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On interpolation of operator, which is the sum of weighted Hardy-Littlewood and Cesaro mean operators

Доведено, що оператори, які є сумою двох середніх вагових Харді-Літтлвуда $\int\limits_0^1 f(xt)\psi(t)dt$ **і Чезаро** $\int\limits_0^1 f(\frac{x}{t})t^{-n}\psi(t)dt$**, обмежені в просторах Лоренця** $\Lambda_{\varphi,a}(\mathbb{R})$**, якщо функції** $f(x)\in\Lambda_{\varphi,a}(\mathbb{R})$ **задовольняють умову** $|f(-x)|=|f(x)|, x>0$**, та незростаючих напівмультиплікативних функцій** ψ**, для яких виконуються такі умови:** $\frac{M_1}{\psi(t)}\le\psi\left(\frac{1}{t}\right)\le\frac{M_2}{\psi(t)}$ **для всіх** $0<t\le 1$**; при деяких** $0<\varepsilon<\frac{1}{2}$**,** $0<\delta<\frac{1}{2}$ **функції** $\psi(t)t^{1-\varepsilon}$**,** $\psi\left(\frac{1}{t}\right)t^{-\delta}$ **монотонно не спадають і функції** $\psi(t)t$**,** $\psi\left(\frac{1}{t}\right)$ **є абсолютно неперервними.**

Доведена основна теорема для функцій $f(x)\in\Lambda_{\varphi,a}(\mathbb{R})$**, які задовольняють умову** $|f(-x)|=|f(x)|, x>0$**, та для незростаючих напівмультиплікативних функцій** ψ**, для яких виконуються умови** $\frac{M_1}{\psi(t)}\le\psi\left(\frac{1}{t}\right)\le\frac{M_2}{\psi(t)}$ **для всіх** $0<t\le 1$**, та для** $0<\varepsilon<\frac{1}{2}, 0<\delta<\frac{1}{2}$ **функції** $\psi(t)t^{1-\varepsilon}, \psi\left(\frac{1}{t}\right)t^{-\delta}$ **монотонно не спадають, а функції** $\psi(t)t, \psi\left(\frac{1}{t}\right)$ **є абсолютно неперервними відносно** $t\in(0,\nu)$**, де** $\nu\in(0,\infty)$**. А саме, що, якщо функція** $\varphi(t)$ **множини** Φ **така, що для деяких** $a\in[1,\infty)$ **виконується умова** $\int\limits_0^1\left[M_\varphi(u^{-1})\right]^{\frac{1}{a}}dM_{\psi(t)t}(u)+\int\limits_1^\infty\left[M_\varphi(u^{-1})\right]^{\frac{1}{a}}dM_{\psi\left(\frac{1}{t}\right)}(u)<\infty$**, то існує така стала** $C>0$**, що для всіх функцій** $f(x)\in\Lambda_{\varphi,a}(0,\infty)$ **має місце нерівність**

$$\left(\int\limits_0^\infty\left[(T_\psi f)^*(t)\right]^a d\varphi(t)\right)^{\frac{1}{a}}\le C\left(\int\limits_0^\infty(f^*(t))^a d\varphi(t)\right)^{\frac{1}{a}},$$

де Φ **є об'єднанням функцій** $\varphi(t)=\mathrm{sign}t$ **і множини додатних, зростаючих, опуклих або вгнутих на нескінченому проміжку** $[0,\infty)$ **функцій** $\varphi(t)$**, які задовольняють умови** $\lim\limits_{t\to+0}\varphi(t)=\varphi(0)=0$**,** $\lim\limits_{t\to\infty}\varphi(t)=\infty$ **та** $\varphi(2t)=O(\varphi(t))$**, коли** $t\to+0$**,** $t\to\infty$**;** $M_\varphi(t)=\sup\limits_{0<s<\infty}\frac{\varphi(st)}{\varphi(s)}$ $(0<t<\infty)$**;** $f^*(t)$ **– не зростаюча перестановка модуля функції** $f(x)$**;** $(T_\psi f)(x)=\int\limits_0^1 f(xt)\psi(t)dt+\int\limits_0^1 f\left(\frac{x}{t}\right)t^{-1}\psi(t)dt$**. Зауважимо, що тоді оператор** $(T_\psi f)^*(x)$ **є оператором слабкого типу** $\left(\psi(t)t,\psi(t)t,\frac{1}{\psi(t)},\frac{1}{\psi(t)}\right)$**.**

Також доведені достатні умови, щоб оператори, що є сумою двох середніх вагових операторів Харді-Літтлвуда та Чезаро, коли $\psi(t)=t^{-\alpha}$**, де** $\alpha\in\left(0,\frac{1}{2}\right)$ **обмежені в просторах Лоренця** $\Lambda_{\varphi,a}(\mathbb{R})$**, якщо функції** $f(x)\in\Lambda_{\varphi,a}(\mathbb{R})$ **задовольняють умову** $|f(-x)|=|f(x)|$**,** $x>0$**.**

Ключові слова: фундаментальна функція, оператори слабкого типу, показники розтягування функцій, простори Лоренця.

Доказано, что операторы, которые есть суммой двух средних весовых Харди-Литтлвуда $\int\limits_0^1 f(xt)\psi(t)dt$ и Чезаро $\int\limits_0^1 f(\frac{x}{t})t^{-n}\psi(t)dt$, ограничены в пространствах Лоренца $\Lambda_{\varphi,a}(\mathbb{R})$, если функции $f(x) \in \Lambda_{\varphi,a}(\mathbb{R})$ удовлетворяют условию $|f(-x)| = |f(x)|, x > 0$, и невозрастающих полумультипликативных функций ψ, для которых выполняются следующие условия: $\frac{M_1}{\psi(t)} \leq \psi\left(\frac{1}{t}\right) \leq \frac{M_2}{\psi(t)}$ для всех $0 < t \leq 1$; при некоторых $0 < \varepsilon < \frac{1}{2}$, $0 < \delta < \frac{1}{2}$ функции $\psi(t)t^{1-\varepsilon}$, $\psi\left(\frac{1}{t}\right)t^{-\delta}$ монотонно не убывают и функции $\psi(t)t$, $\psi\left(\frac{1}{t}\right)$ являются абсолютно непрерывными. Также доказанные достаточные условия, чтобы операторы, которые являются суммой двух средних весовых операторов Харди-Литтлвуда и Чезаро, когда $\psi(t) = t^{-\alpha}$, где $\alpha \in \left(0, \frac{1}{2}\right)$, ограничены в пространствах Лоренца $\Lambda_{\varphi,a}(\mathbb{R})$, если функции $f(x) \in \Lambda_{\varphi,a}(\mathbb{R})$ удовлетворяют условию $|f(-x)| = |f(x)|$, $x > 0$.

Ключевые слова: фундаментальная функция, операторы слабого типа, показатели растяжения функции, пространства Лоренца.

It is proved that operators, which are the sum of weighted Hardy-Littlewood $\int\limits_0^1 f(xt)\psi(t)dt$ and Cesaro $\int\limits_0^1 f(\frac{x}{t})t^{-n}\psi(t)dt$, mean operators, are limited on Lorentz spaces $\Lambda_{\varphi,a}(\mathbb{R})$, if the functions $f(x) \in \Lambda_{\varphi,a}(\mathbb{R})$ satisfy the condition $|f(-x)| = |f(x)|, x > 0$, for such non-increasing semi-multiplicative functions ψ, for which satisfy the next conditions: $\frac{M_1}{\psi(t)} \leq \psi\left(\frac{1}{t}\right) \leq \frac{M_2}{\psi(t)}$ for all $0 < t \leq 1$; at some $0 < \varepsilon < \frac{1}{2}$, $0 < \delta < \frac{1}{2}$ functions $\psi(t)t^{1-\varepsilon}$, $\psi\left(\frac{1}{t}\right)t^{-\delta}$ do not decrease monotonically and functions $\psi(t)t$, $\psi\left(\frac{1}{t}\right)$ are absolutely continuous. Also, there are proved sufficient conditions that the operators, which are the sum of weighted Hardy-Littlewood and Cesaro mean operators, when $\psi(t) = t^{-\alpha}$, where $\alpha \in \left(0, \frac{1}{2}\right)$, on Lorentz spaces $\Lambda_{\varphi,a}(\mathbb{R})$, if the functions $f(x) \in \Lambda_{\varphi,a}(\mathbb{R})$ satisfy the condition $|f(-x)| = |f(x)|$, $x > 0$.

Key words: fundamental function, operators of weak type, index of stretching function, Lorentz spaces.

MSC2010: Pri 41A05, Sec 47A58, 46E30, 46A32

1. Introduction

Let the function $\psi : [0,1] \to [0,\infty]$. be given. In the work of Carton-Lebrun and Fosset [1] the boundedness of the weighted Hardy-Littlewood mean $\int\limits_0^1 f(xt)\psi(t)dt$ in $BMO(\mathbb{R}^n)$. In [1] it is proved that the operator $\int\limits_0^1 f(xt)\psi(t)dt$ is bounded in $BMO(\mathbb{R}^n)$, when the function $t^{1-n}\psi(t)$ is bounded on $[0,1]$. Xiao [2] considered the weighted Hardy-Littlewood mean and the weighted Cesaro $\int\limits_0^1 f(\frac{x}{t})t^{-n}\psi(t)dt$, mean, for an arbitrary Lebesgue measure of the complex-valued function f, given on $\mathbb{R}^n$ and expanded the result of Cardon-Lebrun and Fosset on the boundedness of the weighted Hardy-Littlewood mean in $BMO(\mathbb{R}^n)$. When $\psi = 1, n = 1$, then we have a classical

Hardy-Littlewood mean $(Uf)(x) = \frac{1}{x}\int\limits_0^x f(t)dt, x \neq 0$, and the classical Cesaro mean $(Vf)(x) = \int\limits_x^\infty \frac{f(t)}{t}dt$, $x > 0$ and $(Vf)(x) = -\int\limits_{-\infty}^x \frac{f(t)}{t}dt$, $x < 0$. Hardy considered the boundedness of the operators U and its conjugate V in space $L_p(0,\infty)$ [2].

Our task is to prove that the operators, which are the sum of two weighted Hardy-Littlewood and Cesaro mean operators, are bounded in Lorentz's spaces $\Lambda_{\varphi,a}(\mathbb{R})$ for functions $f(x) \in \Lambda_{\varphi,a}(\mathbb{R})$, such as $|f(-x)| = |f(x)|, x > 0$, and functions ψ, described below. For functions ψ the following conditions $\psi : [0,1] \to [0,\infty]$, are fulfilled , non-increasing semi-multiplicative, that is, such that for any x and t the inequality holds $\psi(xt) \leq \psi(x)\psi(t)$; $\frac{M_1}{\psi(t)} \leq \psi\left(\frac{1}{t}\right) \leq \frac{M_2}{\psi(t)}$ for all $0 < t \leq 1$. In addition to the above conditions for ψ, let for some $0 < \varepsilon < \frac{1}{2}, 0 < \delta < \frac{1}{2}$ the functions $\psi(t)t^{1-\varepsilon}, \psi\left(\frac{1}{t}\right)$ do not downgrade monotonously and the functions $\psi(t)t$, $\psi\left(\frac{1}{t}\right)$ are absolutely continuous relative to t. Also, the problem is to prove sufficient conditions for an operator that is the sum of two weighted Hardy - Littlewood and Cesarro mean operators, where $\psi(t) = t^{-\alpha}$, where $\alpha \in \left(0, \frac{1}{2}\right)$, are bounded in Lorentz spaces $\Lambda_{\varphi,a}(\mathbb{R})$ for such functions $f(x) \in \Lambda_{\varphi,a}(\mathbb{R})$, that $|f(-x)| = |f(x)|$, $x > 0$.

2. Designation and definition

Let Φ be a union of a function $\varphi(t) = \operatorname{sign} t$ and a set of positive, increasing, convex, or curved on infinite interval $[0,\infty)$ of functions $\varphi(t)$, satisfying the conditions $\lim\limits_{t\to+0}\varphi(t) = \varphi(0) = 0$, $\lim\limits_{t\to\infty}\varphi(t) = \infty$ and $\varphi(2t) = O(\varphi(t))$, when $t \to +0$, $t \to \infty$. For the function $\varphi(t)$ of the set Φ we denote through $M_\varphi(t) = \sup\limits_{0<s<\infty}\frac{\varphi(st)}{\varphi(s)}$ $(0 < t < \infty)$ and $\gamma_\varphi, \delta_\varphi$ correspondently, its lower and upper stretch indices [3].

Denote through $S(\mathbb{R}^n)$ the space of real Lebesgue-measurable functions on $\mathbb{R}^n$ and $f^*(t)$ – non-increasing rearrangement of the function $f(x) \in S(\mathbb{R}^n)$ module.

Let $a \in (0,\infty]$ and $\varphi(t)$ be a non-downgrading absolutely continuous function on an infinite interval $[0,\infty)$ such that $\varphi(0) = 0$. The Lorentz space $\Lambda_{\varphi,a}(\mathbb{R}^n)$ consists of functions $f(x) \in S(\mathbb{R}^n)$, for which a quasi-norm $||f||_{\Lambda_{\varphi,a}} = \left(\int\limits_0^\infty (f^*(t))^a d\varphi(t)\right)^{\frac{1}{a}}$ in the case $\varphi(t) \neq \operatorname{sign} t$, $0 < a < \infty$ or a quasi-norm $||f||_{\Lambda_{\varphi,\infty}} = \sup\limits_{0<t<\infty} f^*(t)\varphi(t)$ is finite, if $a = \infty$ [4].

Let the functions $\varphi_0(t), \varphi_1(t) \in \Phi$ be such that $\varphi(t) \neq \operatorname{sign} t$ and $\varphi_0(t)/\varphi_1(t)$ increase on $(0,\infty)$. The space $\Lambda_{\varphi_0,1}(\mathbb{R}^n)+\Lambda_{\varphi_1,1}(\mathbb{R}^n)$ consists of functions $f(x) \in S(\mathbb{R}^n)$, for which the sum $\int\limits_0^1 f^*(t)d\varphi_0(t) + \int\limits_1^\infty f^*(t)d\varphi_1(t)$ is finite. If $\varphi_1(t) = \operatorname{sign} t$ and the conditions $\sup\limits_{0<u<1} M_{\varphi_0}(u)(1 - \ln u) \leq 1$ are fulfilled, then the space of such functions $f(x) \in S(\mathbb{R}^n)$ that $\int\limits_0^1 f^*(t)d\varphi_0(t) + \int\limits_1^\infty f^*(t)t^{-1}dt < \infty$ [4] is denoted $\Lambda_{\varphi_0,1}(\mathbb{R}^n) + L^{\infty 1}(\mathbb{R}^n)$.

Then we will assume the functions $\varphi_0(t), \varphi_1(t), \psi_0(t), \psi_1(t) \in \Phi$, $\varphi_0(t)/\varphi_1(t)$ increase on $(0, \infty)$, the domain of values $\varphi_0(t)/\varphi_1(t)$ coincides with the domain of values $\psi_0(t)/\psi_1(t)$ and $m(t)$ – Lebesgue-measurable, the positive solution of the equation

$$\varphi_0(m(t))/\varphi_1(m(t)) = \psi_0(t)/\psi_1(t).$$

A quasilinear operator T is called a weak type (φ_0, ψ_0) operator, if there $C > 0$ is such that for any $f(x) \in \Lambda_{\varphi_0,1}(\mathbb{R}^n)$ and $t > 0$ the inequality is performed

$$(Tf)^*(t)\psi_0(t) \leq C \left(\int_0^\infty f^*(u) d\varphi_0(u) \right),$$

when $\varphi_0(t) \neq \operatorname{sign} t$, or inequality

$$(Tf)^*(t)\psi_0(t) \leq C \sup_{0<t<\infty} (f^*(t)\varphi_0(t))$$

in the case of $\varphi_0(t) = \operatorname{sign} t$ [3].

A quasilinear operator T is called a weak type operator $(\varphi_0, \psi_0, \varphi_1, \psi_1)$, if there is such $C > 0$, that for any $f(x) \in \Lambda_{\varphi_0,1}(\mathbb{R}^n) + \Lambda_{\varphi_1,1}(\mathbb{R}^n)$ and $t > 0$ the inequality is performed

$$(Tf)^*(t) \leq C \left((\psi_0(t))^{-1} \int_0^{m(t)} f^*(u) d\varphi_0(u) + (\psi_1(t))^{-1} \int_{m(t)}^\infty f^*(u) d\varphi_1(u) \right),$$

when $\varphi_1(t) \neq \operatorname{sign} t$, or inequality

$$(Tf)^*(t) \leq C \left((\psi_0(t))^{-1} \int_0^{m(t)} f^*(u) d\varphi_0(u) + (\psi_1(t))^{-1} \int_{m(t)}^\infty f^*(u) u^{-1} du \right),$$

in the case $\varphi_1(t) = \operatorname{sign} t$ and $\sup\limits_{0<u<1} (M_{\varphi_0}(u)(1 - \ln u)) \leq 1$ [5].

Positive constant M_1, M_2 depend of ψ, denote by C the positive constant, which in each case may be different and does not depend on the essential parameters.

3. Known results

In his work Carton-Lebrun and Fosset [1] on the boundedness of the weighted Hardy-Littlewood mean $\int\limits_0^1 f(xt)\psi(t)dt$ in $BMO(\mathbb{R}^n)$. Xiao [2] expanded the result of Cardon-Lebrun and Fosset on the boundedness of the weighted Hardy-Littlewood mean in $BMO(\mathbb{R}^n)$ and proved that the weighted Hardy-Littlewood mean operator is bounded in $L_p(\mathbb{R}^n)$, where $p \in [1, \infty]$, $\psi : [0, 1] \to [0, \infty)$, if and only if, when

$\int_0^1 t^{-\frac{n}{p}}\psi(t)dt < \infty$ is bounded in $BMO(\mathbb{R}^n)$, $\psi : [0,1] \to [0,\infty)$, then and only then, when $\int_0^1 \psi(t)dt < \infty$. Cesaro's weighted mean operator $\int_0^1 f\left(\frac{x}{t}\right) t^{-n}\psi(t)dt$ is bounded in $L_p(\mathbb{R}^n)$, where $p \in [1,\infty]$, $\psi : [0,1] \to [0,\infty)$, if and only if, when $\int_0^1 t^{-n\left(1-\frac{1}{p}\right)}\psi(t)dt < \infty$. The condition of the boundedness of weighted Cesaro mean $\int_0^1 f\left(\frac{x}{t}\right) t^{-n}\psi(t)dt$ in $BMO(\mathbb{R}^n)$ is $\int_0^1 t^{-n}\psi(t)dt < \infty$.

The following statements are proved below.

Theorem 1. *If $n = 1$, a function $\psi : [0,1] \to [0,\infty)$ is non-increasing and semi-multiplicative and such that $\frac{M_1}{\psi(t)} \le \psi\left(\frac{1}{t}\right) \le \frac{M_2}{\psi(t)}$ for all $0 < t \le 1$. For some $0 < \varepsilon < \frac{1}{2}, 0 < \delta < \frac{1}{2}$ the functions $\psi(t)t^{1-\varepsilon}, \psi\left(\frac{1}{t}\right)t^{-\delta}$ do not downgrade monotonously and the functions $\psi(t)t, \psi\left(\frac{1}{t}\right)$ are absolutely continuous relative to $t \in (0,\nu)$, where $\nu \in (0,\infty)$. If the function $\varphi(t)$ from the set Φ is such that for some $a \in [1,\infty)$ satisfies the condition*

$$\int_0^1 \left[M_\varphi(u^{-1})\right]^{\frac{1}{a}} dM_{\psi(t)t}(u) + \int_1^\infty \left[M_\varphi(u^{-1})\right]^{\frac{1}{a}} dM_{\psi\left(\frac{1}{t}\right)}(u) < \infty,$$

then there is such a constant $C > 0$ that for all functions $f(x) \in \Lambda_{\varphi,a}(0,\infty)$, the inequality is performed

$$\left(\int_0^\infty \left[(T_\psi f)^*(t)\right]^a d\varphi(t)\right)^{\frac{1}{a}} \le C\left(\int_0^\infty (f^*(t))^a d\varphi(t)\right)^{\frac{1}{a}}.$$

Proof. Let's denote through

$$(T_\psi f)(x) = \int_0^1 f(xt)\psi(t)dt + \int_0^1 f\left(\frac{x}{t}\right) t^{-1}\psi(t)dt.$$

Let $x > 0, z > 0, u > 0$. We convert the sum of two integrals by substitution $xt = z, xdt = z; \frac{x}{t} = u, -\frac{dt}{t} = \frac{du}{u}$

$$\int_0^x f(z)\psi\left(\frac{z}{x}\right)\frac{dz}{x} + \int_x^\infty f(u)u^{-1}\psi\left(\frac{x}{u}\right)du.$$

By replacing the variables $z = \tau, dz = d\tau; u = \tau, du = d\tau$, we transform this sum of integrals to the sum and use the semi-multiplicity of function $\psi(t)$

$$\int_0^x f(\tau)\psi\left(\frac{\tau}{x}\right)\frac{d\tau}{x} + \int_x^\infty f(\tau)\psi\left(\frac{x}{\tau}\right)\frac{d\tau}{\tau} \le$$

$$\leq \psi\left(\frac{1}{x}\right)\frac{1}{x}\int_0^x f(\tau)\psi(\tau)d\tau + \psi(x)\int_x^\infty f(\tau)\psi\left(\frac{1}{\tau}\right)\frac{d\tau}{\tau}.$$

Functions $|f(\tau)| \geq 0, \psi\left(\frac{\tau}{x}\right) \geq 0, \psi\left(\frac{x}{\tau}\right) \geq 0.$ then

$$|(T_\psi f)(x)| \leq \int\limits_0^x |f(\tau)|\psi\left(\frac{\tau}{x}\right)\frac{d\tau}{x} + \int\limits_x^\infty |f(\tau)|\psi\left(\frac{x}{\tau}\right)\frac{d\tau}{\tau} \leq$$

$$\leq \psi\left(\frac{1}{x}\right)\frac{1}{x}\int_0^x |f(\tau)|\psi(\tau)d\tau + \psi(x)\int_x^\infty |f(\tau)|\psi\left(\frac{1}{\tau}\right)\frac{d\tau}{\tau} \leq$$

$$\leq \psi\left(\frac{1}{x}\right)\frac{1}{x}\int_0^x |f(\tau)|\frac{\tau\psi(\tau)}{\tau^{\varepsilon+1-\varepsilon}}d\tau + \psi(x)\int_x^\infty |f(\tau)|\psi\left(\frac{1}{\tau}\right)\frac{d\tau}{\tau^{\delta+1-\delta}}.$$

Firstly, we use the condition of the theorem that, for some $0 < \varepsilon < \frac{1}{2}, 0 < \delta < \frac{1}{2}$, the functions $\psi(t)t^{1-\varepsilon}.\psi\left(\frac{1}{t}\right)t^{-\delta}$ do not downgrade monotonously and the functions $\psi(t)t, \psi\left(\frac{1}{t}\right)$ are absolutely continuous relative to $t \in (0, \nu)$, where $\nu \in (0, \infty)$:

$$\int\limits_0^\tau \frac{\psi(t)t}{t}dt \leq \int\limits_0^\tau \frac{\psi(t)t}{t^{\varepsilon+1-\varepsilon}}dt \leq \frac{1}{\varepsilon}\psi(\tau)\tau = \frac{1}{\varepsilon}\int_0^\tau d(\psi(t)t),$$

$$\int\limits_0^\tau \frac{\psi\left(\frac{1}{t}\right)dt}{t} \leq \int\limits_0^\tau \frac{\psi\left(\frac{1}{t}\right)dt}{t^{\delta+1-\delta}} \leq \frac{1}{\delta}\psi\left(\frac{1}{\tau}\right) = \frac{1}{\delta}\int_0^\tau d\left(\psi\left(\frac{1}{t}\right)\right),$$

We use the condition of the theorem $\frac{M_1}{\psi(t)} \leq \psi\left(\frac{1}{t}\right) \leq \frac{M_2}{\psi(t)}$

$$|(Tf)(x)| \leq \frac{M_1}{\varepsilon}\frac{1}{\psi(x)x}\int\limits_0^x |f(\tau)|d(\tau\psi(\tau)) + \frac{M_2}{\delta}\psi(x)\int\limits_x^\infty |f(\tau)|d\left(\psi\left(\frac{1}{\tau}\right)\right) \leq$$

$$\leq \frac{M_1}{\varepsilon}\frac{1}{\psi(x)x}\int_0^x |f(\tau)|d(\tau\psi(\tau)) + \frac{M_2}{\delta}\psi(x)\int_x^\infty |f(\tau)|d\left(\psi\left(\frac{1}{\tau}\right)\right).$$

The function $\frac{\psi(\tau)\tau}{\psi\left(\frac{1}{\tau}\right)}$ is increasing for all $0 < \tau < \infty$.

By the property 13 of non-increasing rearrangements from [3], we obtain

$$|T_\psi f(x)| \leq C\int_0^\infty f^*(\tau)d\min_i\left(\frac{\psi(t)t}{\psi(x)x}, \frac{\psi(x)}{\psi(\tau)}\right) =$$

$$= C\left(\int\limits_0^x f^*(\tau)\psi\left(\frac{\tau}{x}\right)\frac{d\tau}{x} + \int\limits_x^\infty f^*(\tau)\psi\left(\frac{x}{\tau}\right)\frac{d\tau}{\tau}\right).$$

In the right side there is a non-increasing function, by the property 18 of non-increasing rearrangements from [3] we obtain

$$(T_\psi f)^*(x) \le C\left(\psi\left(\frac{1}{x}\right)\frac{1}{x}\int\limits_0^x f^*(\tau)\psi(\tau)d\tau + \psi(x)\int_x^\infty f^*(\tau)\psi\left(\frac{1}{\tau}\right)\frac{d\tau}{\tau}\right).$$

The operator $(T_\psi f)^*(x)$ is a weak type operator $\left(\psi(t)t, \psi(t)t, \frac{1}{\psi(t)}, \frac{1}{\psi(t)}\right)$.
We apply Theorem 6 from work [4], we have

$$\left(\int\limits_0^\infty \left[(T_\psi f)^*(t)\right]^a d\varphi(t)\right)^{\frac{1}{a}} \le C\left(\int\limits_0^\infty (f^*(t))^a d\varphi(t)\right)^{\frac{1}{a}}.$$

Theorem 1 is proved.

Corollary 1. *If $n=1$ the function $f(x)\in\Lambda_{\varphi,a}(0,\infty), \psi(t)=t^{-\alpha}, 0<\alpha<\frac{1}{2}$, the function $\varphi(t)$ of the set Φ is such that for some $a\in[1,\infty)$ satisfies the condition*

$$\int\limits_0^1 \left[M_\varphi(u^{-1})\right]^{\frac{1}{a}} dM_{t^{1-\alpha}}(u) + \int\limits_1^\infty \left[M_\varphi(u^{-1})\right]^{\frac{1}{a}} dM_{t^\alpha}(u) < \infty.$$

Then there is such a constant $C>0$, that for all functions $f(x)\in\Lambda_{\varphi,a}(0,\infty)$ the inequality is performed

$$\left(\int\limits_0^\infty \left[(T_\alpha f)^*(t)\right]^a d\varphi(t)\right)^{\frac{1}{a}} \le C\left(\int\limits_0^\infty (f^*(t))^a d\varphi(t)\right)^{\frac{1}{a}}.$$

The provement follows from Theorem 1, since the function $t^{-\alpha}, \alpha\in(0,\frac{1}{2})$ satisfies the conditions of Theorem 1. Corollary 1 is proved.

Theorem 2. *Suppose $n=1$, $\psi(t)=1$, the function $\varphi(t)$ from the set Φ is such that for some $a\in[1,\infty)$ satisfies the condition*

$$\int\limits_0^1 \left[M_\varphi(u^{-1})\right]^{\frac{1}{a}} du + \int\limits_1^\infty \left[M_\varphi(u^{-1})\right]^{\frac{1}{a}} u^{-1}du < \infty.$$

Then there is such a constant $C>0$ that for all functions $f(x)\in\Lambda_{\varphi,a}(0,\infty)$, the inequality is performed

$$\left(\int\limits_0^\infty \left[(T_0 f)^*(t)\right]^a d\varphi(t)\right)^{\frac{1}{a}} \le C\left(\int\limits_0^\infty (f^*(t))^a d\varphi(t)\right)^{\frac{1}{a}}.$$

Proof. Hardy [6] studied at $n = 1$, $\psi(t) = 1$ the boundedness of operators $\frac{1}{x}\int\limits_0^x f(t)dt, \int\limits_x^\infty \frac{f(t)}{t}dt$. On the other hand, the operator

$$(Tf)(x) = \int\limits_0^1 f(xt)dt + \int\limits_0^1 f\left(\frac{x}{t}\right)dt =$$

$$= \frac{1}{x}\int\limits_0^x f(u)du + \int\limits_x^\infty \frac{f(u)}{u}du \leq \frac{1}{x}\int\limits_0^x f^*(t)dt + \int\limits_x^\infty \frac{f^*(t)dt}{t}$$

is a weak type operator $(t, t, \mathrm{sign}t, \mathrm{sign}t)$ and there is performed $\sup\limits_{0<t<1} t(1-\ln t) < 1$. From this, according to the property of non-increasing rearrangements 13 and 18, we obtain

$$(Tf)^*(t) \leq \frac{1}{t}\int\limits_0^t f^*(\tau)d\tau + \int\limits_t^\infty \frac{f^*(\tau)d\tau}{\tau},$$

Then, as $m(t) = t$, according to Theorem 7 from the paper [4], from the condition

$$\int\limits_0^1 \left[M_\varphi(u^{-1})\right]^{\frac{1}{a}} du + \int\limits_1^\infty \left[M_\varphi(u^{-1})\right]^{\frac{1}{a}} u^{-1}du < \infty$$

Theorem 2 follows.

Let $S(\mathbb{R})$ be given, a Lebesgue measure, m be introduced, which puts in correspondence to each interval (a, b) where the number $m(a, b) = b - a$, where $-\infty \leq a < b \leq \infty$. The distribution function $m_g = m\{x : |g|(x) > \lambda\}$, where $\lambda > 0$, corresponds each function $g(x) \in S(\mathbb{R})$ being given on the negative on the semicolon. On the positive of the semicolon, there is an positive function $g_1(\tau)$, whose distribution function is $m_{g_1} = m\{x : |g_1|(x) > \lambda\}$, where $\lambda > 0$ and such that $m_{|g|}(\lambda) = m_{g_1}(\lambda)$. The rearrangement of the function $g_1(\tau) \in S(\mathbb{R})$ is determined by the formula $g_1^*(t) = \inf\{\lambda : m_{g_1}(\lambda) < t\}$. The norm in space $E(-\infty, 0)$ is given by $||g||_{E(-\infty,0)} = ||g_1^*||_{E(0,\infty)}$.

Denote through $(\overline{T}_\psi f)^*(t)$ a non-increasing rearrangement of the image $|(T_\psi f)(x)|$ module with negative x, and positive numbers we denote respectively by $(T_\psi f)^*(t)$.

Theorem 3. *If* $n = 1$, *a function* $\psi : [0,1] \to [0,\infty)$, *is non-increasing, then it is semimultiplicative and such that* $\frac{M_1}{\psi(t)} \leq \psi\left(\frac{1}{t}\right) \leq \frac{M_2}{\psi(t)}$ *for all* $0 < t \leq 1$, *at some* $0 < \varepsilon < \frac{1}{2}$, $0 < \delta < \frac{1}{2}$ *functions* $\psi(t)t^{1-\varepsilon}, \psi\left(\frac{1}{t}\right)t^{-\delta}$ *do not downgrade monotonically and the functions* $\psi(t)t, \psi\left(\frac{1}{t}\right)$ *are absolutely continuous relative to* $t \in (0, \nu)$, *where*

$\nu \in (0, \infty)$. *If the function $\varphi(t)$ from the set Φ is such that for some $a \in [1, \infty)$ satisfies the condition*

$$\int_0^1 \left[M_\varphi(u^{-1})\right]^{\frac{1}{a}} dM_{\psi(u)u}(u) + \int_1^\infty \left[M_\varphi(u^{-1})\right]^{\frac{1}{a}} dM_{\psi\left(\frac{1}{u}\right)}(u) < \infty.$$

Then there is such a constant $C > 0$ for functions $f(x) \in \Lambda_{\varphi,a}(\mathbb{R})$ such that $|f(-x)| = |f(x)|, x > 0$, the inequality is performe

$$\left(\int_0^\infty \left[(\overline{T}_\psi f)^*(t)\right]^a d\varphi(t)\right)^{\frac{1}{a}} \le \left(\int_0^\infty \left[(T_\psi f)^*(t)\right]^a d\varphi(t)\right)^{\frac{1}{a}} \le C\left(\int_0^\infty (f^*(t))^a d\varphi(t)\right)^{\frac{1}{a}}.$$

Proof. Let $x < 0, t > 0$, we use functions $f(x)$, such that $|f(-|x|)| = |f(|x|)|$, then

$$(T_\psi f)(x) = \int_0^1 f(xt)\psi(t)dt + \int_0^1 f\left(\frac{x}{t}\right) t^{-1}\psi(t)dt =$$

$$= \int_0^1 |f(-|x|t)|\psi(t)dt + \int_0^1 |f\left(\frac{-|x|}{t}\right)| t^{-1}\psi(t)dt =$$

$$= \int_0^1 |f(|x|t)|\psi(t)dt + \int_0^1 |f\left(\frac{|x|}{t}\right)| t^{-1}\psi(t)dt.$$

And for positive $|x|$ we have already proved the statement of the theorem, hence we obtain for negative x:

$$|(T_\psi f)(x)| \le |(T_\psi f)(|x|)|.$$

Using the properties 13 and 18 of non-increasing rearrangements of functions from paper [3], we obtain

$$(\overline{T}_\psi f)^*(t) \le (T_\psi f)^*(t).$$

Then we use Theorem 1 of this paper and Theorem 6 of paper [4]

Theorem 3 is proved.

Corollary 2. *If $n = 1$, $\psi(t) = t^{-\alpha}, 0 < \alpha < \frac{1}{2}$, the function $\varphi(t)$ of the set Φ is such that for some $a \in [1, \infty)$ satisfies the condition*

$$\int_0^1 \left[M_\varphi(u^{-1})\right]^{\frac{1}{a}} dM_{t^{1-\alpha}}(u) + \int_1^\infty \left[M_\varphi(u^{-1})\right]^{\frac{1}{a}} dM_{t^\alpha}(u) < \infty.$$

Then there is such a constant $C > 0$, that for all functions $f(x) \in \Lambda_{\varphi,a}(\mathbb{R})$, for such that $|f(-x)| = |f(x)|, x > 0$, for $t > 0$ the inequality is performed

$$\left(\int_0^\infty \left[(\overline{T}_\psi f)^*(t) \right]^a d\varphi(t) \right)^{\frac{1}{a}} \leq \left(\int_0^\infty \left[(T_\psi f)^*(t) \right]^a d\varphi(t) \right)^{\frac{1}{a}} \leq C \left(\int_0^\infty (f^*(t))^a d\varphi(t) \right)^{\frac{1}{a}}.$$

Proof. The corollary follows from Theorem 3, since the function $\psi(t) = t^{-\alpha}, \alpha \in (0, \frac{1}{2})$ satisfies the conditions of Theorem 3. The corollary 2 is proved.

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