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Estimates of the error of interval quadrature formulas on some classes of differentiable functions

На симетричному класі $H-$ $2\pi-$періодичних неперервних функцій розглядається інтервальна квадратурна формула

$$\int_0^{2\pi} f(t)dt - \sum_{k=1}^{n} c_k \frac{1}{2h} \int_{x_k-h}^{x_k+h} f(t)dt = R_n(f;\vec{c_n};\vec{x_n};h), \tag{0.1}$$

де $\vec{c_n} = \{c_k\}_{k=1}^n$, $c_k \in R$ є коефіцієнти квадратурної формули (0.1), $\vec{x_n} = \{x_k\}_{k=1}^n$, $x_k \in [0, 2\pi)-$ узли, $h \in (0, \pi/n,)$ і $R_n(f;\vec{c_n};\vec{x_n};h)$ є похибкою квадратурної формули (0.1) для функції f. Рівність

$$R_n(H;\vec{c_n};\vec{x_n};h) := \sup_{f\in H} R_n(f;\vec{c_n};\vec{x_n};h)$$

визначає похибку інтервальної квадратурної формули на класі функцій H. Інтервальні квадратурні формули розглядались, наприклад, в роботах Кузьміної А.Л., Шаріпова Р.Н., Бабенко В.Ф., Моторного В.П., Бородачева С.В. та інших математиків. Якщо перейти в (0.1) до границі, коли $h \to 0$, то одержимо звичайну квадратурну формулу.

Задача про оптимізацію інтервальних квадратурних формул (0.1) складається у знаходженні величини

$$E_n(H;\vec{c_0};\vec{x_0};h) = \inf_{;\vec{c_n};\vec{x_n}} R_n(H;\vec{c_n};\vec{x_n};h), \tag{0.2}$$

яка називається похибкою оптимальної інтервальної квадратурної формули на класі H, наборів коефіцієнтів $\vec{c_0} = \{c_k^0\}_{k=1}^n$ і вузлів $\vec{x_0} = \{x_k^0\}_{k=1}^n$, для яких досягається точна нижня межа (0.2). Ця задача розглядалась, наприклад, в роботах [3 - 8]. Виявилося, що у багатьох випадках для класів $2\pi-$періодичних неперервних функцій оптимальною інтервальною квадратурною формулою є формула

$$\int_0^{2\pi} f(t)dt - \frac{\pi}{nh} \sum_{k=0}^{n-1} \int_{-h}^{h} f(t + \frac{2k\pi}{n})dt = R_n(f;\vec{c_0};\vec{x_0};h), \tag{0.3}$$

тобто інтервальна квадратурна формула (0.1), у якій $\vec{c_0} = \{c_k^0\}_{k=1}^n$, $c_k^0 = 2\pi/n$ є коефіцієнтами квадратурної формули (0.1), і $\vec{x_0} = \{x_k^0\}_{k=1}^n$, $x_k^0 = \frac{2\pi k}{n}-$

є вузлами, $h \in (0, \pi/n)$. Очевидно, що інтервальна квадратурна формула (0.3) для функції f збігається з формулою прямокутників для функції Стеклова $f_h(t)$. У даній роботі одержана точна оцінка похибки інтервальної квадратурної формули (0.3) на класі W^rH^ω.

Ключові слова: функція, інтеграл, інтервальна квадратурна формула, похибка, сума

The exact value of error of interval quadrature formulas

$$\int_0^{2\pi} f(t)dt - \frac{\pi}{nh}\sum_{k=0}^{n-1}\int_{-h}^{h} f(t+\frac{2k\pi}{n})dt = R_n(f;\vec{c_0};\vec{x_0};h)$$

obtained for the classes $W^rH^\omega(r = 1, 2, ...)-$ differentiable of periodic functions for which the modulus of continuity of the $r-$th derivative is majorized by the given modulus of continuity $\omega(t)$. This interval quadrature formula coincides with the formula rectangles for the Steklov functions $f_h(t)$ and is optimal for some important classes of functions.

Key words: function, integral, interval quadrature formula, error, sum

MSC2010: Pri 41A55, Sec 26A42

Let $H-$ be a class of $2\pi-$ periodic continuous functions such that together with the function $f(t)$ the class H contains the functions $-f(t), f(t+a), f(t)+a$ for an arbitrary namber a, we denote by H_n subclass $2\pi/n-$ of periodic functions. An interval quadrature formula is called the formula

$$\int_0^{2\pi} f(t)dt - \sum_{k=1}^{n} c_k \frac{1}{2h}\int_{x_k-h}^{x_k+h} f(t)dt = R_n(f;\vec{c_n};\vec{x_n};h), \qquad (1)$$

where $\vec{c_n} = \{c_k\}_{k=1}^n$, $c_k \in R$ are the coefficients of the quadrature (1), $\vec{x_n} = \{x_k\}_{k=1}^n$, $x_k \in [0, 2\pi)-$ are the nodes, $h \in (0, \pi/n,)$ and $R_n(f;\vec{c_n};\vec{x_n};h)$ is the error of the quadrature formula (1) for the function f. Then the quantity

$$R_n(H;\vec{c_n};\vec{x_n};h) := \sup_{f\in H} R_n(f;\vec{c_n};\vec{x_n};h)$$

is the error of the quadrature formula (1)(corresponding to the vectors $\vec{c_n};\vec{x_n};h)$ on the class H.

Interval quadrature formulas were considered, for example, in [1 - 8]. From the point of view of application, interval quadrature formulas are more natural than ordinary quadrature formulas, because the results of measuring a physical quantity, in some cases, due to the installation of measuring instruments, is the averaging of the function that characterizes the measured value. If we take a limit in (1) when $h \to 0$, then we obtain the usual quadrature formula. The problem of optimization of interval quadrature formulas (1) consists in finding the value

$$E_n(H;\vec{c_0};\vec{x_0};h) = \inf_{;\vec{c_n};\vec{x_n}} R_n(H;\vec{c_n};\vec{x_n};h) \qquad (2)$$

which is called the error of the optimal quadrature formula in the class H, and the sets of coefficients $\vec{c_0} = \{c_k^0\}_{k=1}^n$ and nodes $\vec{x_0} = \{x_k^0\}_{k=1}^n$, for which the exact lower limit (2) is reached. The problem of optimization of interval quadrature formulas was considered, for example, in [3 - 8]. It turned out that in many cases for classes of $2\pi-$periodic continuous functions the optimal interval quadrature formula is the formula

$$\int_0^{2\pi} f(t)dt - \frac{\pi}{nh}\sum_{k=0}^{n-1}\int_{-h}^{h} f(t+\frac{2k\pi}{n})dt = R_n(f;\vec{c_0};\vec{x_0};h), \tag{3}$$

where $\vec{c_0} = \{c_k^0\}_{k=1}^n$, $c_k^0 = 2\pi/n$ are the coefficients of the quadrature (1),$\vec{x_0} = \{x_k^0\}_{k=1}^n$, and $x_k^0 = \frac{2\pi(k-1)}{n}-$ are the nodes, $h \in (0, \pi/n)$.

Quadrature formula (3) is exact for any constant, i.e. the integral over the segment $[0, 2\pi]$ is equal to the quadrature sum. So we may assume that the function f in (3) is equel to zero in the average i.e.

$$-\frac{\pi}{nh}\sum_{k=0}^{n-1}\int_{-h}^{h} f(t+\frac{2k\pi}{n})dt = R_n(f;\vec{c_0};\vec{x_0};h), \tag{4}$$

Let $f(t)$ be a $2\pi-$periodic function integrable on the period, let us denote by $f_h(t)$ the Steklov function, and let S_hf be the Steklov operator. that is

$$S_hf = f_h(t) = \frac{1}{2h}\int_{t-h}^{t+h} f(u)du.$$

The quadrature sum in (3) may be represented as

$$\frac{2\pi}{n}\sum_{k=0}^{n-1} f_h(\frac{2k\pi}{n})dt,$$

that is, the interval quadrature formula (3) coincides with the rectangles formula for the function $f_h(t)$.

We introduce the following classes:

H^ω : continuous functions whose moduli of continuity satisfy $\omega(f;t) \le \omega(t)$, where $\omega(t)$ is a given convex modulus of continuity.

$W^rH^\omega(r = 1, 2, ...)-$ the class of $2\pi-$ periodic functions, for which r-th derivative $f^{(r)} \in H^\omega$, where $\omega(t)$ is a given convex modulus of continuity.

$W_1^r(r = 1, 2, ...)$ - the class of all $2\pi-$ periodic functions that have absolutely continuity $(r-1)-$ st derivative and

$$\int_0^{2\pi} |f^{(r)}(x)|dx \le 1$$

$W_\infty^r(r = 1, 2, ...)-$ the a class of all $2\pi-$ periodic functions that have a absolutely continuity derivative $f^{(r-1)}(x)$ and $|f^{(r)}(x)| \le 1$ almost everywhere.

Finally, denote by $W^rH_n^\omega$ the subclass of $2\pi/n-$ periodic functions from W^rH^ω.

In [5] the optimality of the interval quadrature formula (3) on the class W_1^r is proved, and in the paper [6] the optimality of the interval quadrature formula (3) on the class W_∞^r is proved.

In the present paper, we obtain the exact estimate of the error of the interval quadrature formula (3) on the class W^rH^ω.

Theorem 1. *The equality*

$$R_n(H;\vec{c_0};\vec{x_0};h) = R_n(H_n;\vec{c_0};\vec{x_0};h), \tag{4}$$

holds, where H and H_n are the classes of functions defined in above.

Proof. Since H_n is a subset of the set H, the left part does not exceed the right one. On the other hand, due to the properties of the class H (symmetry, shift invariance), for any function $f \in H$, the function

$$\psi_f(t) = \frac{1}{n}\sum_{i=0}^{n-1} f(t + \frac{2\pi i}{n})$$

has the following properties

$$\int_0^{2\pi} \psi_f(t)dt = \int_0^{2\pi} f(t)dt,$$

and the quadrature sums for the functions f and ψ_f coincide. Indeed,

$$\frac{\pi}{nh}\sum_{k=0}^{n-1}\int_{-h}^{h} \psi_f(t + \frac{2k\pi}{n})dt =$$

$$= \frac{\pi}{n^2h}\sum_{k=0}^{n-1}\sum_{i=0}^{n-1}\int_{-h}^{h} f(t + \frac{2\pi(k+i)}{n})dt =$$

$$= \frac{\pi}{n^2h}\sum_{j=0}^{n-1}\sum_{k+i=j(mod n)}\int_{-h}^{h} f(t + \frac{2\pi(k+i)}{n})dt =$$

$$= \frac{\pi}{nh}\sum_{j=0}^{n-1}\int_{-h}^{h} f(t + \frac{2\pi j}{n}). \tag{5}$$

Since the integrals and quadratic sums for the functions $f(t)$ and $\psi_f(t)$ coincide, then the errors of quadrature formula (2) for functions $f(t)$ and $\psi_f(t)$ as well , and this implies equality (4).

Corollary 1. *Due to equations (3 - 5) we have*

$$R_n(W^rH^\omega;\vec{c_0};\vec{x_0};h) = R_n(W^rH_n^\omega;\vec{c_0};\vec{x_0};h) =$$

$$= \sup_{f\in W^rH_n^\omega} \frac{\pi}{nh}\left|\sum_{k=0}^{n-1}\int_{-h}^{h} f(t + \frac{2k\pi}{n})dt\right| \tag{6}$$

Ineed, since the function $f(t)$ is $2\pi/n-$ periodic, the terms in the right-hand side of equation (6) coincide, therefore

$$\sup_{f\in W^rH_n^\omega}\frac{\pi}{nh}|\sum_{k=0}^{n-1}\int_{-h}^{h}f(t+\frac{2k\pi}{n})dt|=\sup_{f\in W^rH_n^\omega}\frac{\pi}{h}|\int_{-h}^{h}f(t)dt|. \quad (7)$$

Denote by I_r (r -natural number) the integration operator, i.e.

$$I_rf=I_rf(x)=\frac{1}{\pi}\int_0^{2\pi}D_r(x-t)f(t)dt,\ \int_0^{2\pi}f(t)dt=0, \quad (8)$$

where $D_r(t)$ -Bernoulli kernel:

$$D_r(t)=\sum_{k=1}^{\infty}\frac{\cos(kt-\frac{r\pi}{2})}{k^r},\ \ r=1,2,....$$

The right-hand side of equation (7) may be represented by the integration operator:

$$\sup_{f\in W^rH_n^\omega}\frac{\pi}{h}|\int_{-h}^{h}f(t)dt|=\sup_{f\in W^rH_n^\omega}\frac{\pi}{h}|I_1f(h)-I_1f(-h)|=$$
$$=\sup_{f\in H_n^\omega}\frac{\pi}{h}|I_{r+1}f(h)-I_{r+1}f(-h).| \quad (9)$$

Thus it is necessary to estimate the modulus of difference $I_{r+1}f(h)-I_{r+1}f(-h)$ of functions f from class $W^{r+1}H_n^\omega$ with step $2h$, i.e. estimate the modulus of continuity of functions f from the class $W^{r+1}H_n^\omega$.

Estimation of values of functions

$$\max_x|f(x)|,f\in K$$

and also their differences

$$\max_x|f(x)-f(x+h)|,f\in K,h>0$$

led to the problem of finding the exact upper bound of the functional

$$\int_a^b\psi(t)f(t)dt \quad (10)$$

on different classes of functions, In particular for the classes $W^rH_1^\omega$. this problem was solved by N.P. Korneichuk and he proposed a new method for estimating the functionalities of the form (10).

Theorem 2. *Let $\psi(t)-$ be an integrable function on the segment $[a,b]$, such that the function $\Psi(x)=\ int_a^x\psi(t)dt$ is strictly monotone on segments $[a,c]$ and $[c,b]$ and $\Psi(b)=0$. If ω is a given convex modulus of continuity, then the equality*

$$\sup_{f\in H^\omega}\int_a^b f(t)\psi(t)dt=\int_a^c|\psi(t)|\omega[\rho(t)-t]dt=$$

$$= \int_c^b |\psi(t)|\omega[t-\rho^{-1}(t)]dt,$$

holds, where the function $\rho(x)$ is defined by the equation

$$\Psi(x) = \Psi(\rho(x)),\ x \in [a,c], \rho(x) \in [c,b]$$

and $\rho^{-1}(x)$ is a function inverse to $\rho(x)$.

Corollary 2. *If the function $\psi(t)$ satisfies the conditions of Theorem 2 and*

$$\psi(a+u) = -\psi(b-u), \tag{11}$$

then $c=(a+b)/2$ and $\rho(t)=a+b-t$, and

$$\sup_{f\in H^\omega} \int_a^b f(t)\psi(t)dt = \int_a^{(a+b)/2} |\psi(t)|\omega[a+b-2t]dt =$$

$$= \int_{(a+b)/2}^b |\psi(t)|\omega[2t-a-b]dt.$$

Note that the conditions of Theorem 2 for the function $\psi(t)$ hold for Bernoulli functions $D_r(t)$.

Really, if $r=2\nu$ is even, then on segments $[0,\pi]$ and $[\pi,2\pi]$ the conditions of the theorem satisfy for the functions $D_{2\nu}(t)$ and if $r=2\nu+1-$ is odd, then for the functions $D_{2\nu+1}(t)$ on the segments $[-\pi/2,\pi/2]$ and $[\pi/2,3\pi/2]$, in addition to the conditions of the Theorem, (11) holds. It is obvious that the conditions of Theorem 2 and condition (11) hold for the corresponding segments and for the functions $D_r(nt)$

In the paper [9] the functions

$$R_r(h,t) = D_r(h-t) - D_r(-h-t)$$

were considered as well, and it was observed that functions $R_{2\nu}(h,t)$ have the same properties as functions $D_{2\nu+1}(h,t)$, and functions $R_{2\nu+1}(h,t)$ behaves similarly to functions $D_{2\nu}(h,t)$. In particular, $R_{2\nu}(h,t)$ is an odd function, and $R_{2\nu+1}(h,t)-$ is even, and Theorem 2 holds for them. It is easy to verify that similar properties have functions

$$K_{\nu,n}(h,t) = D_{2\nu}(n(h-t)) - D_{2\nu}(n(-h-t)).$$

Lemma 1. *The functions $K_{r,n}(h,t)$, if $h\in[0,\pi/2n]$, have the following properties:*

1. $K_{r,n}(h,t)$ are equal to zero on the average.

2. $K_{2\nu,n}(h,t)$ is odd, $K_{2\nu+1,n}(h,t)$ is even. In particular, the functions $K_{2\nu,n}(h,t)$ are equal to zero at the points $k\pi/n, k\in Z$.

3. $\frac{d}{dt}K_{r,n}(h,t) = -K_{r-1,n}(h,t)$.

4. $\int_0^{\pi/n} K_{2\nu+1,n}(h,t)dt = 0$, $\int_{\pi/n}^{2\pi/n} K_{2\nu+1,n}(h,t)dt = 0$.

5. The functions $K_{r,n}(h,t)$ change the sign twice on the period $[0,2\pi/n)$.

6. The functions $K_{2\nu+1,n}(h,t)$ are strictly monotone on the segments $[0,\pi/n]$, $\pi/n,2\pi/n]$.

Proof. Proposition 1 follows from the definition of the functions $K_{r,n}(h,t)$.

We prove the oddness of the function $K_{2\nu,n}(h,t)$. To this end we use the periodicity and evenness of the function $D_{2\ nu}(t)$.

$$K_{2\nu,n}(h,t) = D_{2\nu}(n(h-t)) - D_{2\nu}(n(-h-t)) =$$

$$= D_{2\nu}(n(-h+t)) - D_{2\nu}(n(h+t)) = -K_{2\nu,n}(h,-t)$$

Similarly we prove, the evenness of the function $K_{2\nu+1,n}(h,t)$.

Proposition 3 follows from the definition of the functions $K_{r,n}(h,t)$.

Proposition 4 follows from the Newton-Leibnitz formula and Propositions 3 and 2. To prove 5 we have to check 5 for the functions $K_{1,n}(h,t)$, and then to assume that for some positive integer $r > 1$ the Statement 5 does not hold and to get a contradiction. Proposition 6 follows from the previous one.

Lemma is proved.

Lemma 2. *If $h \in [0, \pi/2n]$, then for the functions $K_{2\nu,n}(h,t)$ the conditions of Theorem 2 and condition (11) are satisfied on segments $[-\pi/2n, \pi/2n]$ and $[\pi/2n, 3\pi/2n]$.*

Proof. The fulfillment of the conditions of Theorem 2 follows from Lemma 1. Condition (11) for functions $K_{2\nu,n}(h,t)$ on segments $[-\pi/2n, \pi/2n]$ means the oddness of these functions and it is proved.

Consider the function $K_{2\nu,n}(h,t)$ on the segment $[\pi/2n, 3\pi/2n]$.

$$K_{2\nu,n}(h,\pi/n-t) = D_{2\nu}(n(h-\pi/n+t)) - D_{2\nu}(n(-x-\pi/n+t)) =$$

$$= D_{2\nu}(n(-h+\pi/n-t)) - D_{2\nu}(n(h+\pi/n-t)) =$$

$$= D_{2\nu}(n(-h-\pi/n-t)) - D_{2\nu}(n(h-\pi/n-t)) = -K_{2\nu,n}(h,\pi/n+t).$$

Lemma is proved.

Consider another integral representation of differential functions. It selected so as to apply N.P. Korneichuk method of the estimation of functionalities of form (10). Let $B_r(t) = \frac{1}{n^{r-1}} D_r(nt)$.

Theorem 3. *Any $2\pi/n$ periodic function, equal to zero on the average, whose $(r-1)$-st derivative is absolutely continuous, may be represented in the form*

$$f(x) = \frac{1}{\pi}\int_0^{2\pi/n} B_r(x-t) f^{(r)}(t)dt, \qquad (12).$$

Proof. First conside the case $r = 1$. Apply the formula of integration by parts

$$\frac{1}{\pi}\int_0^{2\pi/n} B_r(t) f^{(')}(x-t)dt = \frac{-1}{\pi} B_r(t) f(x-t)|_0^{2\pi/n} - \frac{n}{2\pi}\int_0^{2\pi/n} f(x-t)dt.$$

The first term on the right is equal to $f(x)$, and the second is equal to zero because by the condition of the theorem, the function $f(x)$ is equal to zero on the average. Suppose that Theorem holds for a natural number r and show that the statement holds for a natural number $r+1$. Using the equation $\frac{d}{dt}B_{r+1}(t)=B_r(t)$ and integrating the parts we obtain

$$\frac{1}{\pi}\int_0^{2\pi/n} B_{r+1}(x-t)f^{(r+1)}(t)dt=\frac{1}{\pi}B_{r+1}(x-t)f^{(r)}(t)|_0^{2\pi/n}+$$

$$+\frac{1}{\pi}\int_0^{2\pi/n} B_r(x-t)f^{(r)}(t)dt=f(x).$$

Theorem 4. *If r is an even natural number, then for any $h\in[0,\pi/2n]$ the equality*

$$\sup_{f\in W^rH_n^\omega}\max_x|f(x+h)-f(x-h)|=$$

$$=\frac{4}{\pi n^{2\nu-1}}\int_0^{\frac{\pi}{2n}}\sum_{i=1}^{\infty}\frac{\sin(2i+1)nh\sin(2i+1)nt}{(2i+1)^{2\nu}}\omega(2t)dt. \tag{13}$$

holds.

Proof. Due to the periodicity of the function $f(x)$, one may assume that

$$\max_x|f(x+h)-f(x-h)|=|f(h)-f(-h)|$$

We use images (12)

$$|f(h)-f(-h)|=\frac{1}{\pi}|\int_0^{2\pi/n}[B_r(h-t)-B_r(-h-t)]f^{(r)}(t)dt=$$

$$=\frac{1}{\pi n^{2\nu-1}}|\int_0^{2\pi/n}[D_{2\nu}(n(h-t))-D_{2\nu}(n(-h-t))]f^{(2\nu)}(t)dt|.$$

So

$$|f(h)-f(-h)|=\frac{1}{\pi n^{2\nu-1}}|\int_0^{2\pi/n}K_{\nu,n}(h,t)f^{(2\nu)}(t)dt|,$$

where

$$K_{\nu,n}(h,t)=D_{2\nu}(n(h-t))-D_{2\nu}(n(-h-t))=$$

$$=\frac{(-i)^\nu}{\pi}\sum_{k=1}^{\infty}\frac{\cos nk(h-t)-\cos nk(-h-t)}{k^{2\nu}}=$$

$$=\frac{(-i)^\nu 2}{\pi}\sum_{k=1}^{\infty}\frac{\sin nkh\sin nkt}{k^{2\nu}}. \tag{14}$$

By Lemma 2, the conditions of Theorem 1 hold for the function $K_{\nu,n}(h,t)$, as well as condition (11) on the segments $[-\pi/2n,\pi/2n]$ and $[\pi/2n,3\pi/2n]$. Therefore by Theorem 1 we have

$$|f(h)-f(-h)|=\frac{1}{\pi n^{2\nu-1}}|\int_0^{2\pi/n}K_{\nu,n}(h,t)f^{(2\nu)}(t)dt|\le$$

$$\leq \frac{1}{\pi n^{2\nu-1}}\{|\int_{-\pi/2n}^{\pi/2n} K_{\nu,n}(h,t) f^{(2\nu)}(t)dt|+$$

$$+|\int_{\pi/2n}^{3\pi/2n} K_{\nu,n}(h,t) f^{(2\nu)}(t)dt|\} \leq$$

$$\leq \frac{1}{\pi n^{2\nu-1}} \sup_{f\in H_n^\omega} |\int_{-\pi/2n}^{\pi/2n} K_{\nu,n}(h,t) f(t)dt|+$$

$$+\frac{1}{\pi n^{2\nu-1}} \sup_{f\in H_n^\omega} |\int_{\pi/2n}^{3\pi/2n} K_{\nu,n}(h,t) f(t)dt| =$$

$$= \frac{1}{\pi n^{2\nu-1}} \int_{0}^{\pi/2n} |K_{\nu,n}(h,t)|\omega(2t)dt|+$$

$$+\frac{1}{\pi n^{2\nu-1}} \int_{\pi/2n}^{\pi/n} |K_{\nu,n}(h,t)|\omega(2\pi/n-2t)dt. \qquad (15)$$

In the last integral we will make the replacement $\pi/n - t = u$. Then from a chain of inequalities we will get

$$|f(h) - f(-h)| \leq \frac{1}{\pi n^{2\nu-1}}|\int_{0}^{\pi/2n} [K_{\nu,n}(h,t)+$$

$$+K_{\nu,n}(h, \pi/n - t)]\omega(2t)dt. \qquad (16)$$

With this, the sign of equality in (16) holds for an odd $2\pi/n$ periodic function, which $r-$th derivative is equal to $\frac{1}{2}\omega(2t)$ on $[0, \pi/2n]$ and $\omega(2\pi/n - 2t)$ on $x[\pi/2n, \pi/n]$. We denote the extreme function by $f_{n,r}(t)$. Substituting instead of $K_{\nu,n}(h,t)$ the sum (14) in (16), we obtain

$$\sup_{f\in W^{2\nu}H_n^\omega} \omega(f,t) = \frac{4}{\pi n^{2\nu-1}} \int_{0}^{\pi/2n} \sum_{j=0}^{\infty} \frac{\sin n(j+1)h \sin n(j+1)t}{(2j+1)^{2\nu}}.$$

So

$$R_n(W^r H^\omega; \vec{c_0}; \vec{x_0}; h) = \frac{4}{hn^{2\nu-1}} \int_{0}^{\pi/2n} \sum_{j=0}^{\infty} \frac{\sin n(j+1)h \sin n(j+1)t}{(2j+1)^{2\nu}}.$$

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