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The order of the best transfinite interpolation of functions with bounded laplacian with the help of harmonic splines on box partitions

Показано, що похибка найкращої трансфінитної інтерполяції функцій з обмеженим лапласіаном за допомогою кусково-гармонічних сплайнів на прямокутних розбиттях, які складаються з N елементів, має порядок N^{-2} , коли $N \rightarrow \infty$.

Ключові слова: трансфінитна інтерполяція, найкраще наближення, кусково-гармонічний сплайн.

Показано, что погрешность наилучшей трансфинитной интерполяции функций с ограниченным лапласианом с помощью кусочно-гармонических сплайнов на прямоугольных разбиениях, состоящих из N элементов, имеет порядок N^{-2} , когда $N \rightarrow \infty$.

Ключевые слова: трансфинитная интерполяция, наилучшее приближение, кусочно-гармонический сплайн.

We show that the error of the best transfinite interpolation of functions with bounded laplacian with the help of harmonic splines on box partitions comprising N elements has the order N^{-2} as $N \rightarrow \infty$.

Key words: transfinite interpolation, best approximation, harmonic spline.

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Depending on the information available on the multivariate function a variety of interpolation schemes exist in Approximation Theory. *Transfinite interpolation* is the special type of interpolation schemes that consists in constructing the function coinciding with the given function along some manifolds. It was first considered by D. Mangeron [1] in 1932 and was developed further in papers [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] (see also surveys [14, 15, 16] and references therein). Transfinite interpolation finds applications in computer tomography, finite elements methods, geometric modeling, cartography and many others (see [17, 18, 19, 12]).

In certain important situations we may assume that the manifolds, along which the values of the interpolated function are known, split the function domain of definition into sub-domains with the Lipschitz boundary. The natural method of transfinite interpolation in such case is delivered by *harmonic splines* that coincide with the harmonic continuation of the function from the boundary of every element in the partition into its interior. Properties of transfinite interpolation by harmonic splines

were studied in [8, 10, 11, 12, 13], where the problem of the best transfinite interpolation of multivariate functions on box partitions was considered. In this paper we continue the investigation of this problem and find the exact order of error of the best transfinite interpolation with the help of harmonic splines on box partitions. Let us proceed to the rigorous settings of the problem.

Let $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be a compact set. By $C(\Omega)$ we denote the standard space of continuous on Ω functions $f : \Omega \rightarrow \mathbb{R}$. For $1 \leq s \leq \infty$, we denote by $L_s(\Omega)$ the space of measurable functions $f : \Omega \rightarrow \mathbb{R}$ endowed with the standard norm

$$\|f\|_{L_s(\Omega)} := \begin{cases} \left(\int_{\Omega} |f(\mathbf{x})|^s d\mathbf{x} \right)^{\frac{1}{s}}, & 1 \leq s < \infty, \\ \text{esssup} \{ |f(\mathbf{x})| : \mathbf{x} \in \Omega \}, & s = \infty. \end{cases}$$

As usually, by Δ we denote d -variate Laplace operator:

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}.$$

Let $b_1, \dots, b_d > 0$ be given and denote by $\Pi = [0, b_1] \times \dots \times [0, b_d]$ the d -box in $\mathbb{R}^d$ those sides are parallel to the coordinate axes. A finite set of d -boxes P is called *box partition* of Π if $\Pi = \bigcup_{\Omega \in P} \Omega$ and $\Omega' \cap \Omega'' = \emptyset$, for every two distinct d -boxes $\Omega', \Omega'' \in P$.

For a function $f \in C(\Pi)$ and a box partition P of Π , we construct the harmonic spline $S_P f$ in the following way. For every $\Omega \in P$, by $u = S_P f|_{\Omega}$ we denote the solution of the Dirichlet problem for the Laplace equation:

$$\begin{cases} \Delta u(\mathbf{x}) = 0, & \mathbf{x} \in \text{int } \Omega, \\ u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{cases}$$

From the construction it is easy to see that the spline $S_P f$ is continuous on Π and coincide with the function f on the set $\bigcup_{\Omega \in P} \partial\Omega$. For $1 \leq q \leq \infty$, we consider the class

$$W_q^{\Delta} = W_q^{\Delta}(\Pi) := \{f \in C^2(\Pi) : \|\Delta f\|_{L_q(\Pi)} \leq 1\}.$$

Next, for a box partition P of Π , we define the *error of transfinite interpolation* of functions $f \in W_q^{\Delta}$ by harmonic splines:

$$\mathcal{E}_P(W_q^{\Delta})_s := \sup_{f \in W_q^{\Delta}} \|f - S_P f\|_{L_s(\Pi)}. \quad (1)$$

The quantity (1) was studied in papers [10, 11] on the special class of box partitions P that consist of d -boxes obtained by splitting Π with the help of a finite number of $(d-1)$ -variate hyperplanes are orthogonal to the coordinate axes. Repeating the proof of Theorems 1 and 2 in [11], we can establish the following representation for $\mathcal{E}_P(W_q^{\Delta})_s$ in terms of the norms of the Green function.

Proposition 1. Let P be an arbitrary box partition of Π , $1 \leq s < \infty$, $s' = s/(s-1)$. Then

$$\mathcal{E}_P(W_\infty^\Delta)_s = \mathcal{E}_P(W_{s'}^\Delta)_1 = \left(\sum_{\Omega \in P} \left\| \int_{\Omega} G_\Omega(\cdot, \mathbf{y}) d\mathbf{y} \right\|_{L_s(\Omega)}^s \right)^{1/s},$$

where G_Ω is the Green function in the Dirichlet problem for the Laplace equation on the set Ω . Moreover,

$$\mathcal{E}_P(W_\infty^\Delta)_\infty = \mathcal{E}_P(W_1^\Delta)_1 = \max_{\Omega \in P} \sup_{\mathbf{x} \in \Omega} \int_{\Omega} G_\Omega(\mathbf{x}, \mathbf{y}) d\mathbf{y}.$$

For $N \in \mathbb{N}$, by $\mathcal{P}_N := \mathcal{P}_N(\Pi)$ we denote the set of all box partitions of Π comprising N elements. The problem of the best transfinite interpolation of functions from the class W_q^Δ by harmonic splines on box partitions $P \in \mathcal{P}_N$ can be stated as follows.

Problem 1. Find the quantity

$$\mathcal{E}_N(W_q^\Delta)_s := \inf_{P \in \mathcal{P}_N} \mathcal{E}_P(W_q^\Delta)_s, \quad (2)$$

and find optimal partitions $P^* \in \mathcal{P}_N$ (if any exists) such that

$$\mathcal{E}_N(W_q^\Delta)_s = \mathcal{E}_{P^*}(W_q^\Delta)_s.$$

V. F. Babenko and T. Yu. Leskevych [11] considered this problem for a special class of partitions. To formulate their result we denote by $\mathbf{m} = (m_1, \dots, m_d)$ positive integer multi-index, *i.e.* $m_k \in \mathbb{N}$, $k = 1, 2, \dots, d$, and denote $|\mathbf{m}| := m_1 m_2 \dots m_d$. Let $\mathcal{P}_{\mathbf{m}}$ be the set of box partitions of Π comprising of $|\mathbf{m}|$ d -boxes that are obtained from Π by slicing it with the help of $(m_1 - 1)$ hyperplanes orthogonal to the axis Ox_1 , $(m_2 - 1)$ hyperplanes orthogonal to the axis Ox_2 , etc. From definition it is clear that $\mathcal{P}_{\mathbf{m}} \subset \mathcal{P}_{|\mathbf{m}|}$. Also, we denote by $P_{\mathbf{m}}^*$ the regular box partition from $\mathcal{P}_{\mathbf{m}}$ comprising of $|\mathbf{m}|$ equal d -boxes. The following proposition was established in [11] (see arguments right above Corollaries 2 and 4 in [11]).

Proposition 2. Let $\mathbf{m} = (m_1, \dots, m_d)$ be positive integer multi-index. Then

$$\mathcal{E}_{\mathbf{m}}(W_\infty^\Delta)_\infty := \inf_{P \in \mathcal{P}_{\mathbf{m}}} \mathcal{E}_P(W_\infty^\Delta)_\infty = \mathcal{E}_{P_{\mathbf{m}}^*}(W_\infty^\Delta)_\infty.$$

Moreover, if $d = 2$ then $\mathcal{E}_{\mathbf{m}}(W_\infty^\Delta)_1 = \mathcal{E}_{P_{\mathbf{m}}^*}(W_\infty^\Delta)_1$.

It was natural to expect that the order of the quantity $\mathcal{E}_N(W_\infty^\Delta)_s$, $1 \leq s \leq \infty$, degrades with the increase of the number of dimensions, as always happen in the case of approximating isotropic classes of functions with the help of polynomial splines (see [20]). However, in [13] it was established (see Theorem 1) that $\mathcal{E}_N(W_\infty^\Delta)_s = \mathcal{O}(N^{-2})$ as $N \rightarrow \infty$. Also, it was shown that this order is achieved on box partitions $P_N^{*,j}$, $j = 1, \dots, d$, obtained by splitting Π into equal d -boxes with the help of $N - 1$ hyperplanes orthogonal to the axis Ox_j . In this paper we will prove that N^{-2} is the correct order of quantity $\mathcal{E}_N(W_\infty^\Delta)_s$ as $N \rightarrow \infty$. The main result of this paper is contained in the following statement.

Theorem 1. *Let $d \in \mathbb{N}$, $1 \leq s \leq \infty$ and $s' = s/(s-1)$. Then there exist constants $C_1, C_2 > 0$ such that, for every $N \in \mathbb{N}$,*

$$C_1 \leq N^2 \cdot \mathcal{E}_N(W_\infty^\Delta)_s = N^2 \cdot \mathcal{E}_N(W_{s'}^\Delta)_1 \leq C_2. \quad (3)$$

Proof. By Theorem 1 in [13] there exists a constant $C_2^* > 0$ such that

$$\begin{aligned} \limsup_{N \rightarrow \infty} N^2 \mathcal{E}_N(W_\infty^\Delta)_s &= \limsup_{N \rightarrow \infty} N^2 \mathcal{E}_N(W_{s'}^\Delta)_1 \\ &\leq \min_{j=1, \dots, d} \left(\lim_{N \rightarrow \infty} N^2 \mathcal{E}_{P_j^*}(W_\infty^\Delta)_s \right) = C_2^*. \end{aligned} \quad (4)$$

Hence, there exists a constant $C_2 > 0$ such that the right hand inequality in (3) holds true for every $N \in \mathbb{N}$.

Let us now establish the lower estimate for $N^2 \cdot \mathcal{E}_N(W_\infty^\Delta)_s$. First, we consider the case $s = \infty$. Let $P \in \mathcal{P}_N$ be a box partition of Π . Let us show that some box Ω in P contains a box obtained from $\frac{1}{N} \cdot \Pi$ with the help of a translation. To this end we assume to the contrary that, for every $\Omega \in P$, there exists index $j_\Omega \in \{1, \dots, d\}$ such that the length of the side of Ω that is parallel to the axis Ox_{j_Ω} does not exceed $\frac{b_{j_\Omega}}{N}$. Then we can estimate the volume of any box $\Omega \in P$ as follows:

$$\mu\Omega < \frac{b_1 \cdot \dots \cdot b_d}{b_{j_\Omega}} \cdot \frac{b_{j_\Omega}}{N} = \frac{\mu\Pi}{N}.$$

However, by definition of the partition $P \in \mathcal{P}_N$ on Π we have

$$\mu\Pi = \mu \left(\bigcup_{\Omega \in P} \Omega \right) < \sum_{\Omega \in P} \mu\Omega \leq \frac{\mu\Pi}{N} \cdot \#P \leq \mu\Pi,$$

which leads to a contradiction. Therefore, some box $\Omega^* \in P$ contains a box Π^* obtained from $\frac{1}{N} \cdot \Pi$ with the help of a translation.

Now, without loss of generality we assume that $b_1 = \min_{j=1, \dots, d} b_j$. Also, let $\mathbf{s}$ be the center of the box Π^* and B be the ball with the center at the point $\mathbf{s}$ and radius $\frac{b_1}{N}$. In addition, we define a function $\tau : [0, +\infty) \rightarrow \mathbb{R}$:

$$\tau(t) := \begin{cases} \frac{b_1^2}{16dN^2} - \frac{t^2}{2d}, & t \in \left[0, \frac{b_1}{4N}\right], \\ \frac{1}{2d} \left(\frac{b_1}{2N} - t \right)^2, & t \in \left[\frac{b_1}{4N}, \frac{b_1}{2N}\right], \\ 0, & t \geq \frac{b_1}{2N}. \end{cases}$$

Let us consider the function $\varphi_P : \Pi \rightarrow \mathbb{R}$ where:

$$\varphi_P(\mathbf{x}) := \tau(\|\mathbf{x}\|), \quad \mathbf{x} \in \Pi.$$

Obviously, $\varphi_P \in W_\infty^\Delta$ and φ_P vanishes on the boundary of every box $\Omega \in P$. Hence, $S_P \varphi_P \equiv 0$ and, as a result, we obtain

$$\mathcal{E}_N(W_\infty^\Delta)_\infty = \inf_{P \in \mathcal{P}_N} \mathcal{E}_P(W_\infty^\Delta)_\infty \geq \inf_{P \in \mathcal{P}_N} \|\varphi_P\|_{L_\infty(B)} = \frac{b_1^2}{4dN^2}.$$

Consequently,

$$N^2 \cdot \mathcal{E}_N(W_\infty^\Delta)_\infty \geq \frac{1}{4d} \cdot \left(\min_{j=1, \dots, d} b_j \right)^2. \quad (5)$$

Let us consider the case when $s \in [1, \infty)$. Let $P \in \mathcal{P}_N$ be a box partition of Π . For every box $\Omega \in P$, by a_Ω we denote the smallest side of Ω , and by Ω^* the set of points in Ω having the distance at least $\frac{1}{4} a_\Omega$ to $\partial\Omega$, *i.e.*

$$\Omega^* = \left\{ \mathbf{x} \in \Omega : \inf_{\mathbf{t} \in \partial\Omega} \|\mathbf{x} - \mathbf{t}\| = \frac{1}{4} a_\Omega \right\}.$$

Let us estimate from below the Lebesgue measure of the set Ω^* . To this end we observe that Ω^* is the box whose sides are parallel to the sides of Ω . If b is the length of a side of box Ω , then the length of the parallel side of Ω^* is equal to $b - \frac{1}{2} a_\Omega$. Since a_Ω is the length of the smallest side of Ω , we conclude that $b - \frac{1}{2} a_\Omega \geq \frac{b}{2}$. Hence,

$$\mu\Omega^* \geq \frac{\mu\Omega}{2^d}. \quad (6)$$

Now, we let $P \in \mathcal{P}_N$ be a partition of Π . Without loss of generality we assume that $b_1 = \min_{j=1, \dots, d} b_j$. Also, we denote by $I = I(P)$ the set consisting of d -boxes $\Omega \in P$ such that $\mu\Omega \geq \frac{\mu\Pi}{2N}$. It is not difficult to see that

$$\sum_{\Omega \in I} \mu\Omega = \mu\Pi - \sum_{\Omega \in P \setminus I} \mu\Omega \geq \mu\Pi - \frac{\mu\Pi}{2N} \cdot \#(P \setminus I) \geq \frac{\mu\Pi}{2}. \quad (7)$$

Also, it is clear that $a_\Omega \geq \frac{b_1}{2N}$, for every $\Omega \in I$.

Now, for $a > 0$, we consider the function $\tau_a : [0, +\infty) \rightarrow \mathbb{R}$:

$$\tau_a(t) := \begin{cases} \frac{a^2}{64d} - \frac{t^2}{8d}, & t \in \left[0, \frac{a}{8}\right], \\ \frac{1}{2d} \left(\frac{a}{4} - t\right)^2, & t \in \left[\frac{a}{8}, \frac{a}{4}\right], \\ 0, & t \geq \frac{a}{4}, \end{cases}$$

and define the function $\psi_P : \Pi \rightarrow \mathbb{R}$ as follows:

$$\psi_P(\mathbf{x}) := \tau_{a_\Omega}(\text{dist}(\mathbf{x}; \Omega^*)), \quad \mathbf{x} \in \Omega, \quad \Omega \in P,$$

where $\text{dist}(\mathbf{x}; A) := \inf_{\mathbf{t} \in A} \|\mathbf{x} - \mathbf{t}\|$ stands for the Euclidean distance from a point $\mathbf{x}$ to a set $A \subset \mathbb{R}^d$. Evidently, $\psi_P \in W_\infty^\Delta$ and $S_P \psi_P \equiv 0$. It is also clear that, for every $\Omega \in P$ and every point $\mathbf{x} \in \Omega^*$, we have $\psi_P(\mathbf{x}) \geq \frac{1}{64d} a_\Omega^2$. Taking into account (6), we obtain

$$\|\psi_P\|_{L_s(\Omega)}^s \geq \|\psi_P\|_{L_s(\Omega^*)}^s \geq \left(\frac{a_\Omega^2}{64d} \right)^s \frac{\mu\Omega}{2^d}$$

As a result, by definition of the set I and inequality (7), we have

$$\begin{aligned} \mathcal{E}_N^s(W_\infty^\Delta)_s &= \inf_{P \in \mathcal{P}_N} \mathcal{E}_P^s(W_\infty^\Delta)_s \geq \inf_{P \in \mathcal{P}_N} \|\psi_P\|_{L_s(\Pi)}^s = \inf_{P \in \mathcal{P}_N} \sum_{\Omega \in P} \|\psi_P\|_{L_s(\Omega)}^s \\ &\geq \frac{1}{2^d (64d)^s} \inf_{P \in \mathcal{P}_N} \sum_{\Omega \in P} a_\Omega^{2s} \mu \Omega \geq \frac{1}{2^d (64d)^s} \inf_{P \in \mathcal{P}_N} \sum_{\Omega \in I} a_\Omega^{2s} \mu \Omega \\ &\geq \frac{1}{2^d (64d)^s} \cdot \frac{b_1^{2s}}{(2N)^s} \inf_{P \in \mathcal{P}_N} \sum_{\Omega \in I} \mu \Omega \geq \frac{1}{2^d (64d)^s} \cdot \frac{b_1^{2s}}{(2N)^{2s}} \cdot \frac{\mu \Pi}{2}. \end{aligned}$$

Hence,

$$N^2 \cdot \mathcal{E}_N(W_\infty^\Delta)_s = N^2 \cdot \mathcal{E}_N(W_{s'}^\Delta)_1 \geq \frac{b_1^2 (\mu \Pi)^{\frac{1}{s}}}{2^{\frac{d+1}{s}} \cdot 256d}. \quad (8)$$

Therefore, combining (4), (5) and (8), we finish the proof of theorem.

Remark 1. Theorem 1 implies somewhat more general result. For a box partition $P \in \mathcal{P}_N$, we denote $\partial P := \bigcup_{\Omega \in P} \partial \Omega$ and call a mapping $\Phi : C(\partial P) \rightarrow C(\Omega)$ *method of recovery*. For $1 \leq q, s \leq \infty$, define the error of the best transfinite interpolation on box partitions as follows

$$\mathfrak{E}_N(W_q^\Delta)_s := \inf_{P \in \mathcal{P}_N} \inf_{\Phi : C(\partial P) \rightarrow C(\Omega)} \mathcal{E}_P(W_q^\Delta)_s.$$

From Theorem 2.3.1 in [21] it follows that, for $1 \leq s \leq \infty$,

$$\mathfrak{E}_N(W_\infty^\Delta)_s \geq \inf_{P \in \mathcal{P}_N} \sup_{f \in W_\infty^\Delta : \forall \mathbf{x} \in \partial P \Rightarrow f(\mathbf{x})=0} \|f\|_{L_s(\Pi)}.$$

Since the functions φ_P and ψ_P constructed in the proof of Theorem 1 vanish on ∂P , we conclude that there exists a constant $C_1 > 0$ such that $N^2 \cdot \mathfrak{E}_N(W_\infty^\Delta)_s > C_1$, for every $N \in \mathbb{N}$. Taking into account the trivial estimate $\mathfrak{E}_N(W_\infty^\Delta)_s \leq \mathcal{E}_N(W_\infty^\Delta)_s$, we obtain that $\mathfrak{E}_N(W_\infty^\Delta)_s$ has the order N^{-2} as $N \rightarrow \infty$. Similar result for the quantity $\mathfrak{E}_N(W_s^\Delta)_1$ can be obtained using the same idea.

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