alpha(t)&=\left[\left(I^{\gamma,\psi}_{0^+,gH}(\lambda\mathfrak{f})^{\alpha,l}\right)(t),\left(I^{\gamma,\psi}_{0^+,gH}(\lambda\mathfrak{f})^{\alpha,r}\right)(t)\right]\\&=\left[\lambda\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,l}\right)(t),\left(\lambda I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,r}\right)(t)\right]\\&=\lambda\left[\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,l}\right)(t),\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,r}\right)(t)\right]=\lambda\left[I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}\right]^\alpha(t).\end{aligned}$$

Therefore, $\left(I^{\gamma,\psi}_{0^+,gH}(\lambda\mathfrak{f})\right)(t)=\lambda\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}\right)(t)$.

2) Using the same way as 1 we have:

$$\begin{aligned}&\left[I^{\gamma,\psi}_{0^+,gH}(\mathfrak{f}+\mathfrak{h})\right]_\alpha(t)\\&=\left[\left(I^{\gamma,\psi}_{0^+,gH}(\mathfrak{f}+\mathfrak{h})_{\alpha,l}\right)(t),\left(I^{\gamma,\psi}_{0^+,gH}(\mathfrak{f}+\mathfrak{h})_{\alpha,r}\right)(t)\right]\\&=\left[\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,l}\right)(t)+\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{h}_{\alpha,l}\right)(t),\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,r}\right)(t)+\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{h}_{\alpha,r}\right)(t)\right]\\&=\left[\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,l}\right)(t),\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,r}\right)(t)\right]+\left[\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{h}_{\alpha,l}\right)(t),\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{h}_{\alpha,r}\right)(t)\right]\\&=\left[I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}\right]_\alpha(t)+\left[I^{\gamma,\psi}_{0^+,gH}\mathfrak{h}\right]_\alpha(t),\end{aligned}$$

and,

$$\begin{aligned}&\left[I^{\gamma,\psi}_{0^+,gH}(\mathfrak{f}+\mathfrak{h})\right]^\alpha(t)\\&=\left[\left(I^{\gamma,\psi}_{0^+,gH}(\mathfrak{f}+\mathfrak{h})^{\alpha,l}\right)(t),\left(I^{\gamma,\psi}_{0^+,gH}(\mathfrak{f}+\mathfrak{h})^{\alpha,r}\right)(t)\right]\\&=\left[\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,l}\right)(t)+\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{h}^{\alpha,l}\right)(t),\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,r}\right)(t)+\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{h}^{\alpha,r}\right)(t)\right]\\&=\left[\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,l}\right)(t),\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,r}\right)(t)\right]+\left[\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{h}^{\alpha,l}\right)(t),\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{h}^{\alpha,r}\right)(t)\right]\\&=\left[I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}\right]^\alpha(t)+\left[I^{\gamma,\psi}_{0^+,gH}\mathfrak{h}\right]^\alpha(t).\end{aligned}$$

Hence, $\left(I^{\gamma,\psi}_{0^+,gH}(\mathfrak{f}+\mathfrak{h})\right)(t)=\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}\right)(t)+\left(I^{\gamma,\psi}_{0^+,gH}\mathfrak{h}\right)(t)$.

3) Let $\gamma, \delta \in (0,1)$, we have:

$$\left[I^{\gamma,\psi}_{0^+,gH} I^{\delta,\psi}_{0^+,gH} \mathfrak{f}\right]_\alpha (t) = \left[\left(I^{\gamma,\psi}_{0^+,gH} I^{\delta,\psi}_{0^+,gH} \mathfrak{f}_{\alpha,l}\right)(t), \left(I^{\gamma,\psi}_{0^+,gH} I^{\delta,\psi}_{0^+,gH} \mathfrak{f}_{\alpha,r}\right)(t)\right]$$
$$= \left[\left(I^{\gamma+\delta,\psi}_{0^+,gH} \mathfrak{f}_{\alpha,l}\right)(t), \left(I^{\gamma+\delta,\psi}_{0^+,gH} \mathfrak{f}_{\alpha,r}\right)(t)\right] = \left[I^{\gamma+\delta,\psi}_{0^+,gH} \mathfrak{f}\right]_\alpha (t),$$

and,

$$\left[I^{\gamma,\psi}_{0^+,gH} I^{\delta,\psi}_{0^+,gH} \mathfrak{f}\right]^\alpha (t) = \left[\left(I^{\gamma,\psi}_{0^+,gH} I^{\delta,\psi}_{0^+,gH} \mathfrak{f}^{\alpha,l}\right)(t), \left(I^{\gamma,\psi}_{0^+,gH} I^{\delta,\psi}_{0^+,gH} \mathfrak{f}^{\alpha,r}\right)(t)\right]$$
$$= \left[\left(I^{\gamma+\delta,\psi}_{0^+,gH} \mathfrak{f}^{\alpha,l}\right)(t), \left(I^{\gamma+\delta,\psi}_{0^+,gH} \mathfrak{f}^{\alpha,r}\right)(t)\right] = \left[I^{\gamma+\delta,\psi}_{0^+,gH} \mathfrak{f}\right]^\alpha (t),$$

Then, $\left(I^{\gamma,\psi}_{0^+,gH} I^{\delta,\psi}_{0^+,gH} \mathfrak{f}\right)(t) = \left(I^{\gamma+\delta,\psi}_{0^+,gH} \mathfrak{f}\right)(t)$, $\forall \gamma, \delta \in (0,1)$.

Now, we move on to give the definition of the fractional derivative $\psi-$Caputo and demonstrate some of its properties.

Definition 11. Let $\mathfrak{f} \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$ be an intuitionnistic fuzzy-valued function, then the intuitionistic fuzzy $\psi-$Caputo fractional derivative of order $0 < \gamma < 1$ of $\mathfrak{f}$ is given by:

$${}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (\psi(t) - \psi(s))^{\gamma-1} \odot \mathfrak{f}'_{gH}(s) ds \ , \ \forall t > 0.$$

Theorem 3. *Let $\mathfrak{f} \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$ be an intuitionnistic fuzzy-valued function, then the intuitionistic fuzzy $\psi-$Caputo fractional derivative of order $0 < \gamma < 1$ of $\mathfrak{f}$ can be expressed in the following way:*

i- If $\mathfrak{f}$ is $[(i) - gH]-$differentiable, then:

$$\left[{}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}(t)\right]_\alpha = \left[{}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}_{\alpha,l}(t) \ , \ {}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}_{\alpha,r}(t)\right],$$

$$\left[{}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}(t)\right]^\alpha = \left[{}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}^{\alpha,l}(t) \ , \ {}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}^{\alpha,r}(t)\right].$$

ii- If $\mathfrak{f}$ is $[(ii) - gH]-$differentiable, then:

$$\left[{}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}(t)\right]_\alpha = \left[{}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}_{\alpha,r}(t) \ , \ {}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}_{\alpha,l}(t)\right],$$

$$\left[{}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}(t)\right]^\alpha = \left[{}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}^{\alpha,r}(t) \ , \ {}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}^{\alpha,l}(t)\right].$$

With,

$${}^C D^{\gamma,\psi}_{0^+,gH} \mathfrak{f}_{\alpha,l}(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (\psi(t) - \psi(s))^{\gamma-1} (\mathfrak{f}_{\alpha,l})'(s) ds,$$

$$^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,r}(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (\psi(t)-\psi(s))^{\gamma-1}(\mathfrak{f}_{\alpha,r})'(s)ds,$$

$$^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,l}(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (\psi(t)-\psi(s))^{\gamma-1}(\mathfrak{f}^{\alpha,l})'(s)ds,$$

$$^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}^{\alpha,r}(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (\psi(t)-\psi(s))^{\gamma-1}(\mathfrak{f}^{\alpha,r})'(s)ds.$$

Proof. Let $\mathfrak{f} \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$ be an intuitionistic fuzzy-valued function that is ψ-Caputo (gH)-differentiable of order $0 < \gamma < 1$.

By definition, the intuitionistic fuzzy ψ-Caputo fractional derivative is given by:

$$^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}(t) = I^{1-\gamma,\psi}_{0^+,gH}\left(\frac{d}{dt}\mathfrak{f}(t)\right),$$

where $I^{1-\gamma,\psi}_{0^+,gH}$ denotes the generalized Hukuhara ψ-fractional integral of order $1-\gamma$.

Since $\mathfrak{f}$ is an intuitionistic fuzzy-valued function, it can be represented at each level α by intervals:

$$[\mathfrak{f}(t)]_\alpha = [\mathfrak{f}_{\alpha,l}(t), \mathfrak{f}_{\alpha,r}(t)], \quad [\mathfrak{f}(t)]^\alpha = [\mathfrak{f}^{\alpha,l}(t), \mathfrak{f}^{\alpha,r}(t)].$$

Case (i): If $\mathfrak{f}$ is $(i)-gH$-differentiable, the fractional derivative acts on the endpoints of the intervals in the natural order, so:

$$\left[^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}(t)\right]_\alpha = \left[^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,l}(t), \quad ^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,r}(t)\right].$$

Case (ii): If $\mathfrak{f}$ is $(ii)-gH$-differentiable, the bounds are reversed in the application of the derivative, so:

$$\left[^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}(t)\right]_\alpha = \left[^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,r}(t), \quad ^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}_{\alpha,l}(t)\right].$$

By definition of the generalized Caputo fractional derivative ψ, for each real differentiable function (the bounds of $\mathfrak{f}$), we have:

$$^{C}D^{\gamma,\psi}_{0^+,gH}h(t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t (\psi(t)-\psi(s))^{-\gamma}h'(s)\,ds,$$

with $h = \mathfrak{f}_{\alpha,l}, \mathfrak{f}_{\alpha,r}, \mathfrak{f}^{\alpha,l}, \mathfrak{f}^{\alpha,r}$.

Thus, the fractional fuzzy intuitionistic derivative ψ-Caputo is calculated directly on the bounds of the levels α according to the type of differentiability (i) or (ii), which completes the demonstration.

Example 2. We consider the function $\psi(t) = t$ and $\gamma = \dfrac{1}{2}$.
For the function $\mathfrak{f}$ we consider the following example: $[\mathfrak{f}(t)]_\alpha = [1, t]$ and $[\mathfrak{f}(t)]^\alpha = \left[t^2, -2t\right]$.

Then, the intuitionistic fuzzy $\psi-$Caputo fractional derivative of $\mathfrak{f}$ using the previous theorem is:

$$ {}^{C}D^{\frac{1}{2},t}_{0^+,gH}\mathfrak{f}_{\alpha,l}(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s))^{\frac{1}{2}-1}(1)^{'}ds = 0, $$

$$ {}^{C}D^{\frac{1}{2},t}_{0^+,gH}\mathfrak{f}_{\alpha,r}(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s))^{\frac{1}{2}-1}(s)^{'}ds = \frac{2\sqrt{t}}{\Gamma(\frac{1}{2})} = \frac{2}{\sqrt{\pi}}\sqrt{t}, $$

$$ {}^{C}D^{\frac{1}{2},t}_{0^+,gH}\mathfrak{f}^{\alpha,l}(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s))^{\frac{1}{2}-1}(s^2)^{'}ds = \frac{8\sqrt{t^3}}{3\Gamma(\frac{1}{2})} = \frac{8}{3\sqrt{\pi}}\sqrt{t^3}, $$

$$ {}^{C}D^{\frac{1}{2},t}_{0^+,gH}\mathfrak{f}^{\alpha,r}(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s))^{\frac{1}{2}-1}(-2s)^{'}ds = \frac{-4\sqrt{t}}{\Gamma(\frac{1}{2})} = \frac{-4}{\sqrt{\pi}}\sqrt{t}. $$

Therefore, if $\mathfrak{f}$ is $[(i)-gH]-$differentiable, then:

$$ \left({}^{C}D^{\frac{1}{2},t}_{0^+,gH}\mathfrak{f}(t)\right) = \left[0\ ,\ \frac{2}{\sqrt{\pi}}\sqrt{t}\ ;\ \frac{8}{3\sqrt{\pi}}\sqrt{t^3}\ ,\ \frac{-4}{\sqrt{\pi}}\sqrt{t}\right]. $$

And, if $\mathfrak{f}$ is $[(ii)-gH]-$differentiable, then:

$$ \left({}^{C}D^{\frac{1}{2},t}_{0^+,gH}\mathfrak{f}(t)\right) = \left[\frac{2}{\sqrt{\pi}}\sqrt{t}\ ,\ 0\ ;\ \frac{-4}{\sqrt{\pi}}\sqrt{t}\ ,\ \frac{8}{3\sqrt{\pi}}\sqrt{t^3}\right]. $$

Remark 3. Let $\mathfrak{f} \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$, then:

$$ {}^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}(t) = I^{1-\gamma,\psi}_{0^+,gH}\left(\frac{\mathfrak{f}'_{gH}(t)}{\psi'(t)}\right)\ ,\ \forall t > 0\ ,\ \forall \gamma \in (0,1). $$

Theorem 4. *Let $\mathfrak{f} \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$ be an intuitionnistic fuzzy-valued function, then the intuitionistic fuzzy $\psi-$Caputo fractional derivative check the following properties:*

i- ${}^{C}D^{\gamma,\psi}_{0^+,gH}I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}(t) = \mathfrak{f}(t)\ ,\ \forall q \in (0,1).$

ii- ${}^{C}D^{\gamma,\psi}_{0^+,gH}\ {}^{C}D^{\delta,\psi}_{0^+,gH}\mathfrak{f}(t) = {}^{C}\ D^{\gamma+\delta,\psi}_{0^+,gH}\mathfrak{f}(t)\ ,\ \forall \gamma,\delta \in (0,1).$

iii- ${}^{C}D^{n-\gamma,\psi}_{0^+,gH}\ {}^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}(t) = {}^{C}\ D^{n,\psi}_{0^+,gH}\mathfrak{f}(t)\ ,\ \forall \gamma \in (0,1)\ ,\ \forall n \in \mathbb{N}.$

Proof. Let $\mathfrak{f} \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$ is an intuitionistic fuzzy-valued function,

i- Let $\gamma \in (0,1)$, then:
${}^{C}D^{\gamma,\psi}_{0^+,gH}I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}(t) = {}^{C}\ D^{\gamma,\psi}_{0^+,gH}F(t)$, with $F(t) = I^{\gamma,\psi}_{0^+,gH}\mathfrak{f}(t)$.

According to remark 3, we have: ${}^C D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}(t) = I^{1-\gamma,\psi}_{0^+,gH}\left(\frac{\mathfrak{f}'_{gH}(t)}{\psi'(t)}\right)$.

Then,

$$\begin{aligned}
{}^C D^{\gamma,\psi}_{0^+,gH}F(t) &= I^{1-\gamma,\psi}_{0^+,gH}\left(\frac{F'_{gH}(t)}{\psi'(t)}\right)\\
&= I^{1-\gamma,\psi}_{0^+,gH}I^{\gamma,\psi}_{0^+,gH}\left(\frac{\mathfrak{f}'_{gH}(t)}{\psi'(t)}\right)\\
&= I^{1,\psi}_{0^+,gH}\left(\frac{\mathfrak{f}'_{gH}(t)}{\psi'(t)}\right)\\
&= \frac{1}{\Gamma(1)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{1-1}\odot\left(\frac{\mathfrak{f}'_{gH}(s)}{\psi'(s)}\right)ds\\
&= \int_0^t \mathfrak{f}'_{gH}(s)ds\\
&= f(t),
\end{aligned}$$

ii- Let $\gamma,\delta\in(0,1)$, then:

$$ {}^C D^{\gamma,\psi}_{0^+,gH}\ {}^C D^{\delta,\psi}_{0^+,gH}\mathfrak{f}(t) = {}^C D^{\gamma,\psi}_{0^+,gH}F(t), $$

with, $F(t) = {}^C D^{\delta,\psi}_{0^+,gH}\mathfrak{f}(t)$.
We pose: $1-p = 1-(p+q)+q$, then,

$$\begin{aligned}
F(t) &= {}^C D^{\delta,\psi}_{0^+,gH}\mathfrak{f}(t)\\
&= I^{1-\delta,\psi}_{0^+,gH}\left(\frac{\mathfrak{f}'_{gH}(t)}{\psi'(t)}\right)\\
&= I^{1-(\delta+\gamma)+\gamma,\psi}_{0^+,gH}\left(\frac{\mathfrak{f}'_{gH}(t)}{\psi'(t)}\right)\\
&= I^{1-(\delta+\gamma),\psi}_{0^+,gH}I^{\gamma,\psi}_{0^+,gH}\left(\frac{\mathfrak{f}'_{gH}(t)}{\psi'(t)}\right)\\
&= I^{\gamma,\psi}_{0^+,gH}I^{1-(\delta+\gamma),\psi}_{0^+,gH}\left(\frac{\mathfrak{f}'_{gH}(t)}{\psi'(t)}\right)\\
&= I^{\gamma,\psi}_{0^+,gH}\ {}^C D^{\gamma+\delta,\psi}_{0^+,gH}\mathfrak{f}(t)
\end{aligned}$$

Therefore,

$$\begin{aligned}
{}^C D^{\gamma,\psi}_{0^+,gH}\ {}^C D^{\delta,\psi}_{0^+,gH}\mathfrak{f}(t) &= {}^C D^{\gamma,\psi}_{0^+,gH}F(t)\\
&= {}^C D^{\gamma,\psi}_{0^+,gH}I^{\gamma,\psi}_{0^+,gH}\ {}^C D^{\gamma+\delta,\psi}_{0^+,gH}\mathfrak{f}(t)\\
&= {}^C D^{\gamma+\delta,\psi}_{0^+,gH}\mathfrak{f}(t),
\end{aligned}$$

iii- Let $\gamma \in (0,1)$ and $n \in \mathbb{N}$, then:

$$\begin{aligned} {}^{C}D^{n-\gamma,\psi}_{0^+,gH}\, {}^{C}D^{\gamma,\psi}_{0^+,gH}\mathfrak{f}(t) &= {}^{C}D^{n-\gamma+\gamma,\psi}_{0^+,gH}\mathfrak{f}(t) \\ &= {}^{C}D^{n,\psi}_{0^+,gH}\mathfrak{f}(t), \end{aligned}$$

Definition 12. Let $\mathfrak{f} \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$ is an intuitionistic fuzzy-valued function, then the intuitionistic fuzzy $\psi-$Laplace transform of $\mathfrak{f}$ is represented by:

$$\mathcal{L}_\psi(\mathfrak{f}(t))(\lambda) = \int_0^\infty \psi^{'}(t)e^{-\lambda(\psi(t)-\psi(0))}\mathfrak{f}(t)dt \ , \ \forall\lambda > 0.$$

Now, we observe that the $\psi-$Laplace transform of an intuitionistic fuzzy-valued function $\mathfrak{f}(t)$ can be represented via its $\alpha-$level sets as follows:

$$\begin{aligned} \left[\mathcal{L}_\psi(\mathfrak{f}(t))(\lambda)\right]_\alpha &= \left[\mathcal{L}_\psi(\mathfrak{f}_{\alpha,l}(t))(\lambda), \mathcal{L}_\psi(\mathfrak{f}_{\alpha,r}(t))(\lambda)\right], \\ \left[\mathcal{L}_\psi(\mathfrak{f}(t))(\lambda)\right]^\alpha &= \left[\mathcal{L}_\psi(\mathfrak{f}^{\alpha,l}(t))(\lambda), \mathcal{L}_\psi(\mathfrak{f}^{\alpha,r}(t))(\lambda)\right]. \end{aligned}$$

With,

$$\begin{aligned} \mathcal{L}_\psi(\mathfrak{f}_{\alpha,l}(t))(\lambda) &= \int_0^\infty \psi^{'}(t)e^{-\lambda(\psi(t)-\psi(0))}\mathfrak{f}_{\alpha,l}(t)dt, \\ \mathcal{L}_\psi(\mathfrak{f}_{\alpha,r}(t))(\lambda) &= \int_0^\infty \psi^{'}(t)e^{-\lambda(\psi(t)-\psi(0))}\mathfrak{f}_{\alpha,r}(t)dt, \\ \mathcal{L}_\psi(\mathfrak{f}^{\alpha,l}(t))(\lambda) &= \int_0^\infty \psi^{'}(t)e^{-\lambda(\psi(t)-\psi(0))}\mathfrak{f}^{\alpha,l}(t)dt, \\ \mathcal{L}_\psi(\mathfrak{f}^{\alpha,r}(t))(\lambda) &= \int_0^\infty \psi^{'}(t)e^{-\lambda(\psi(t)-\psi(0))}\mathfrak{f}^{\alpha,r}(t)dt. \end{aligned}$$

Theorem 5. *Let $\mathfrak{f}$ and $\mathfrak{h}$ be two intuitionistic fuzzy-valued functions, a and b be two constants, then:*

$$\mathcal{L}_\psi(a \odot \mathfrak{f}(t) \oplus b \odot \mathfrak{h}(t))(\lambda) = a \odot \mathcal{L}_\psi(\mathfrak{f}(t))(\lambda) \oplus b \odot \mathcal{L}_\psi(\mathfrak{h}(t))(\lambda) \ , \ \forall\lambda > 0.$$

Proof. Let $\mathfrak{f}$ and $\mathfrak{h}$ be two intuitionistic fuzzy-valued functions, a and b be two constants.
For $\lambda > 0$ we have:

$$\begin{aligned} &\mathcal{L}_\psi(a \odot \mathfrak{f}(t) \oplus b \odot \mathfrak{h}(t))(\lambda) \\ &= \int_0^\infty \psi^{'}(t)e^{-\lambda(\psi(t)-\psi(0))}(a \odot \mathfrak{f}(t) \oplus b \odot \mathfrak{h}(t))dt \\ &= \int_0^\infty \psi^{'}(t)e^{-\lambda(\psi(t)-\psi(0))}(a \odot \mathfrak{f}(t)) \oplus \int_0^\infty \psi^{'}(t)e^{-\lambda(\psi(t)-\psi(0))}(b \odot \mathfrak{h}(t))dt \\ &= a \odot \int_0^\infty \psi^{'}(t)e^{-\lambda(\psi(t)-\psi(0))}\mathfrak{f}(t)dt \oplus b \odot \int_0^\infty \psi^{'}(t)e^{-\lambda(\psi(t)-\psi(0))}\mathfrak{h}(t)dt \\ &= a \odot \mathcal{L}_\psi(\mathfrak{f}(t))(\lambda) \oplus b \odot \mathcal{L}_\psi(\mathfrak{h}(t))(\lambda), \end{aligned}$$

4. Intuitionistic fuzzy fractional differential equations

4.1. Intuitionistic fuzzy nonlinear differential equation

Let the following intuitionistic fuzzy nonlinear differential equation:

$$\begin{cases} {}^{C}D^{\gamma,\psi}_{0^+,gH}\varphi(t) = \mathcal{F}(t,\varphi(t)) \quad , \quad t \in I = [0,T] \\ \varphi(0) = \varphi_0 \in \mathbb{F}, \end{cases} \tag{4.1}$$

with, $\gamma \in (0,1)$ and $\mathcal{F} : I \times \mathbb{F} \longrightarrow \mathbb{F}$ a continuous function with an intuitionistic fuzzy value.
For $\alpha \in [0,1]$ we have:

$$[\varphi(t)]_\alpha = [\varphi_{\alpha,l}(t), \varphi_{\alpha,r}(t)] \ , \ [\varphi(t)]^\alpha = \left[\varphi^{\alpha,l}(t), \varphi^{\alpha,r}(t)\right],$$

and,

$$[\mathcal{F}(t,\varphi(t))]_\alpha = [\mathcal{F}_{\alpha,l}(t,\varphi(t)), \mathcal{F}_{\alpha,r}(t,\varphi(t))] \ ,$$
$$[\mathcal{F}(t,\varphi(t))]^\alpha = \left[\mathcal{F}^{\alpha,l}(t,\varphi(t)), \mathcal{F}^{\alpha,r}(t,\varphi(t))\right].$$

Then, we find the following problem:

$$\begin{cases} {}^{C}D^{\gamma,\psi}_{0^+,gH}\varphi_{\alpha,l}(t) = \mathcal{F}_{\alpha,l}(t,\varphi_{\alpha,l}(t)) \quad , \quad t \in I = [0,T] \\ \varphi_{\alpha,l}(0) = \varphi_{0,\alpha,l} \in \mathbb{F}, \\ {}^{C}D^{\gamma,\psi}_{0^+,gH}\varphi_{\alpha,r}(t) = \mathcal{F}_{\alpha,r}(t,\varphi_{\alpha,r}(t)) \quad , \quad t \in I = [0,T] \\ \varphi_{\alpha,r}(0) = \varphi_{0,\alpha,r} \in \mathbb{F}, \\ {}^{C}D^{\gamma,\psi}_{0^+,gH}\varphi^{\alpha,l}(t) = \mathcal{F}^{\alpha,l}(t,\varphi^{\alpha,l}(t)) \quad , \quad t \in I = [0,T] \\ \varphi^{\alpha,l}(0) = \varphi_0^{\alpha,l} \in \mathbb{F}, \\ {}^{C}D^{\gamma,\psi}_{0^+,gH}\varphi^{\alpha,r}(t) = \mathcal{F}^{\alpha,r}(t,\varphi^{\alpha,r}(t)) \quad , \quad t \in I = [0,T] \\ \varphi^{\alpha,r}(0) = \varphi_0^{\alpha,r} \in \mathbb{F}, \end{cases} \tag{4.2}$$

Definition 13. We say that $\varphi(t)$ is a mild solution of problem 4.1 if:

$$\varphi(t) = \varphi_0 \oplus \frac{1}{\Gamma(\gamma)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\gamma-1} \odot \mathcal{F}(s,\varphi(s))ds \ , \ \forall t \in I,$$

and we obtain,

$$\varphi_{\alpha,l}(t) = \varphi_{0,\alpha,l} + \frac{1}{\Gamma(\gamma)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\gamma-1} \mathcal{F}_{\alpha,l}(s,\varphi_{\alpha,l}(s))ds \ , \ \forall t \in I,$$

$$\varphi_{\alpha,r}(t) = \varphi_{0,\alpha,r} + \frac{1}{\Gamma(\gamma)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\gamma-1} \mathcal{F}_{\alpha,r}(s,\varphi_{\alpha,r}(s))ds \ , \ \forall t \in I,$$

$$\varphi^{\alpha,l}(t) = \varphi_0^{\alpha,l} + \frac{1}{\Gamma(\gamma)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\gamma-1} \mathcal{F}^{\alpha,l}(s,\varphi^{\alpha,l}(s))ds \ , \ \forall t \in I,$$

$$\varphi^{\alpha,r}(t) = \varphi_0^{\alpha,r} + \frac{1}{\Gamma(\gamma)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\gamma-1} \mathcal{F}^{\alpha,r}(s,\varphi^{\alpha,r}(s))ds \ , \ \forall t \in I.$$

We pose the following hypotheses:

(H_1) The function $\mathcal{F} : I \times \mathbb{F} \longrightarrow \mathbb{F}$ is continuous and there exists a positive constant $\mathcal{K}$ such that:

$$d_\infty\left(\mathcal{F}(t,\varphi_1(t)),\mathcal{F}(t,\varphi_2(t))\right) \leq \mathcal{K} d_\infty\left(\varphi_1(t),\varphi_2(t)\right)\ ,\ \forall \varphi_1,\varphi_2 \in C^{\mathbb{F}}.$$

(H_2) The quantity verifies the following condition:

$$\frac{\mathcal{K}}{\Gamma(\gamma+1)}(\psi(T)-\psi(0))^{\gamma} < 1.$$

Theorem 6. *Suppose that the two hypotheses* (H_1) *and* (H_2) *are verified, then problem* 4.1 *admits a unique mild solution.*

Proof. Let the operator $\mathcal{P} : C^{\mathbb{F}} \longrightarrow C^{\mathbb{F}}$ defined by:

$$\mathcal{P}(\varphi(t)) = \varphi_0 \oplus \frac{1}{\Gamma(\gamma)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1} \odot \mathcal{F}(s,\varphi(s))ds$$

Let $\varphi \in C^{\mathbb{F}}$, $t \in I$ and ε a very small number, we have:

$$\begin{aligned}
&d_\infty\left(\mathcal{P}(\varphi(t+\varepsilon)),\mathcal{P}(\varphi(t))\right)\\
&\leq d_\infty(\frac{1}{\Gamma(\gamma)}\int_0^{t+\varepsilon} \psi^{'}(s)(\psi(t+\varepsilon)-\psi(s))^{\gamma-1} \odot \mathcal{F}(s,\varphi(s))ds,\\
&\frac{1}{\Gamma(\gamma)}\int_0^{t} \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1} \odot \mathcal{F}(s,\varphi(s))ds)\\
&\leq d_\infty\left(\frac{1}{\Gamma(\gamma)}\int_0^{\varepsilon} \psi^{'}(s)(\psi(t+\varepsilon)-\psi(s))^{\gamma-1} \odot \mathcal{F}(s,\varphi(s))ds,\tilde{0}\right)\\
&+ d_\infty(\frac{1}{\Gamma(\gamma)}\int_\varepsilon^{t+\varepsilon} \psi^{'}(s)(\psi(t+\varepsilon)-\psi(s))^{\gamma-1} \odot \mathcal{F}(s,\varphi(s))ds,\\
&\frac{1}{\Gamma(\gamma)}\int_0^{t} \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1} \odot \mathcal{F}(s,\varphi(s))ds)\\
&\leq d_\infty\left(\frac{1}{\Gamma(\gamma)}\int_0^{\varepsilon} \psi^{'}(s)(\psi(t+\varepsilon)-\psi(s))^{\gamma-1} \odot \mathcal{F}(s,\varphi(s))ds,\tilde{0}\right)\\
&+ d_\infty(\frac{1}{\Gamma(\gamma)}\int_0^{t} \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1} \odot \mathcal{F}(s,\varphi(s))ds,\\
&\frac{1}{\Gamma(\gamma)}\int_0^{t} \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1} \odot \mathcal{F}(s,\varphi(s))ds),
\end{aligned}$$

and we have,

$$d_\infty\left(\frac{1}{\Gamma(\gamma)}\int_0^{\varepsilon} \psi^{'}(s)(\psi(t+\varepsilon)-\psi(s))^{\gamma-1} \odot \mathcal{F}(s,\varphi(s))ds,\tilde{0}\right) \longrightarrow 0,$$

$$d_\infty(\frac{1}{\Gamma(\gamma)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1}\odot \mathcal{F}(s,\varphi(s))ds,$$
$$\frac{1}{\Gamma(\gamma)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1}\odot \mathcal{F}(s,\varphi(s))ds) \longrightarrow 0.$$

Let $\varphi_1,\varphi_2 \in C^{\mathbb{F}}$, then:

$$\begin{aligned}
&d_\infty\left(\mathcal{P}(\varphi_1(t)),\mathcal{P}(\varphi_2(t))\right)\\
&\quad\leq \frac{1}{\Gamma(\gamma)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1} d_\infty\left(\mathcal{F}(s,\varphi_1(s)),\mathcal{F}(s,\varphi_2(s))\right)ds\\
&\quad\leq \frac{\mathcal{K}}{\Gamma(\gamma)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1} d_\infty\left(\varphi_1(s),\varphi_2(s)\right)ds\\
&\quad\leq \frac{\mathcal{K}}{\Gamma(\gamma+1)}(\psi(T)-\psi(0))^{\gamma}\sup_{s\in I} d_\infty\left(\varphi_1(s),\varphi_2(s)\right),
\end{aligned}$$

Since, according to hypothesis (H_2) we have:

$$\frac{\mathcal{K}}{\Gamma(\gamma+1)}(\psi(T)-\psi(0))^{\gamma} < 1.$$

Then, the operator $\mathcal{P}$ is a contraction, therefore admits a unique fixed point φ which is the unique mild solution of the problem 4.1.

4.2. Intuitionistic fuzzy fractional linear evolution problem

Let the following Intuitionistic fuzzy fractional linear evolution problem:

$$\begin{cases} {}^{C}D^{\gamma,\psi}_{0^+,gH}\varphi(t) = \mathcal{A}(t)\varphi(t) \quad , \quad t\in I=[0,T]\\ \varphi(0)=\varphi_0\in\mathbb{F}, \end{cases} \tag{4.3}$$

with, $\gamma\in(0,1)$ and $\mathcal{A}(t)$ a bounded linear operator.
For $\alpha\in[0,1]$, we have:

$$[\varphi(t)]_\alpha = [\varphi_{\alpha,l}(t),\varphi_{\alpha,r}(t)] \ , \ [\varphi(t)]^\alpha = \left[\varphi^{\alpha,l}(t),\varphi^{\alpha,r}(t)\right],$$

Definition 14. We say that $\varphi(t)$ is a mild solution of problem 4.3 if and only if:

i- $\varphi(t)$ is continuous and $\varphi(t)\in D(\mathcal{A}(t))$, $\forall t\in I$.

ii- $\varphi(t)$ is given by the following formula:

$$\varphi(t) = \varphi_0 \oplus \frac{1}{\Gamma(\gamma)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1}\odot \mathcal{A}(s)\varphi(s)ds,$$

that's to say, for all $\alpha \in [0,1]$:

$$\varphi_{\alpha,l}(t) = \varphi_{0,\alpha,l} + \frac{1}{\Gamma(\gamma)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1}\mathcal{A}(s)\varphi_{\alpha,l}(s)ds,$$

$$\varphi_{\alpha,r}(t) = \varphi_{0,\alpha,r} + \frac{1}{\Gamma(\gamma)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1}t\mathcal{A}(s)\varphi_{\alpha,r}(s)ds,$$

$$\varphi^{\alpha,l}(t) = \varphi_0^{\alpha,l} + \frac{1}{\Gamma(\gamma)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1}\mathcal{A}(s)\varphi^{\alpha,l}(s)ds,$$

$$\varphi^{\alpha,r}(t) = \varphi_0^{\alpha,r} + \frac{1}{\Gamma(\gamma)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1}\mathcal{A}(s)\varphi^{\alpha,r}(s)ds.$$

We pose the following hypotheses:

(H_1) $\mathcal{A}(t)$ a bounded linear operator in $\mathbb{F}$.

(H_2) The function $t \longmapsto \mathcal{A}(t)$ is continuous.

(H_3) Let $\mathcal{A}_0 = \sup_{s\in I} d_\infty \left(\mathcal{A}(s), \tilde{0}\right)$.Then, we have the following condition:

$$\frac{\mathcal{A}_0}{\Gamma(\gamma+1)}(\psi(T)-\psi(0))^\gamma < 1,$$

Theorem 7. *Suppose that the hypotheses $(H_1) - (H_3)$ are verified, then problem 4.3 admits a unique mild solution.*

Proof. Let the operator $\mathcal{R} : C^{\mathbb{F}} \longrightarrow C^{\mathbb{F}}$ defined by:

$$\mathcal{R}(\varphi(t)) = \varphi_0 \oplus \frac{1}{\Gamma(\gamma)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1} \odot \mathcal{A}(s)\varphi(s)ds$$

Suppose that both hypotheses (H_1) and (H_2) are verified.
Let $\varphi_1, \varphi_2 \in C^{\mathbb{F}}$, then:

$$\begin{aligned}
&d_\infty \left(\mathcal{R}(\varphi_1(t)), \mathcal{R}(\varphi_2(t))\right) \\
&\quad \le \frac{1}{\Gamma(\gamma)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1} d_\infty \left(\mathcal{A}(s)\varphi_1(s), \mathcal{A}(s)\varphi_2(s)\right) ds \\
&\quad \le \frac{\mathcal{A}_0}{\Gamma(\gamma)} \int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\gamma-1} d_\infty \left(\varphi_1(s), \varphi_2(s)\right) ds \\
&\quad \le \frac{\mathcal{A}_0}{\Gamma(\gamma+1)}(\psi(T)-\psi(0))^\gamma \sup_{s\in I} d_\infty \left(\varphi_1(s), \varphi_2(s)\right),
\end{aligned}$$

Since, according to hypothesis (H_3) we have:

$$\frac{\mathcal{A}_0}{\Gamma(\gamma+1)}(\psi(T)-\psi(0))^\gamma < 1.$$

Then, the operator $\mathcal{R}$ is a contraction, therefore admits a unique fixed point φ which is the unique mild solution of the problem 4.3.

5. Application

5.1. Application 1

We consider the following example of the fractional differential equation 4.1:

$$\begin{cases} {}^{C}D_{0^+,gH}^{\frac{1}{2},ln(t+1)}\varphi(t) = \mathcal{F}(t,\varphi(t)) \quad , \quad t \in I = [0,T] \\ \varphi(0) = \varphi_0 \in \mathbb{F}, \end{cases} \tag{5.1}$$

with, $\gamma = \dfrac{1}{2}$, $\psi(t) = ln(t+1)$, $T = 1$ and the fuction $\mathcal{F}(t,\varphi(t)) = \dfrac{2}{3}\varphi(t)$.
For hypothesis (H_1), we have:
Let $\varphi_1, \varphi_2 \in C^{\mathbb{F}}$, then:

$$\begin{aligned} d_\infty\left(\mathcal{F}(t,\varphi_1(t)), \mathcal{F}(t,\varphi_2(t))\right) &= d_\infty\left(\frac{2}{3}\varphi_1(t), \frac{2}{3}\varphi_2(t)\right) \\ &\leq \frac{2}{3} d_\infty\left(\varphi_1(t), \varphi_2(t)\right), \end{aligned}$$

Hence, $\mathcal{K} = \dfrac{2}{3}$.
For hypothesis (H_2), we have:

$$\begin{aligned} \frac{\mathcal{K}}{\Gamma(\gamma+1)}(\psi(T)-\psi(0))^\gamma &= \frac{\frac{2}{3}}{\Gamma(\frac{1}{2}+1)}(\psi(1)-\psi(0))^{\frac{1}{2}} \\ &= \frac{4}{3\sqrt{\pi}}(ln(2)-ln(1))^{\frac{1}{2}} \\ &\simeq 0,62 < 1, \end{aligned}$$

Therefore, according to theorem 6, problem 5.1 admits a unique mild solution.

5.2. Application 2

We pose the following intuitionistic fuzzy evolution problem:

$$\begin{cases} {}^{C}D_{0^+,gH}^{\frac{1}{2},\sqrt{t}}\varphi(t) = \mathcal{A}(t)\varphi(t) \quad , \quad t \in I = [0,1] \\ \varphi(0) = \varphi_0 \in \mathbb{F}, \end{cases} \tag{5.2}$$

with, $\gamma = \dfrac{1}{2}$, $\psi(t) = \sqrt{t}$, $T = 1$ and the operator $\mathcal{A}(t) = \dfrac{1}{4}e^t id$, where id is the identity function defined on $\mathbb{F}$.
We have $t \longmapsto e^t$ is continuous, then the function $t \longmapsto \mathcal{A}(t)$ is continuous.
Therefore, the hypotheses (H_1) and (H_2) are verified.

We have: $\mathcal{A}_0 = \dfrac{1}{4}$.
Then, for hypothesis (H_3), we have:

$$\begin{aligned}
\frac{\mathcal{A}_0}{\Gamma(\gamma+1)}(\psi(T)-\psi(0))^{\gamma} &= \frac{\frac{1}{4}}{\Gamma(\frac{1}{2}+1)}(\psi(1)-\psi(0))^{\frac{1}{2}} \\
&= \frac{\frac{1}{4}}{\frac{1}{2}\sqrt{\pi}} \\
&\simeq 0,28 < 1,
\end{aligned}$$

Then, according to the theorem 7 the problem 5.2 admits a unique mild solution.

6. Conclusion

In this document, we have given the definition of the fractional derivative $\psi-$Caputo in the intuitionistic fuzzy case and the same for the $\psi-$fractional integral and the $\psi-$Laplace transform and demonstrated some of their properties. After which we have studied the existence and the uniqueness of the solution of the intuitionistic fuzzy nonlinear differential equation and the intuitionistic fuzzy linear evolution problem under the $\psi-$Caputo derivative, and at the end we proposed an example of application for each of the two problems.

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