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Error bounds for numerical differentiation using kernels of finite smoothness

Dedicated to the bright memory of Vitalii Pavlovych Motornyi

Abstract. We provide improved error bounds for kernel-based numerical differentiation in terms of growth functions when kernels are of a finite smoothness, such as polyharmonic splines, thin plate splines or Wendland kernels. In contrast to existing literature, the new estimates take into account the Hölder class smoothness of kernel's derivatives, which helps to improve the order of the estimate. In addition, the new estimates apply to certain deficient point sets, relaxing a standard assumption that an approximation with conditionally positive definite kernels must rely on determining sets for polynomials.
Key words: numerical differentiation, meshless methods, kernel-based method, growth function, deficient sets, polyharmonic splines, thin plate splines, Wendland kernels

Анотація. Ми пропонуємо покращені межі похибки для чисельного диференціювання на основі ядер у термінах функцій зростання, коли ядра мають скінченну гладкість, такі як полігармонічні сплайни, сплайни тонкої пластини або ядра Вендленда. На відміну від наявних результатів у літературі, нові оцінки враховують гладкість похідних ядра за класом Гельдера, що допомагає покращити порядок оцінки. Крім того, нові оцінки застосовуються до деяких дефіцитних множин точок, послаблюючи стандартне припущення, що наближення з умовно додатно визначеними ядрами повинно спиратися на визначаючі множини для поліномів.
Ключові слова: чисельне диференціювання, безсіткові методи, метод на основі ядра, функція зростання, дефіцитні множини, полігармонічні сплайни, сплайни тонкої пластини, ядра Вендленда

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1. Introduction

Let

$$K : \Omega \times \Omega \to \mathbb{R}$$

be a symmetric conditionally positive definite (cpd) kernel on $\Omega \subset \mathbb{R}^d$ of order $s \geq 0$, see for example [9], and D a linear differential operator of order $k \geq 0$ in d real variables

$$Df = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha f, \quad \partial^\alpha := \frac{\partial^{|\alpha|}}{\partial \mathbf{x}^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_d, \tag{1.1}$$

with variable coefficients a_α. In particular, for $k = 0$ and $a_{(0,\dots,0)} = 1$, D is the identity operator $Df = f$.

For the operator D or any other operator or functional A applicable to real-valued function of d variables, we will use the notation $A'K$ and $A''K$ to denote the result of applying A to the first, respectively, second d-dimensional argument of the kernel K. If one operator is applied to the first and another to the second argument of K, then we write for example $A'B''K$ when first B was applied to the second argument, and then A to the first. For the partial derivative operators ∂^α that are applied repeatedly to the first and second arguments of K, we use a simplified notation

$$\partial^{\alpha,\beta} K(\mathbf{x},\mathbf{y}) := \frac{\partial^{|\alpha|}}{\partial \mathbf{x}^\alpha} \frac{\partial^{|\beta|}}{\partial \mathbf{y}^\beta} K(\mathbf{x},\mathbf{y}) = \frac{\partial^{|\alpha|+|\beta|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d} \partial y_1^{\alpha_1} \cdots \partial y_d^{\alpha_d}} K(\mathbf{x},\mathbf{y})$$

whenever the ordering of partial differentiation does not influence the result.

Given a point $\mathbf{z} \in \mathbb{R}^d$ and a finite set $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \Omega$, a *kernel-based numerical differentiation formula*

$$Df(\mathbf{z}) \approx \sum_{j=1}^N w_j^* \, f(\mathbf{x}_j) \tag{1.2}$$

with weight vector $\mathbf{w}^* = [w_j^*]_{j=1}^N$ is obtained by solving the linear system

$$\sum_{j=1}^N w_j K(\mathbf{x}_i, \mathbf{x}_j) + \sum_{j=1}^M v_j p_j(x_i) = D'K(\mathbf{z}, \mathbf{x}_i), \qquad 1 \leq i \leq N, \tag{1.3}$$

$$\sum_{j=1}^N w_j p_i(\mathbf{x}_j) \qquad = Dp_i(\mathbf{z}), \qquad 1 \leq i \leq M, \tag{1.4}$$

with unknowns $w_1 \dots, w_N$ and $v_1 \dots, v_M$, where $p_1, \dots, p_M$ is a basis for the space Π_s^d of d-variate polynomials of total degree at most $s-1$, such that $M = \binom{d+s-1}{d}$. In the case $s = 0$ the kernel is strictly positive definite and we set $\Pi_s^d = \{0\}$ and $M = 0$.

It has been shown in [3] that the weights $w_j^* = w_j$, $j = 1, \dots, N$, are uniquely determined by (1.3)–(1.4) as soon as (1.4) is consistent, that is, there is at least one vector $\mathbf{w} \in \mathbb{R}^N$ satisfying (1.4). This extends the more restrictive condition that the set $\mathbf{X}$ is a determining set for Π_s^d, in which case the right hand side of formula (1.2) expresses the values at $\mathbf{z}$ of D applied to the kernel-based interpolant of f, see [4]. Usually, (1.4) has many solutions, but only one of them satisfies (1.3). The coefficients v_j may not be uniquely determined, but they are not needed for numerical differentiation.

Accurate numerical differentiation formulas of type (1.2) are needed in the meshless finite difference methods for partial differential equations, such as RBF-FD, see for example [7]. Asymptotic behavior of the error of (1.2) when s is fixed and $N \to \infty$ is usually estimated in terms of so-called *fill distance* of the set $\mathbf{X}$ in Ω [9]. Applications to meshless finite difference methods however requires estimates applicable to N not significantly exceeding M [1]. Such estimates using a *growth function* instead of the fill distance have been proposed in [4]. However, as we will see in Section 3 the results of [4] provide suboptimal error estimates in the case when the kernels are of a finite smoothness. In this paper we improve the error estimates of [4] by taking into account the fact that the highest order continuous partial derivatives of such kernels belong to certain Hölder smoothness classes. Moreover, we show that the error bounds remain valid under the less restrictive assumption of solvability of (1.4), which allows to apply them to certain deficient sets as demonstrated numerically in [3].

The paper is organized as follows. Section 2 is devoted to the main general results that are subsequently applied in Section 3 to the particularly important translation-invariant kernels of finite smoothness, with full details provided for polyharmonic splines, thin plate splines and Wendland kernels.

2. Main results

We start by summarizing some basic facts about kernels and numerical differentiation formulas that will be needed in what follows.

Theorems 1 and 2 in [3] imply the following statement.

Proposition 1. Let D be a linear differential operator of order k and let K be a cpd kernel of order s. The kernel–based numerical differentiation formula (1.2) is well defined as long as $D'K(\mathbf{z}, \mathbf{x}_i)$ exists for all $i = 1 \dots, N$, and the linear system (1.4) is consistent. The formula (1.2) is exact for all functions f of the form

$$\sum_{i=1}^{N} a_i K(\cdot, \mathbf{x}_i) + \sum_{i=1}^{M} b_i p_i, \quad a_i, b_i \in \mathbb{R}, \tag{2.1}$$

satisfying

$$\sum_{j=1}^{N} a_j p_i(\mathbf{x}_j) = 0, \quad 1 \le i \le M. \tag{2.2}$$

In particular, (1.2) is exact for all polynomials in Π_s^d.

Note that exactness for polynomials is obvious from (1.4).

Let $\mathcal{F}_K$ denote the *native space* of the kernel K [8, 9]. It is a semi-Hilbert space of real valued functions on Ω arising as completion of the space of functions of the form (2.1) satisfying (2.2). Moreover, $\mathcal{F}_K$ is the direct sum of a Hilbert space and the polynomial space $\Pi_s^d \subset \ker \mathcal{F}_K$. We denote by $(f, g)_K$ and $\|f\|_K = (f, f)_K^{1/2}$ the inner product and the (semi-)norm of $\mathcal{F}_K$. We also denote by $\|\lambda\|_K$ the standard norm of any bounded linear functional $\lambda \in \mathcal{F}_K^*$.

In the case $s = 0$ the strictly positive definite kernel K is the reproducing kernel of $\mathcal{F}_K$. Otherwise, K possesses a weaker reproduction property

$$\lambda f = (f, \lambda'' K)_K \quad \text{for all } \lambda \in \mathcal{L}(\Pi_s^d, \Omega) \tag{2.3}$$

where $\mathcal{L}(\Pi_s^d, \Omega)$ consists of all linear functionals $\lambda \in \mathcal{F}_K^*$ of the form $\lambda f = \sum_{j=1}^N a_j f(\mathbf{x}_j)$ for arbitrary $\mathbf{X} \subset \Omega$ and coefficients a_j such that (2.2) holds, that is $\lambda p = 0$ for all $p \in \Pi_s^d$. This property follows for example from [8, Theorem 6.1] since for the auxiliary kernel Ψ used there the difference $\lambda'' K - \lambda'' \Psi$ belongs to Π_s^d and hence it is orthogonal to any element f of $\mathcal{F}_K$.

Using (2.3) we can argue as in [4, Section 2] to prove the following result.

Proposition 2. Let D be a linear differential operator of order k, and K a cpd kernel of order s. The error functional

$$\varepsilon_{\mathbf{w}} : f \mapsto Df(\mathbf{z}) - \sum_{j=1}^N w_j f(\mathbf{x}_j) \tag{2.4}$$

of any numerical differentiation formula satisfying $\varepsilon_{\mathbf{w}} p = 0$ for all $p \in \Pi_s^d$ is continuous on $\mathcal{F}_K$ if $\partial^{\alpha,\beta} K(\mathbf{z}, \mathbf{z})$ exists for all $|\alpha|, |\beta| \leq k$. Moreover,

$$\|\varepsilon_{\mathbf{w}}\|_K^2 = \varepsilon'_{\mathbf{w}} \varepsilon''_{\mathbf{w}} K. \tag{2.5}$$

Note that the only improvement in comparison to [4, Lemma 4] is that we do not require from $\mathbf{X}$ to be a determining set for Π_s^d.

Similarly, without changing its proof we obtain a slight improvement of [4, Lemma 6]:

Proposition 3. Let D be a linear differential operator of order k, and K a cpd kernel of order s on $\Omega \subset \mathbb{R}^d$. For $\mathbf{z} \in \Omega$ and $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \Omega$, assume that

(a) $\partial^{\alpha,\beta} K(\mathbf{z}, \mathbf{z})$ exists for all $|\alpha|, |\beta| \leq k$,

(b) $D' K(\mathbf{z}, \mathbf{x}_i)$ exists for all $i = 1 \dots, N$, and

(c) the linear system (1.4) is consistent.

Then the kernel-based numerical differentiation formula (1.2) satisfies

$$\Big| Df(\mathbf{z}) - \sum_{j=1}^N w_j^* f(\mathbf{x}_j) \Big| \leq P_{K,D,\mathbf{X},\mathbf{z}} \, \|f\|_K \quad \text{for all } f \in \mathcal{F}_K, \tag{2.6}$$

where $P_{K,D,\mathbf{X},\mathbf{z}}$ is the *power function* defined by

$$P_{K,D,\mathbf{X},\mathbf{z}}^2 = \min \Big\{ Q_{K,D,\mathbf{X},\mathbf{z}}(\mathbf{w}) : \mathbf{w} \in \mathbb{R}^N, \ \varepsilon_{\mathbf{w}} p = 0 \text{ for all } p \in \Pi_s^d \Big\}, \tag{2.7}$$

with

$$Q_{K,D,\mathbf{X},\mathbf{z}}(\mathbf{w}) := \varepsilon'_{\mathbf{w}} \varepsilon''_{\mathbf{w}} K. \tag{2.8}$$

Note that the formula (1.2) is well defined by Proposition 1.

In view of Proposition 2, $Q_{K,D,\mathbf{X},\mathbf{z}}(\mathbf{w})$ is the square of the worst case error of numerical differentiation using weight vectors $\mathbf{w}$ satisfying the polynomial exactness condition $\varepsilon_{\mathbf{w}} p = 0$ for all $p \in \Pi^d_s$, which we will briefly refer to as $\varepsilon_{\mathbf{w}}(\Pi^d_s) = 0$, for functions in the unit ball of $\mathcal{F}_K$,

$$Q_{K,D,\mathbf{X},\mathbf{z}}(\mathbf{w}) = \sup_{\substack{f\in\mathcal{F}_K\\ \|f\|_K\le 1}} \Big|Df(\mathbf{z}) - \sum_{j=1}^N w_j f(x_j)\Big|^2 \quad \text{if } \varepsilon_{\mathbf{w}}(\Pi^d_s) = 0. \tag{2.9}$$

Moreover, Proposition (3) shows that the kernel-based numerical differentiation formula (1.2) provides the *optimal recovery* of $Df(\mathbf{z})$ for f in the native space $\mathcal{F}_K$ among all numerical differentiation formulas exact for polynomials in Π^d_s, and the power function gives the error of the optimal recovery,

$$P_{K,D,\mathbf{X},\mathbf{z}} = \inf_{\substack{w\in\mathbb{R}^N\\ \varepsilon_{\mathbf{w}}(\Pi^d_s)=0}} \sup_{\substack{f\in\mathcal{F}_K\\ \|f\|_K\le 1}} \Big|Df(\mathbf{z}) - \sum_{j=1}^N w_j f(x_j)\Big|. \tag{2.10}$$

Remark 1. We can easily compute $Q_{K,D,\mathbf{X},\mathbf{z}}(\mathbf{w})$ for any weight vector $\mathbf{w}$,

$$Q_{K,D,\mathbf{X},\mathbf{z}}(\mathbf{w}) = D'D''K(\mathbf{z},\mathbf{z}) - 2\sum_{j=1}^N w_j D'K(\mathbf{z},\mathbf{x}_j) + \sum_{i,j=1}^N w_i w_j K(\mathbf{x}_i,\mathbf{x}_j), \tag{2.11}$$

where we used the identity $D''K(\mathbf{x}_j,\mathbf{z}) = D'K(\mathbf{z},\mathbf{x}_j)$ that holds in view of the symmetry of K. For $\mathbf{w} = \mathbf{w}^*$ the formula simplifies further to

$$P^2_{K,D,\mathbf{X},\mathbf{z}} = Q_{K,D,\mathbf{X},\mathbf{z}}(\mathbf{w}^*) = D'D''K(\mathbf{z},\mathbf{z}) - \sum_{j=1}^N w^*_j D'K(\mathbf{z},\mathbf{x}_j) \tag{2.12}$$

since

$$D'K(\mathbf{z},\mathbf{x}_j) = \sum_{i=1}^N w^*_i K(\mathbf{x}_j,\mathbf{x}_i) + \sum_{i=1}^M v_i p_i(x_j)$$

and therefore

$$\sum_{j=1}^N w^*_j D'K(\mathbf{z},\mathbf{x}_j) = \sum_{i,j=1}^N w^*_i w^*_j K(\mathbf{x}_i,\mathbf{x}_j)$$

by (1.3)–(1.4).

We now extend the upper bound for $Q_{K,D,\mathbf{X},\mathbf{z}}(\mathbf{w})$ given in [4, Lemma 7] to kernels with weaker smoothness assumptions and all sets $\mathbf{X}$ for which (1.4) is consistent.

Let

$$S_{\mathbf{z},\mathbf{X}} := \bigcup_{i=1}^N [\mathbf{z},\mathbf{x}_i].$$

For any function $U : S_{\mathbf{z},\mathbf{X}} \times S_{\mathbf{z},\mathbf{X}} \to \mathbb{R}$, we set

$$\big|U\big|_{\mathbf{z},\mathbf{X},\gamma} := \sup_{\substack{\mathbf{x},\mathbf{y}\in S_{\mathbf{z},\mathbf{X}}\\ \mathbf{x}\neq\mathbf{z},\,\mathbf{y}\neq\mathbf{z}}} \frac{\big|U(\mathbf{x},\mathbf{y})-U(\mathbf{x},\mathbf{z})-U(\mathbf{z},\mathbf{y})+U(\mathbf{z},\mathbf{z})\big|}{\|\mathbf{x}-\mathbf{z}\|_2^\gamma\,\|\mathbf{y}-\mathbf{z}\|_2^\gamma}, \quad 0<\gamma\le 1,$$

and for $r\in\mathbb{N}$, $0<\gamma\le 1$, we set

$$|U|_{\mathbf{z},\mathbf{X},r+\gamma} := \frac{1}{(\gamma+1)^2\cdots(\gamma+r)^2}\Big(\sum_{|\alpha|=r}\sum_{|\beta|=r}\binom{r}{\alpha}\binom{r}{\beta}\big|\partial^{\alpha,\beta}U\big|^2_{\mathbf{z},\mathbf{X},\gamma}\Big)^{1/2},$$

where $\binom{r}{\alpha} := \frac{r!}{\alpha!}$, $\alpha! := \alpha_1!\cdots\alpha_d!$, whenever U is defined on $\Omega\times\Omega$ for a domain Ω containing $S_{\mathbf{z},\mathbf{X}}$, and derivatives $\partial^{\alpha,\beta}U(\mathbf{x},\mathbf{y})$, with $|\alpha|=r$ and $|\beta|=r$ exist for all $\mathbf{x},\mathbf{y}\in S_{\mathbf{z},\mathbf{X}}$.

Proposition 4. Let D be a linear differential operator of order k, and K a cpd kernel of order s on $\Omega\subset\mathbb{R}^d$. Given $\mathbf{z}\in\Omega$, $\mathbf{X}=\{\mathbf{x}_1,\dots,\mathbf{x}_N\}$, with $S_{\mathbf{z},\mathbf{X}}\subset\Omega$, and $\mathbf{w}\in\mathbb{R}^N$, assume that for some integer $q\ge\max\{s,k+1\}$ and m such that $k\le m\le q-1$,

(a) K possesses continuous partial derivatives $\partial^{\alpha,\beta}K(\mathbf{x},\mathbf{y})$ for all $|\alpha|,|\beta|\le m$ and all $\mathbf{x},\mathbf{y}\in S_{\mathbf{z},\mathbf{X}}$,

(b) $|K|_{\mathbf{z},\mathbf{X},m+\gamma}<\infty$, for some $\gamma\in(0,1]$, and

(c) $\mathbf{w}$ satisfies $Dp(\mathbf{z})=\sum_{j=1}^N w_j p(\mathbf{x}_j)$ for all $p\in\Pi_q^d$.

Then

$$Q_{K,D,\mathbf{X},\mathbf{z}}(\mathbf{w}) \le \Big(\sum_{i=1}^N |w_i|\,\|\mathbf{x}_i-\mathbf{z}\|_2^{m+\gamma}\Big)^2 |K|_{\mathbf{z},\mathbf{X},m+\gamma}. \tag{2.13}$$

Proof. Observe that $Q_{K,D,\mathbf{X},\mathbf{z}}(\mathbf{w})$, given by (2.8), is well defined because all assumptions of Proposition 3 are satisfied. In particular, (1.4) is consistent thanks to (c) because $q\ge s$.

We adapt in part the proofs of [4, Lemma 7] and [5, Lemma 2]. Denote by $T_{q,\mathbf{z}}f$ the Taylor polynomial of degree m of a function $f:\Omega\to\mathbb{R}$ centered at $\mathbf{z}$,

$$T_{m,\mathbf{z}}f(\mathbf{x}) = \sum_{|\alpha|\le m}\frac{(\mathbf{x}-\mathbf{z})^\alpha}{\alpha!}\partial^\alpha f(\mathbf{z}).$$

As in [5, Lemma 2], we see that for any point $\mathbf{x}\in S_{\mathbf{z},\mathbf{X}}$ the remainder

$$R_{m,\mathbf{z}}f(\mathbf{x}) = f(\mathbf{x}) - T_{m,\mathbf{z}}f(\mathbf{x})$$

can be represented as

$$R_{m,\mathbf{z}}f(\mathbf{x}) = m\sum_{|\alpha|=m}\frac{(\mathbf{x}-\mathbf{z})^\alpha}{\alpha!}\int_0^1(1-t)^{m-1}[\partial^\alpha f(\mathbf{z}+t(\mathbf{x}-\mathbf{z}))-\partial^\alpha f(\mathbf{z})]\,dt,$$

as long as f is m times continuously differentiable everywhere in $S_{\mathbf{z},\mathbf{X}}$. When an expression $U(\mathbf{x},\mathbf{y})$ depends on two d-dimensional variables $\mathbf{x}$ and $\mathbf{y}$, we write

$$T^{\mathbf{x}}_{m,\mathbf{z}}U(\mathbf{x},\mathbf{y}) = \sum_{|\alpha|\le m} \frac{(\mathbf{x}-\mathbf{z})^\alpha}{\alpha!}\partial^{\alpha,0}U(\mathbf{z},\mathbf{y}),$$

respectively,

$$T^{\mathbf{y}}_{m,\mathbf{z}}U(\mathbf{x},\mathbf{y}) = \sum_{|\alpha|\le m} \frac{(\mathbf{y}-\mathbf{z})^\alpha}{\alpha!}\partial^{0,\alpha}U(\mathbf{x},\mathbf{z}),$$

for the Taylor polynomial in $\mathbf{x}$, respectively, $\mathbf{y}$, and use a similar notation $R^{\mathbf{x}}_{m,\mathbf{z}}U(\mathbf{x},\mathbf{y}) = U(\mathbf{x},\mathbf{y}) - T^{\mathbf{x}}_{m,\mathbf{z}}U(\mathbf{x},\mathbf{y})$ and $R^{\mathbf{y}}_{m,\mathbf{z}}U(\mathbf{x},\mathbf{y}) = U(\mathbf{x},\mathbf{y}) - T^{\mathbf{y}}_{m,\mathbf{z}}U(\mathbf{x},\mathbf{y})$ for the respective remainders.

Following the proof of [4, Lemma 7], we consider the Boolean sum $B_{q,\mathbf{z}}(\mathbf{x},\mathbf{y})$ of Taylor polynomials of $K(\mathbf{x},\mathbf{y})$,

$$\begin{aligned}
B_{m,\mathbf{z}}(\mathbf{x},\mathbf{y}) &:= T^{\mathbf{x}}_{m,\mathbf{z}}K(\mathbf{x},\mathbf{y}) + T^{\mathbf{y}}_{m,\mathbf{z}}K(\mathbf{x},\mathbf{y}) - T^{\mathbf{x}}_{m,\mathbf{z}}T^{\mathbf{y}}_{m,\mathbf{z}}K(\mathbf{x},\mathbf{y})\\
&= \sum_{|\alpha|\le m} \frac{(\mathbf{x}-\mathbf{z})^\alpha}{\alpha!}\partial^{\alpha,0}K(\mathbf{z},\mathbf{y}) + \sum_{|\alpha|\le m} \frac{(\mathbf{y}-\mathbf{z})^\alpha}{\alpha!}\partial^{0,\alpha}K(\mathbf{x},\mathbf{z})\\
&\quad - \sum_{|\alpha|,|\beta|\le m} \frac{(\mathbf{x}-\mathbf{z})^\alpha(\mathbf{y}-\mathbf{z})^\beta}{\alpha!\beta!}\partial^{\alpha,\beta}K(\mathbf{z},\mathbf{z}),
\end{aligned}$$

and conclude that

$$K(\mathbf{x},\mathbf{y}) - B_{m,\mathbf{z}}(\mathbf{x},\mathbf{y}) = R(\mathbf{x},\mathbf{y}) := R^{\mathbf{x}}_{m,\mathbf{z}}R^{\mathbf{y}}_{m,\mathbf{z}}K(\mathbf{x},\mathbf{y}) \quad \text{for all } \mathbf{x},\mathbf{y}\in\Omega_{\mathbf{z}},$$

and, thanks to Assumption (c),

$$Q_{K,D,\mathbf{X},\mathbf{z}}(\mathbf{w}) = \sum_{i,j=1}^N w_iw_jR(\mathbf{x}_i,\mathbf{x}_j). \tag{2.14}$$

The above remainder formula for the Taylor polynomial implies that

$$R(\mathbf{x}_i,\mathbf{x}_j) = m\sum_{|\alpha|=m}\frac{(\mathbf{x}_i-\mathbf{z})^\alpha}{\alpha!}\int_0^1(1-t)^{m-1}\big[\partial^{\alpha,0}\tilde R(\mathbf{x}_i^t,\mathbf{x}_j) - \partial^{\alpha,0}\tilde R(\mathbf{z},\mathbf{x}_j)\big]\,dt,$$

where $\mathbf{x}_i^t := \mathbf{z} + t(\mathbf{x}_i-\mathbf{z})$, and

$$\tilde R(\mathbf{x},\mathbf{y}) := R^{\mathbf{y}}_{m,\mathbf{z}}K(\mathbf{x},\mathbf{y}) = K(\mathbf{x},\mathbf{y}) - T^{\mathbf{y}}_{m,\mathbf{z}}K(\mathbf{x},\mathbf{y})$$

satisfies

$$\partial^{\alpha,0}\tilde R(\mathbf{x},\mathbf{x}_j) = m\sum_{|\beta|=m}\frac{(\mathbf{x}_j-\mathbf{z})^\beta}{\beta!}\int_0^1(1-s)^{m-1}\big[\partial^{\alpha,\beta}K(\mathbf{x},\mathbf{x}_j^s) - \partial^{\alpha,\beta}K(\mathbf{x},\mathbf{z})\big]\,ds.$$

Since

$$\begin{aligned}\big|\partial^{\alpha,\beta}K(\mathbf{x}_i^t,\mathbf{x}_j^s)-\partial^{\alpha,\beta}K(\mathbf{x}_i^t,\mathbf{z})-\partial^{\alpha,\beta}K(\mathbf{z},\mathbf{x}_j^s)+\partial^{\alpha,\beta}K(\mathbf{z},\mathbf{z})\big|&\\ \le \|\mathbf{x}_i^t-\mathbf{z}\|_2^\gamma\,\|\mathbf{x}_j^s-\mathbf{z}\|_2^\gamma\,\big|\partial^{\alpha,\beta}K\big|_{\mathbf{z},\mathbf{X},\gamma}&\\ =(ts)^\gamma\|\mathbf{x}_i-\mathbf{z}\|_2^\gamma\,\|\mathbf{x}_j-\mathbf{z}\|_2^\gamma\,\big|\partial^{\alpha,\beta}K\big|_{\mathbf{z},\mathbf{X},\gamma},&\end{aligned}$$

and

$$\int_0^1(1-t)^{m-1}t^\gamma dt=\frac{(m-1)!}{(\gamma+1)\cdots(\gamma+m)},$$

we obtain

$$\begin{aligned}\big|R(\mathbf{x}_i,\mathbf{x}_j)\big|\le\Big(\frac{m!}{(\gamma+1)\cdots(\gamma+m)}\Big)^2\|\mathbf{x}_i-\mathbf{z}\|_2^\gamma\,\|\mathbf{x}_j-\mathbf{z}\|_2^\gamma&\\ \sum_{|\alpha|=m}\frac{|(\mathbf{x}_i-\mathbf{z})^\alpha|}{\alpha!}\sum_{|\beta|=m}\frac{|(\mathbf{x}_j-\mathbf{z})^\beta|}{\beta!}\,\big|\partial^{\alpha,\beta}K\big|_{\mathbf{z},\mathbf{X},\gamma}.&\end{aligned}$$

By the multinomial theorem,

$$\sum_{|\alpha|=m}\frac{(\mathbf{x}-\mathbf{z})^{2\alpha}}{\alpha!}=\frac{1}{m!}\|\mathbf{x}-\mathbf{z}\|_2^{2m}.$$

Hence, it follows by the Cauchy-Schwarz inequality, that

$$\begin{aligned}&\sum_{|\alpha|=m}\frac{|(\mathbf{x}_i-\mathbf{z})^\alpha|}{\alpha!}\sum_{|\beta|=m}\frac{|(\mathbf{x}_j-\mathbf{z})^\beta|}{\beta!}\,\big|\partial^{\alpha,\beta}K\big|_{\mathbf{z},\mathbf{X},\gamma}\\ &\qquad\le\frac{\|\mathbf{x}_i-\mathbf{z}\|_2^m\|\mathbf{x}_j-\mathbf{z}\|_2^m}{m!}\Big(\sum_{|\alpha|=m}\sum_{|\beta|=m}\frac{\big|\partial^{\alpha,\beta}K\big|_{\mathbf{z},\mathbf{X},\gamma}^2}{\alpha!\beta!}\Big)^{1/2},\end{aligned}$$

which implies

$$\big|R(\mathbf{x}_i,\mathbf{x}_j)\big|\le\|\mathbf{x}_i-\mathbf{z}\|_2^{m+\gamma}\|\mathbf{x}_j-\mathbf{z}\|_2^{m+\gamma}|K|_{\mathbf{z},\mathbf{X},m+\gamma}.$$

In view of this, (2.13) follows from (2.14).

Recall from [5] that the *growth function*

$$\rho_{q,D}(\mathbf{z},\mathbf{X},\mu):=\sup\{Dp(\mathbf{z}):p\in\Pi_q^d,\ |p(\mathbf{x}_j)|\le\|\mathbf{x}_j-\mathbf{z}\|_2^\mu,\ j=1,\dots,N\},$$

where $\mu\ge0$, coincides with

$$\inf\Big\{\sum_{j=1}^N|w_j|\|\mathbf{x}_j-\mathbf{z}\|_2^\mu:\mathbf{w}\in\mathbb{R}^N,\ Dp(\mathbf{z})=\sum_{j=1}^N w_jp(\mathbf{x}_j)\text{ for all }p\in\Pi_q^d\Big\}.$$

Therefore, Propositions 3 and 4 imply the following estimate of the error of the kernel-based numerical differentiation.

Theorem 1. *Let D be a linear differential operator of order k, and K a cpd kernel of order s on $\Omega \subset \mathbb{R}^d$. Given $\mathbf{z} \in \Omega$ and $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$, with $S_{\mathbf{z},\mathbf{X}} \subset \Omega$, assume that the linear system* (1.4) *is consistent, and for some $r \geq k$ and $\gamma \in (0, 1]$, the kernel K possesses continuous partial derivatives $\partial^{\alpha,\beta} K(\mathbf{x}, \mathbf{y})$ for all $|\alpha|, |\beta| \leq r$ and all $\mathbf{x}, \mathbf{y} \in S_{\mathbf{z},\mathbf{X}}$, and $|K|_{\mathbf{z},\mathbf{X},r+\gamma} < \infty$. Then for all $f \in \mathcal{F}_K$ the kernel-based numerical differentiation formula* (1.2) *satisfies the error bound*

$$\Big| Df(\mathbf{z}) - \sum_{j=1}^{N} w_j^* f(\mathbf{x}_j) \Big| \leq \rho_{q,D}(\mathbf{z}, \mathbf{X}, r + \gamma)\, |K|_{\mathbf{z},\mathbf{X},r+\gamma}^{1/2}\, \|f\|_K, \tag{2.15}$$

where $q = \max\{s, r + 1\}$.

Since $\rho_{q,D}(\mathbf{z}, \mathbf{X}, \mu) = \sigma\, h_{\mathbf{z},\mathbf{X}}^{\mu-k}$, with a scale-independent factor σ, where

$$h_{\mathbf{z},\mathbf{X}} := \max_{\mathbf{x}\in\mathbf{X}} \|\mathbf{z} - \mathbf{x}\|_2.$$

see [5, Section 4.2], we infer that (2.15) gives an $\mathcal{O}(h_{\mathbf{z},\mathbf{X}}^{r+\gamma-k})$ error bound for constellations $\mathbf{z}, \mathbf{X}$ in a good geometric position. Note that $\rho_{q,D}(\mathbf{z}, \mathbf{X}, \mu)$ can be efficiently estimated by least squares methods, see [5, Sections 5 and 6].

By applying Proposition 4 with $k \leq m \leq r - 1$ instead of $m = r$, and choosing $q = \max\{s, m + 1\}$, and taking into account that $|K|_{\mathbf{z},\mathbf{X},q} < \infty$ if $\partial^{\alpha,\beta} K(\mathbf{x}, \mathbf{y})$ are continuous for all $|\alpha|, |\beta| \leq q$ and all $\mathbf{x}, \mathbf{y} \in S_{\mathbf{z},\mathbf{X}}$, we obtain the following statement.

Corollary 1. *Under the hypotheses of Theorem 1, for all $f \in \mathcal{F}_K$ and all q satisfying* $\max\{s, k + 1\} \leq q \leq r$,

$$\Big| Df(\mathbf{z}) - \sum_{j=1}^{N} w_j^* f(\mathbf{x}_j) \Big| \leq \rho_{q,D}(\mathbf{z}, \mathbf{X}, q)\, |K|_{\mathbf{z},\mathbf{X},q}^{1/2}\, \|f\|_K. \tag{2.16}$$

Note that Corollary 1 makes the same smoothness assumptions about K as [4, Theorem 9], and only relaxes the condition on $\mathbf{X}$ that does not anymore have to be a determining set for Π_s^d, and uses a slightly different seminorm $|K|_{\mathbf{z},\mathbf{X},q}$ of K. In the same time, the estimate (2.15) of Theorem 1 provides a higher approximation order of numerical differentiation for certain kernels K, as we will demonstrate in the next section.

3. Applications to translation-invariant kernels

We now consider the *translation-invariant kernels* $K(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y})$, for a function $\Phi : \mathbb{R}^d \to \mathbb{R}$, which in particular covers the *radial basis functions* $\Phi(\mathbf{x}) = \varphi(\|\mathbf{x}\|_2)$, where $\varphi : [0, \infty) \to \mathbb{R}$.

Denote by $C^{r,\gamma}(\Omega)$, $r = 0, 1, \dots$, $0 < \gamma \leq 1$, $\Omega \subset \mathbb{R}^d$, the *Hölder space* that consists of all r times continuously differentiable functions f on Ω such that

$$|f|_{C^{r,\gamma}(\Omega)} := \max_{|\alpha|=r} |\partial^\alpha f|_{\Omega,\gamma} < \infty,$$

where

$$|f|_{C^{0,\gamma}(\Omega)} := \sup_{\mathbf{x},\mathbf{y}\in\Omega,\, \mathbf{x}\neq\mathbf{y}} \frac{|f(\mathbf{x})-f(\mathbf{y})|}{\|\mathbf{x}-\mathbf{y}\|^\gamma}, \quad 0<\gamma\le 1.$$

Given $\mathbf{z}\in\mathbb{R}^d$ and $\mathbf{X}=\{\mathbf{x}_1,\dots,\mathbf{x}_N\}\subset\mathbb{R}^d$, we denote by $B_{\mathbf{z},\mathbf{X}}$ the closed ball in $\mathbb{R}^d$ centered at the origin and with radius equal to the diameter of the set $S_{\mathbf{z},\mathbf{X}}$.

Lemma 1. *Assume that $K(\mathbf{x},\mathbf{y})=\Phi(\mathbf{x}-\mathbf{y})$, and $\Phi\in C^{r,\gamma}(B_{\mathbf{z},\mathbf{X}})$. Then K possesses continuous partial derivatives $\partial^{\alpha,\beta}K(\mathbf{x},\mathbf{y})$ for all $|\alpha|,|\beta|\le m$, where $m=\lfloor r/2\rfloor$, and*

$$|\partial^{\alpha,\beta}K|_{\mathbf{z},\mathbf{X},(r-2m+\gamma)/2} \le \max\{2,\sqrt{d}\}|\Phi|_{C^{r,\gamma}(B_{\mathbf{z},\mathbf{X}})}, \quad |\alpha|=|\beta|=m, \tag{3.1}$$

$$|K|_{\mathbf{z},\mathbf{X},(r+\gamma)/2} \le C_{d,r}|\Phi|_{C^{r,\gamma}(B_{\mathbf{z},\mathbf{X}})}, \tag{3.2}$$

where

$$C_{d,r} = \begin{cases} \frac{2d^{r/2}}{(\gamma/2+1)^2\cdots(\gamma/2+\lfloor r/2\rfloor)^2} & \text{if } r \text{ is even,}\\ \frac{d^{r/2}}{((1+\gamma)/2+1)^2\cdots((1+\gamma)/2+\lfloor r/2\rfloor)^2} & \text{if } r \text{ is odd.}\end{cases} \tag{3.3}$$

Proof. The first statement follows because

$$\partial^{\alpha,\beta}K(\mathbf{x},\mathbf{y}) = \frac{\partial^{|\alpha|}}{\partial\mathbf{x}^\alpha}\frac{\partial^{|\beta|}}{\partial\mathbf{y}^\beta}\Phi(\mathbf{x}-\mathbf{y}) = (-1)^{|\beta|}\partial^{\alpha+\beta}\Phi(\mathbf{x}-\mathbf{y}).$$

To show (3.1) and (3.2), we consider separately the cases of an even and an odd r.

Let $r=2m$. Then $(r+\gamma)/2=m+\gamma/2$. Let $|\alpha|=|\beta|=m$, hence $|\alpha+\beta|=r$. For $U(\mathbf{x},\mathbf{y})=\partial^{\alpha+\beta}\Phi(\mathbf{x}-\mathbf{y})$, let

$$\Delta_{\mathbf{z},\mathbf{x},\mathbf{y}}U = U(\mathbf{x},\mathbf{y})-U(\mathbf{x},\mathbf{z})-U(\mathbf{z},\mathbf{y})+U(\mathbf{z},\mathbf{z}).$$

As soon as $\mathbf{x},\mathbf{y}\in S_{\mathbf{z},\mathbf{X}}$ we have

$$\begin{aligned}
|\Delta_{\mathbf{z},\mathbf{x},\mathbf{y}}U| &\le \left|\partial^{\alpha+\beta}\Phi(\mathbf{x}-\mathbf{y})-\partial^{\alpha+\beta}\Phi(\mathbf{x}-\mathbf{z})\right| + \left|\partial^{\alpha+\beta}\Phi(\mathbf{z}-\mathbf{y})-\partial^{\alpha+\beta}\Phi(0)\right|\\
&\le 2|\partial^{\alpha+\beta}\Phi|_{C^{0,\gamma}(B_{\mathbf{z},\mathbf{X}})}\,\|\mathbf{y}-\mathbf{z}\|_2^\gamma, \quad \text{and}\\
|\Delta_{\mathbf{z},\mathbf{x},\mathbf{y}}U| &\le \left|\partial^{\alpha+\beta}\Phi(\mathbf{x}-\mathbf{y})-\partial^{\alpha+\beta}\Phi(\mathbf{z}-\mathbf{y})\right| + \left|\partial^{\alpha+\beta}\Phi(\mathbf{x}-\mathbf{z})-\partial^{\alpha+\beta}\Phi(0)\right|\\
&\le 2|\partial^{\alpha+\beta}\Phi|_{C^{0,\gamma}(B_{\mathbf{z},\mathbf{X}})}\,\|\mathbf{x}-\mathbf{z}\|_2^\gamma,
\end{aligned}$$

since all points $\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{z},\mathbf{z}-\mathbf{y},0$ belong to $B_{\mathbf{z},\mathbf{X}}$. It follows that

$$\begin{aligned}
|\Delta_{\mathbf{z},\mathbf{x},\mathbf{y}}U| &\le 2|\partial^{\alpha+\beta}\Phi|_{C^{0,\gamma}(B_{\mathbf{z},\mathbf{X}})}\min\{\|\mathbf{x}-\mathbf{z}\|_2^\gamma,\|\mathbf{y}-\mathbf{z}\|_2^\gamma\}\\
&\le 2|\partial^{\alpha+\beta}\Phi|_{C^{0,\gamma}(B_{\mathbf{z},\mathbf{X}})}\,\|\mathbf{x}-\mathbf{z}\|_2^{\gamma/2}\|\mathbf{y}-\mathbf{z}\|_2^{\gamma/2},
\end{aligned}$$

which implies (3.1).

Let now $r=2m+1$ and $|\alpha|=|\beta|=m$. Then $|\alpha+\beta|=r-1$, and hence $U(\mathbf{x},\mathbf{y})=\partial^{\alpha+\beta}\Phi(\mathbf{x}-\mathbf{y}) =: \Phi_{\alpha,\beta}(\mathbf{x}-\mathbf{y})$ is continuously differentiable, with its partial derivatives of the first order in $C^{0,\gamma}(B_{\mathbf{z},\mathbf{X}})$. Therefore, with $\mathbf{y}^s := \mathbf{z} + s(\mathbf{y}-\mathbf{z}) \in S_{\mathbf{z},\mathbf{X}}$, we have

$$U(\mathbf{x},\mathbf{y}) - U(\mathbf{x},\mathbf{z}) = -\int_0^1 \nabla\Phi_{\alpha,\beta}(\mathbf{x}-\mathbf{y}^s)\cdot(\mathbf{y}-\mathbf{z})\,ds,$$
$$U(\mathbf{z},\mathbf{y}) - U(\mathbf{z},\mathbf{z}) = -\int_0^1 \nabla\Phi_{\alpha,\beta}(\mathbf{z}-\mathbf{y}^s)\cdot(\mathbf{y}-\mathbf{z})\,ds,$$

such that

$$\begin{aligned}|\Delta_{\mathbf{z},\mathbf{x},\mathbf{y}}U| &= \Big|\int_0^1 \big(\nabla\Phi_{\alpha,\beta}(\mathbf{x}-\mathbf{y}^s) - \nabla\Phi_{\alpha,\beta}(\mathbf{z}-\mathbf{y}^s)\big)\cdot(\mathbf{y}-\mathbf{z})\,ds\Big|\\ &\le \sqrt{d}|\Phi|_{C^{r,\gamma}(B_{\mathbf{z},\mathbf{X}})}\,\|\mathbf{x}-\mathbf{z}\|_2^\gamma\|\mathbf{y}-\mathbf{z}\|.\end{aligned}$$

Similarly to the above, we obtain

$$|\Delta_{\mathbf{z},\mathbf{x},\mathbf{y}}U| \le \sqrt{d}|\Phi|_{C^{r,\gamma}(B_{\mathbf{z},\mathbf{X}})}\,\|\mathbf{x}-\mathbf{z}\|_2\|\mathbf{y}-\mathbf{z}\|^\gamma.$$

Hence

$$|\Delta_{\mathbf{z},\mathbf{x},\mathbf{y}}U| \le \sqrt{d}|\Phi|_{C^{r,\gamma}(B_{\mathbf{z},\mathbf{X}})}\,\|\mathbf{x}-\mathbf{z}\|_2^{(1+\gamma)/2}\|\mathbf{y}-\mathbf{z}\|^{(1+\gamma)/2},$$

which implies (3.1) also in this case.

The estimate (3.2) follows from the above using the definition of $|K|_{\mathbf{z},\mathbf{X},m+\theta}$, where we take $\theta = \gamma/2$ if r is even, and $\theta = (1+\gamma)/2$ if r is odd.

Using this lemma we immediately obtain from Theorem 1 the following error bound for translation-invariant kernels.

Corollary 2. *Let D be a linear differential operator of order k, and $K(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y})$ a translation-invariant cpd kernel of order s on $\mathbb{R}^d$. Given $\mathbf{z} \in \mathbb{R}^d$ and $\mathbf{X} = \{\mathbf{x}_1,\dots,\mathbf{x}_N\} \subset \mathbb{R}^d$, assume that the linear system* (1.4) *is consistent, and $\Phi \in C^{r,\gamma}(B_{\mathbf{z},\mathbf{X}})$ for some $r \ge 2k$ and $\gamma \in (0,1]$. Then for all $f \in \mathcal{F}_K$ the kernel-based numerical differentiation formula* (1.2) *satisfies the error bound*

$$\Big|Df(\mathbf{z}) - \sum_{j=1}^N w_j^* f(\mathbf{x}_j)\Big| \le \rho_{q,D}(\mathbf{z},\mathbf{X},\tfrac{r+\gamma}{2})\,C_{d,r}^{1/2}|\Phi|^{1/2}_{C^{r,\gamma}(B_{\mathbf{z},\mathbf{X}})}\,\|f\|_K, \tag{3.4}$$

where $q = \max\{s, \lfloor r/2\rfloor + 1\}$, and $C_{d,r}$ is given by (3.3).

Remark 2. Clearly, using either [4, Theorem 9] or Corollary 1 we could only obtain a bound in terms of the growth function $\rho_{q,D}(\mathbf{z},\mathbf{X},q)$ with $q \le m = \lfloor r/2\rfloor$, and hence at best an $\mathcal{O}(h_{\mathbf{z},\mathbf{X}}^{\lfloor r/2\rfloor-k})$ estimate of the numerical differentiation error, rather than $\mathcal{O}(h_{\mathbf{z},\mathbf{X}}^{(r+\gamma)/2-k})$ by Corollary 2.

We now apply Corollary 2 to several families of radial basis functions with finite smoothness. Their properties can be found for example in the books [2, 6, 9].

Polyharmonic spline: $\Phi(\mathbf{x}) = \Phi_{\mathrm{PHS},\nu}(\mathbf{x}) := (-1)^{\lceil \nu/2 \rceil}\|\mathbf{x}\|_2^\nu$, $\nu > 0$, $\nu \notin 2\mathbb{N}$, conditionally positive definite of order $s \geq \lceil \nu/2 \rceil$. It can be easily shown that $\Phi_{\mathrm{PHS},\nu} \in C^{r,\gamma}(B)$ for any ball $B \subset \mathbb{R}^d$ centered at the origin, where $r = \lceil \nu \rceil - 1$, $\gamma = \nu - r$.

Corollary 3. *Let D be a linear differential operator of order k, and $K(\mathbf{x},\mathbf{y}) = \Phi_{PHS,\nu}(\mathbf{x}-\mathbf{y})$ for some $\nu > 2k$, $\nu \notin 2\mathbb{N}$, and let $s \geq \lceil \nu/2 \rceil$. Given $\mathbf{z} \in \mathbb{R}^d$ and $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathbb{R}^d$, assume that the linear system* (1.4) *is consistent. Then for all $f \in \mathcal{F}_K$ the kernel-based numerical differentiation formula* (1.2) *satisfies the error bound*

$$\Big|Df(\mathbf{z}) - \sum_{j=1}^N w_j^* f(\mathbf{x}_j)\Big| \leq \rho_{s,D}(\mathbf{z},\mathbf{X},\tfrac{\nu}{2})\, C_{d,r}^{1/2} |\Phi_{PHS,\nu}|^{1/2}_{C^{r,\gamma}(B_{\mathbf{z},\mathbf{X}})}\, \|f\|_K,$$

where $r = \lceil \nu \rceil - 1$, $\gamma = \nu - r$, and $C_{d,r}$ is given by (3.3).

Thin plate splines: $\Phi(\mathbf{x}) = \Phi_{\mathrm{TPS},n}(\mathbf{x}) := (-1)^{n+1}\|\mathbf{x}\|_2^{2n}\log\|\mathbf{x}\|_2$, $n \in \mathbb{N}$, conditionally positive definite of order $s \geq n+1$. As shown for example in [9, p. 186], $\Phi_{\mathrm{TPS},n} \in C^{2n-1,\gamma}(B)$ for any ball $B \subset \mathbb{R}^d$ centered at the origin and every $0 < \gamma < 1$.

Corollary 4. *Let D be a linear differential operator of order k, and $K(\mathbf{x},\mathbf{y}) = \Phi_{TPS,n}(\mathbf{x}-\mathbf{y})$ for some $n \geq k+1$, and let $s \geq n+1$. Given $\mathbf{z} \in \mathbb{R}^d$ and $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathbb{R}^d$, assume that the linear system* (1.4) *is consistent. Then for all $f \in \mathcal{F}_K$ the kernel-based numerical differentiation formula* (1.2) *satisfies for any $0 < \gamma < 1$ the error bound*

$$\Big|Df(\mathbf{z}) - \sum_{j=1}^N w_j^* f(\mathbf{x}_j)\Big| \leq \rho_{s,D}(\mathbf{z},\mathbf{X},n-\tfrac{1-\gamma}{2})\, C_{d,2n-1}^{1/2} |\Phi_{TPS,n}|^{1/2}_{C^{2n-1,\gamma}(B_{\mathbf{z},\mathbf{X}})}\, \|f\|_K,$$

where $C_{d,2n-1}$ is given by (3.3).

Wendland's compactly supported radial basis functions: $\Phi(\mathbf{x}) = \Phi_{d,n}(\mathbf{x}) = \varphi_{d,n}(\|\mathbf{x}\|_2)$, $n = 0, 1, \dots$, see [9, Section 9.4] for a definition. They are strictly positive definite, and hence we can use any $s \geq 0$. It is shown in [9] that $\Phi_{d,n}$ is $2n$ times continuously differentiable in $\mathbb{R}^d$. Since $\varphi_{d,n}$ are piecewise polynomials, it follows that $\Phi_{d,n} \in C^{2n,1}(B)$ for any ball $B \subset \mathbb{R}^d$ centered at the origin.

Corollary 5. *Let D be a linear differential operator of order k, and $K(\mathbf{x},\mathbf{y}) = \Phi_{d,n}(\mathbf{x}-\mathbf{y})$ for some $n \geq k$, and let $s \geq 0$. Let $\mathbf{z} \in \mathbb{R}^d$ and $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathbb{R}^d$. If $s > 0$, we assume that the linear system* (1.4) *is consistent. Then for all $f \in \mathcal{F}_K$ the kernel-based numerical differentiation formula* (1.2) *satisfies the error bound*

$$\Big|Df(\mathbf{z}) - \sum_{j=1}^{N} w_j^* f(\mathbf{x}_j)\Big| \le \rho_{q,D}(\mathbf{z}, \mathbf{X}, n + \tfrac{1}{2})\, C_{d,2n}^{1/2} |\Phi_{d,n}|_{C^{2n,1}(B_{\mathbf{z},\mathbf{X}})}^{1/2} \|f\|_K,$$

where $q = \max\{s, n+1\}$, and $C_{d,2n}$ is given by (3.3)*.*

References

1. *Bayona V., Flyer N., Fornberg B., Barnett G.A.*: On the role of polynomials in RBF-FD approximations: II. Numerical solution of elliptic PDEs. Journal of Computational Physics 2017; 332: pp. 257–273. doi:10.1016/j.jcp.2016.12.008
2. *Buhmann M.D.*: Radial basis functions. Cambridge University Press, 2003. doi:10.1017/CBO9780511543241
3. *Davydov O.*: Approximation with conditionally positive definite kernels on deficient sets. Approximation Theory XVI: Nashville 2019, edited by M. N. Gregory E. Fasshauer and L. L. Schumaker. Springer, 2021; pp. 27–38. doi:10.1007/978-3-030-57464-2_3
4. *Davydov O., Schaback R.*: Error bounds for kernel-based numerical differentiation. Numerische Mathematik 2016; 132(2): pp. 243–269. doi:10.1007/s00211-015-0722-9
5. *Davydov O., Schaback R.*: Minimal numerical differentiation formulas. Numerische Mathematik 2018; 140(3): pp. 555–592. doi:10.1007/s00211-018-0973-3
6. *Fasshauer G.*: Meshfree approximation methods with MATLAB. World Scientific Publishing, 2007. doi:10.1142/6437
7. *Fornberg B., Flyer N.*: A primer on radial basis functions with applications to the Geosciences. SIAM, 2015. doi:10.1137/1.9781611974041
8. *Schaback R.*: Native Hilbert spaces for radial basis functions I. New Developments in Approximation Theory: 2nd International Dortmund Meeting (IDoMAT), Germany 1998, edited by M. W. Müller, M. D. Buhmann, D. H. Mache, and M. Felten. Basel, 1999; pp. 255–282.
9. *Wendland H.*: Scattered Data Approximation. Cambridge University Press, 2004. doi:10.1017/CBO9780511617539

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