Researches in Mathematics https://vestnmath.dnu.dp.ua
Res. Math. **28**(1), 2020, p. 29–42 ISSN (Print) 2664-4991
doi:10.15421/242004 ISSN (Online) 2664-5009

UDK 517.5

S. E. Sinem*, E. Melih, Ç. Selim*****

* Akdeniz University, Antalya 07058. *E-mail: sinemsezer@akdeniz.edu.tr*
** Akdeniz University, Antalya 07058. *E-mail: eryigit@akdeniz.edu.tr*
*** A.E.V.T.A.High School Urfa 63900. *E-mail: selim855@hotmail.com*

The Rate of Convergence of Truncated Hypersingular Integrals Generated by the Generalized Poisson Semigroup

Добре відомо, що оператори дробового інтегрування, зокрема потенціали Рісса та Бесселя, відіграють фундаментальну роль в аналізі та його застосуваннях. Ці потенціали інтерпретують як «від'ємні» дробові степені $(-\Delta)$ та $(E-\Delta)$, де $\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$ — лапласіан, а E — тотожній оператор.

У той час як локальна поведінка ядер потенціалів Рісса й Бесселя однакова, поведінка ядра потенціала Рісса в нескінченності не така гарна, як у потенціала Бесселя. Існують, однак, інші оператори дробового інтегрування, поведінки котрих, грубо кажучи, знаходяться посередині між потенціалами Рісса й Бесселя. Ці потенціали, які ми називаємо потенціалами Флетта, були введені Т.М. Флеттом у його фундаментальній роботі (1971). Потенціали Флетта $\mathcal{F}^{\alpha} f$ та узагальнені потенціали Флетта $\mathcal{F}^{\alpha}_{\nu} f$ функції f визначаються в термінах перетворень Фур'є та Фур'є-Бесселя. Узагальнені потенціали Флетта допускають певне інтегральне представлення.

Використовуючи узагальнений інтеграл Пуассона (напівгруповий), породжений узагальненим оператором зсуву $V_t f$, ми можемо визначити видозмінений узагальнений інтеграл Пуассона $V_t^M f$.

У цій роботі, по-перше, застосовуючи техніки з робіт Рубіна (1986, 1987), ми вводимо «нові узагальнені зрізані гіперсингулярні інтегральні оператори $\mathbb{D}^{\alpha}_{\varepsilon} f, (\varepsilon > 0)$», породжені узагальненою пуассонівською напівгрупою; у визначенні цих операторів вживано скінченні різниці порядку $l \in \mathbb{N}$ із кроком $\tau \in \mathbb{R}$ функції $g\left(t\right), \left(t \in \mathbb{R}^1\right)$. Відтак, використовуючи ці узагальнені зрізані гіперсингулярні інтегральні оператори, ми отримуємо теорему, яка дає нам деякі співвідношення між «порядком $L_{p,\nu}$-гладкості» функції φ та «швидкістю $L_{p,\nu}$-збіжності» сімейств $\mathbb{D}^{\alpha}_{\varepsilon} \mathcal{F}^{\alpha}_{\nu} \varphi$ до φ при $\varepsilon \to 0^+$.

Ключові слова: потенціали Флетта, зрізані гіперсингулярні інтеграли, швидкість збіжності, узагальнена пуассонівська напівгрупа

We introduce a family of Balakrishnan-Rubin type hypersingular integrals depending on a parameter ε and generated by the Generalized Poisson semigroup. Then the rate of convergence of these families of truncated hypersingular integrals, which converge to $L_{p,\nu}$–function φ as ε tends to 0, is obtained.

Key words: Flett potentials, truncated hypersingular integrals, rate of convergence, generalized Poisson semigroup.

MSC2010: Pri 26A33 , Sec 41A35,44A35

1. Introduction

The fundamental role of fractional integral operators, in particular, Riesz and Bessel potentials, in analysis and its applications is well known. For a very smooth and small at infinity (sufficiently good) function f on $\mathbb{R}^n$, the Riesz potentials $I^\alpha f$ and Bessel potentials $J^\alpha f$ of order α are defined in terms of the Fourier transform by

$$(I^\alpha f)^\frown(x) = |x|^{-\alpha} \widehat{f}(x), \quad (x \in \mathbb{R}^n, \ \alpha > 0), \tag{1.1}$$

$$(J^\alpha f)^\frown(x) = \left(1 + |x|^2\right)^{-\frac{\alpha}{2}} \widehat{f}(x), \quad (x \in \mathbb{R}^n, \ \alpha > 0), \tag{1.2}$$

where the identity is to be understood in the sense of the distribution theory. (For details, see [21, 22, 23, 24]).

These potentials are interpreted as a "negative" fractional powers of the $(-\Delta)$ and $(E - \Delta)$ where $\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$ is Laplacean and E is the identity operator. Further these potentials have the following integral representation, respectively:

$$(I^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} |y|^{\alpha - n} f(x - y)\, dy, 0 < \alpha < n,$$

$$\gamma_n(\alpha) = \pi^{\frac{n}{2}} 2^\alpha \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)};$$

$$(J^\alpha f)(x) = \frac{1}{\beta_n(\alpha)} \int_{\mathbb{R}^n} G_\alpha(y) f(x - y)\, dy, \ \alpha > 0,$$

$$G_\alpha(y) = \int_0^\infty e^{-t - \frac{|y|^2}{4t}} t^{\frac{\alpha - n}{2} - 1} dt,$$

$$\beta_n(\alpha) = 2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right).$$

While the local behaviours of the kernels of Riesz and Bessel potentials are the same, the behaviour of the kernel of the Riesz potential at infinity is not as good as that of the Bessel potential. There are, however, other fractional integral operators whose behaviours are roughly midway between the Riesz and Bessel potentials. These potentials, which we call the Flett potentials, are introduced by T.M. Flett in his fundamental paper [10] (see also [23, p.541-542]).

The Flett potentials $\mathcal{F}^\alpha f$ and the generalized Flett potentials $\mathcal{F}_\nu^\alpha f$ of a function f are defined in terms of the Fourier and Fourier-Bessel transforms by the following formulas:

$$F(\mathcal{F}^\alpha f)(x) = (1 + |x|)^{-\alpha} F(f)(x), \ (x \in \mathbb{R}^n, \ \alpha > 0), \tag{1.3}$$

$$F_\nu\left(\mathcal{F}_\nu^\alpha f\right)(x)=(1+|x|)^{-\alpha}F_\nu(f)(x),\ \left(x\in\mathbb{R}_+^n,\ \alpha>0\right). \tag{1.4}$$

These potentials are interpreted as a negative fractional powers of the operator $\left(E+(-\Delta)^{\frac{1}{2}}\right)$ and $\left(E+(-\Delta_\nu)^{\frac{1}{2}}\right)$ respectively. Here

$$\Delta_\nu=\sum_{k=1}^{n-1}\left(\frac{\partial^2}{\partial x_k^2}+\frac{2\nu}{x_n}\cdot\frac{\partial}{\partial x_n}\right),(\nu>0),$$

is the singular generalized Laplace-Bessel differential operator.

Approximation properties of the potentials such as Riesz, Bessel, Flett, generalized Riesz, generalized Bessel potentials have been studied by T. Kurokawa [15], A.D. Gadjiev, A. Aral and I.A. Aliev [11], I.A. Aliev and M. Eryiğit [6], I.A. Aliev and S. Çobanoğlu [7], M. Eryiğit and S. Çobanoğlu [8] and S. Sezer Evcan, M. Eryiğit and S. Çobanoğlu [9].

In the paper [20] (see, also [19], [21, p.217-222]), B. Rubin introduced some family of "truncated hypersingular integrals" $D_\varepsilon^\alpha f$ and $\mathfrak{D}_\varepsilon^\alpha f,(\varepsilon>0)$, generated by the Gauss-Weierstrass semigroup, and proved that under some conditions on a function $\varphi\in L_p(\mathbb{R}^n)$ and parameter $\alpha>0$, the expressions $D_\varepsilon^\alpha I^\alpha\varphi$ and $\mathfrak{D}_\varepsilon^\alpha J^\alpha\varphi$ converge to φ as $\varepsilon\to0^+$, pointwise (a.e.) and in the L_p-norm.

In this paper, using by the techniques in the papers [19] and [20] we introduce "a new truncated hypersingular integral operators $\mathbb{D}_\varepsilon^\alpha f,(\varepsilon>0)$" generated by generalized Poisson semigroup. By making use of these integral operators we find some relationships between the "order of $L_{p,\nu}$-smoothness" of function φ and the "rate of $L_{p,\nu}$-convergence" of the families $\mathbb{D}_\varepsilon^\alpha\mathcal{F}_\nu^\alpha\varphi$ to φ as $\varepsilon\to0^+$.

2. Preliminaries, definitions and auxiliary lemmas

Let $L_p\equiv L_p(\mathbb{R}^n)$ be the space of functions on $\mathbb{R}^n$ with the norm

$$\|f\|_p=\left(\int_{\mathbb{R}^n}|f(x)|^p\,dx\right)^{\frac{1}{p}},1\le p<\infty;\ dx=dx_1...dx_n,$$

$$\|f\|_\infty=ess\sup_{x\in\mathbb{R}^n}|f(x)|.$$

Let $\mathbb{R}_+^n=\{x\mid x=(x_1,x_2,...,x_{n-1},x_n)\in\mathbb{R}^n,x_n>0\}$ and

$$L_{p,\nu}\equiv L_{p,\nu}\left(\mathbb{R}_+^n\right)=\left\{f:\|f\|_{p,\nu}=\left(\int_{\mathbb{R}_+^n}|f(x)|^p\,x_n^{2\nu}dx\right)^{\frac{1}{p}}<\infty\right\}$$

where $1\le p<\infty$, $\nu>0$ and $x_n^{2\nu}dx=x_n^{2\nu}dx_1...dx_n$

We denote by Δ_ν the generalized singular Laplace-Bessel differential operator

$$\Delta_\nu=\sum_{k=1}^{n-1}\left(\frac{\partial^2}{\partial x_k^2}+\frac{2\nu}{x_n}\cdot\frac{\partial}{\partial x_n}\right).$$

The Fourier transform and Fourier-Bessel transform of a function f are defined by the following formulas:

$$(Ff)(\xi)=\int_{\mathbb{R}^n} e^{i\xi\cdot x} f(x)\,dx, \xi\cdot x=\xi_1x_1+...+\xi_nx_n,$$

$$(F_\nu f)(\xi)=\int_{\mathbb{R}^n_+} f(x)\left(\prod_{k=1}^{n} j_{\nu-\frac{1}{2}}(x_k\xi_k)\right)x^{2\nu}dx$$

where $j_\lambda(\tau)=2^\lambda\Gamma(\lambda+1)J_\lambda(\tau)\tau^{-\lambda}$, $J_\lambda(\tau)$ is the Bessel function of the first kind. (see for details, [14], [16]).

The generalized translation operator of a function f is defined by

$$T^y f(x)=\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)}\int_0^\pi f\left(x'-y';\sqrt{x_n^2-2x_ny_n\cos\alpha_n+y_n^2}\right)\sin^{2\nu-1}\alpha d\alpha$$

where $x'=(x_1,x_2,...,x_{n-1})$

Note that T^y is a combination of the ordinary translation in x' and Bessel translation in x_n which is closely connected with the Bessel operator

$$B_t=\frac{d^2}{dt^2}+\frac{2\nu}{t}\cdot\frac{d}{dt},\quad(0<t<\infty),$$

(for details see [16]).

Note also that for $1\le p\le\infty$ (see [18])

$$\|T^y f\|_{p,\nu}\le\|f\|_{p,\nu}\ ,\tag{2.1}$$

$$\|T^y f-f\|_{p,\nu}\to 0 \text{ as } |y|\to 0.\tag{2.2}$$

In the case $p=\infty$ we identify $L_{\infty,\nu}$ with $C_0=C_0\left(\mathbb{R}^n_+\right)$ - the corresponding space of continuous functions vanishing at infinity.

The generalized Poisson kernel is defined as

$$p_\nu(y,t)=\frac{2}{\pi^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n+1}{2}+\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)}\cdot\frac{t}{\left(|y|^2+t^2\right)^{\frac{n+1}{2}+\nu}}.\tag{2.3}$$

It is known that (see [29])

$$\|p_\nu(\cdot,t)\|_{1,\nu}=1,\ \ (\forall t>0).\tag{2.4}$$

Now, we define the Generalized ν-maximal function of f as

$$(M_\nu f)(y)=\sup_{r>0}\frac{1}{r^{n+2\nu}\omega(n,\nu)}\int_{B_r^+}|T^y f(x)|\,x^{2\nu}dx,\tag{2.5}$$

where $B_r^+=\left\{y:y\in\mathbb{R}^n_+,|y|\le r\right\}$ and $\omega(n,\nu)=\int_{B_1^+}y^{2\nu}dy$.

Note that for $f \in L_{p,\nu}$

$$\|M_\nu f\|_{p,\nu} \leq c\,\|f\|_{p,\nu}\,, (1 < p \leq \infty)\,, \tag{2.6}$$

(see [1, 12, 13]).

We denote by $(V_t f)(x), (t > 0)$ the Generalized Poisson integral (semigroup) generated by the generalized translation operator:

$$(V_t f)(x) = \int_{\mathbb{R}^n_+} p_\nu(y,t)\,(T^y f(x))\, y^{2\nu} dy. \tag{2.7}$$

Lemma 1. *([29, p.685])*

Let $f \in L_{p,\nu}$ and $V_t f$ is defined as in (2.7). Then

i)

$$\|V_t f\|_{p,\nu} \leq \|f\|_{p,\nu}\ ,\ (1 \leq p \leq \infty, \forall t > 0)\,, \tag{2.8}$$

ii)

$$\sup_{x\in\mathbb{R}^n_+} |V_t f(x)| \leq c t^{-(n+2\nu)/p}\,\|f\|_{p,\nu}\,, \tag{2.9}$$

here, $1 \leq p < \infty$ and c being independent of t,

iii)

$$\sup_{t>0} |V_t f(x)| \leq M_\nu f(x)\,, \tag{2.10}$$

iv)

$$V_t(V_s f)(x) = (V_{t+s} f)(x)\,; (t, s > 0)\,, \tag{2.11}$$

v)

$$\lim_{t\to 0^+} V_t f(x) = f(x)\,, \tag{2.12}$$

where the limit being understood in the $L_{p,\nu}$ $(1 \leq p < \infty)$-norm or pointwise for almost all $x \in \mathbb{R}^n_+$. If $f \in C_0$, then the convergence is uniform on $\mathbb{R}^n_+$.

By making use of generalized Poisson integral $V_t f$, we define modified generalized Poisson integral as

$$\left(V_t^M f\right)(x,t) = e^{-t}\,(V_t f)(x)\,, (0 < t < \infty)\,. \tag{2.13}$$

For $t = 0$ we set

$$(V_0 f)(x) = \left(V_0^M f\right)(x, 0) = f(x)\,.$$

The generalized Flett potentials $\mathcal{F}^\alpha_\nu$ initially defined by (1.4), have the following integral representations via the generalized Poisson integral:

$$(\mathcal{F}^\alpha_\nu f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t}\,(V_t f)(x)\,dt, \ \text{([27, p.120]).} \tag{2.14}$$

Furthermore, considering the notion $V_t^M f$ defined as (2.13) we can write the formula (2.14) as:

$$(\mathcal{F}_\nu^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \left(V_t^M f\right)(x,t)\, dt. \tag{2.15}$$

We note that the formula (2.15) gives the relation between the generalized Flett potentials and the modified generalized Poisson integral.

In addition, it is not difficult to show that

$$\|\mathcal{F}_\nu^\alpha f\|_{p,\nu} \leq \|f\|_{p,\nu} \;\;, 1 \leq p \leq \infty, \forall \alpha > 0. \tag{2.16}$$

The finite difference with order $l \in \mathbb{N}$ and step $\tau \in \mathbb{R}^1$ of the function $g(t), (t \in \mathbb{R}^1)$ is defined by

$$\Delta_\tau^l [g](t) = \sum_{k=0}^{l} \binom{l}{k} (-1)^k g(t + k\tau). \tag{2.17}$$

Using this finite difference and modified generalized Poisson semigroup $\left(V_t^M f\right)$, we introduce the following "Balakrishnan-Rubin type truncated integral" (cf. [21, p.220]).

Definition 1. Let $f \in L_{p,\nu}\left(\mathbb{R}_+^n\right), 1 \leq p < \infty, \alpha > 0$ and $l > \alpha, (l \in \mathbb{N})$. Then the construction

$$\begin{aligned} (\mathbb{D}_\varepsilon^\alpha f)(x) &= \frac{1}{\varkappa_l(\alpha)} \int_\varepsilon^\infty \Delta_\tau^l \left[\left(V_\cdot^M f\right)(x)\right](0) \frac{d\tau}{\tau^{1+\alpha}} \\ &= \frac{1}{\varkappa_l(\alpha)} \int_\varepsilon^\infty \left[\sum_{k=0}^{l} \binom{l}{k} (-1)^k e^{-k\tau} (V_{k\tau} f)(x)\right] \frac{d\tau}{\tau^{1+\alpha}}, \varepsilon > 0 \end{aligned} \tag{2.18}$$

will be called a "generalized truncated integrals " with parameter $\varepsilon > 0$. Here the normalized coefficient $\varkappa_l(\alpha)$ is defined by

$$\varkappa_l(\alpha) = \int_0^\infty \left(1 - e^{-t}\right)^l t^{-1-\alpha} dt.$$

As shown in the following lemma, there is a close connection between the construction (2.18) and the generalized Flett potential $(\mathcal{F}_\nu^\alpha f)$.

Lemma 2. *(cf. [1])*

Let $\varphi \in L_{p,\nu}\left(\mathbb{R}_+^n\right), (1 \leq p < \infty)$ and $0 < \alpha < \frac{n+2\nu}{p}$. Then for any $\varepsilon > 0$ and for a.e. $x \in \mathbb{R}_+^n$

$$(\mathbb{D}_\varepsilon^\alpha \mathcal{F}_\nu^\alpha \varphi)(x) = \int_0^\infty K_\alpha^{(l)}(\eta)\, e^{-\varepsilon\eta} (V_{\varepsilon\eta}\varphi)(x)\, d\eta. \tag{2.19}$$

Here the function $K_\alpha^{(l)}(\eta)$ is defined as

$$K_\alpha^{(l)}(\eta) = \frac{1}{\Gamma(1+\alpha)\chi_l(\alpha)} \cdot \frac{1}{\eta} \sum_{k=0}^{l} \binom{l}{k} (-1)^k (\eta - k)_+^\alpha,$$

with $(\eta - k)_+^\alpha = \left\{ \begin{array}{ll} (\eta - k)^\alpha, & if\ \eta > k \\ 0, & if\ \eta \leq k \end{array} \right\}.$

The following lemma gives some properties of the function $K_{\alpha}^{(l)}(\eta)$ that will be used later.

Lemma 3. *(see [21, p.158] and [23, p.125])*
i)

$$K_{\alpha}^{(l)}(\eta) \in L_1(0,\infty) \ \ and \ \int_0^{\infty} K_{\alpha}^{(l)}(\eta)\, d\eta = 1,$$

ii)

$$K_{\alpha}^{(l)}(\eta) = \left\{ \begin{array}{ll} O(\eta^{\alpha-1}), & if\ \eta \to 0^+ \\ O(\eta^{\alpha-l-1}), & if\ \eta \to \infty \end{array} \right\}.$$

The following definition and the subsequent lemma play crucical role in the sequel.

Definition 2. Let $\rho \in (0,1)$ be a fixed parameter and the function $\mu(r)$, $(0 \le r \le \rho)$ be continuous and strictly increasing on $[0,\rho]$ and $\mu(0) = 0$. We say that a function $\varphi \in L_{p,\nu}\left(\mathbb{R}_+^n\right), (1 \le p < \infty)$ has μ-smoothness property in $L_{p,\nu}$-sense if

$$\mathcal{M}_{\mu} \equiv \sup_{0<r\le\rho} \frac{1}{r^{n+2\nu}\mu(r)} \int_{B_r^+} \|T^x\varphi(t) - \varphi(t)\|_{p,\nu}\, x^{2\nu}dx < \infty. \tag{2.20}$$

It is clear that if $\mu(r)$ is the $L_{p,\nu}$-modulus of continuity of φ, i.e.

$$\mu(r) = \sup_{|x|\le r} \|T^x\varphi(t) - \varphi(t)\|_{p,\nu}$$

then the condition (2.20) is satisfied.

Remark 1. From now on it will be assumed that $\mu(t) \ge at, (0 \le t \le \rho)$, for some $a > 0$ and $\mu(t) = \mu(\rho)$ for $\rho \le t < \infty$.

Lemma 4. *(cf. [6]; see also [2, 28]) Let a function $\varphi \in L_{p,\nu}\left(\mathbb{R}_+^n\right), (1 \le p < \infty)$ has the μ-smoothness property in the $L_{p,\nu}$-sense. Let, further, the function $\psi(r), (0 \le r < \rho)$ be decreasing, nonnegative and continuously differentiable on $[0,\rho]$. Then,*

$$\begin{aligned} &\int_{B_r^+} \|T^x\varphi(t) - \varphi(t)\|_{p,\nu}\, \psi(|x|)\, x^{2\nu}dx \\ \le\ & \mathcal{M}_{\mu}\left[\rho^{n+2\nu}\mu(\rho)\,\psi(\rho) + \int_0^{\rho} r^{n+2\nu}\mu(r)\,(-\psi'(r))\,dr\right]. \end{aligned} \tag{2.21}$$

Proof. Set $h(x) = \|T^x\varphi(t) - \varphi(t)\|_{p,\nu}$ and $x = r\theta$, where $r = |x|$. Then

$$I \equiv \int_{B_r^+} h(x)\,\psi(|x|)\,x^{2\nu}dx = \int_0^{\rho} r^{n-1+2\nu}\psi(r)\left(\int_{|\theta|=1} h(r\theta)\,d\sigma(\theta)\right)dr$$

If we define

$$\lambda(r) = \int_{|\theta|=1} h(r\theta)d\sigma(\theta) \text{ and } \Omega(r) = \int_0^{r} \lambda(t)t^{n-1+2\nu}dt, \tag{2.22}$$

then we have

$$I \equiv \int_0^\rho \psi(r)\,\lambda(r) r^{n-1+2\nu} dr = \int_0^\rho \psi(r)\, d\Omega(r)$$
$$= \psi(r)\,\Omega(r)\,|_0^\rho - \int_0^\rho \Omega(r)\psi'(r)\,dr$$
$$= \psi(\rho)\,\Omega(\rho) + \int_0^\rho \Omega(r)(-\psi'(r))dr.$$

It follows from (2.20) that

$$\Omega(r) = \int_0^r \lambda(t) t^{n-1+2\nu} dt = \int_{|x|\le r} h(x) x^{2\nu} dx$$
$$= \int_{|x|\le r} \|T^x\varphi(t) - \varphi(t)\|_{p,\nu} x^{2\nu} dx \le r^{n+2\nu}\mu(r)\mathcal{M}_\mu.$$

Hence,

$$I \le \mathcal{M}_\mu \left[\rho^{n+2\nu}\mu(\rho)\psi(\rho) + \int_0^\rho r^{n+2\nu}\mu(r)(-\psi'(r))dr\right].$$

Lemma 5. *Let $p_\nu(x,\varepsilon)$ be the generalized Poisson kernel defined as in (2.3), i.e. for $x \in \mathbb{R}_+^n$,*

$$p_\nu(x,\varepsilon) = \frac{2}{\pi^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n+1}{2}+\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)} \cdot \frac{\varepsilon}{\left(|x|^2+\varepsilon^2\right)^{\frac{n+1}{2}+\nu}}$$

Then there exist $c > 0$ such that

$$\int_{|x|\le r} \|T^x\varphi(t) - \varphi(t)\|_{p,\nu} p_\nu(x,\varepsilon) x^{2\nu} dx \le c\mathcal{M}_\mu \left[\varepsilon + \int_0^\infty \mu(\varepsilon t) \frac{dt}{1+t^2}\right]. \qquad (2.23)$$

Proof. Let $a_{n,\nu} = \frac{2}{\pi^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n+1}{2}+\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)}$ and we put $\psi(|x|) = p_\nu(x,\varepsilon)$ in (2.21).

$$\int_{|x|\le\rho} \|T^x\varphi(t) - \varphi(t)\|_{p,\nu} p_\nu(x,\varepsilon) x^{2\nu} dx$$
$$\le \mathcal{M}_\mu \left[\rho^{n+2\nu}\mu(\rho) \frac{a_{n,\nu}\cdot\varepsilon}{(\rho^2+\varepsilon^2)^{\frac{n+1}{2}+\nu}} + \right.$$
$$\left. + \int_0^\rho r^{n+2\nu}\mu(r)\left(-\frac{a_{n,\nu}\cdot\varepsilon}{(r^2+\varepsilon^2)^{\frac{n+1}{2}+\nu}}\right)' dr\right].$$

Since

$$-\psi'(r) = \left(-\frac{a_{n,\nu}\cdot\varepsilon}{(r^2+\varepsilon^2)^{\frac{n+1}{2}+\nu}}\right)' = c_1 \frac{\varepsilon r}{(r^2+\varepsilon^2)^{\frac{n+3}{2}+\nu}};$$

here, $c_1 = 2a_{n,\nu}(n+1)+\nu$ and

$$\rho^{n+2\nu}\mu(\rho)\frac{a_{n,\nu}\cdot\varepsilon}{(\rho^2+\varepsilon^2)^{\frac{n+1}{2}+\nu}} \le c_2\varepsilon, \quad \left(c_2 = a_{n,\nu}\frac{\mu(\rho)}{\rho}\right).$$

We have for $\rho < 1$ and $c = \max\{c_1, c_2\}$,

$$\begin{aligned}
&\int_{|x|\le\rho} \|T^x\varphi(t)-\varphi(t)\|_{p,\nu}\, p_\nu(x,\varepsilon)\, x^{2\nu}dx \\
\le\ & c\mathcal{M}_\mu\left[\varepsilon + \int_0^\rho \varepsilon\frac{r^{n+1+2\nu}}{(r^2+\varepsilon^2)^{\frac{n+3}{2}+\nu}}\mu(r)\,dr\right] \\
...(\ r =\ & \varepsilon t,\ dr = \varepsilon dt \text{ (changing variables) })... \\
=\ & c\mathcal{M}_\mu\left[\varepsilon + \int_0^{\frac{\rho}{\varepsilon}} \frac{\varepsilon(\varepsilon t)^{n+1+2\nu}}{(\varepsilon^2t^2+\varepsilon^2)^{\frac{n+3}{2}+\nu}}\mu(\varepsilon t)\,\varepsilon dt\right] \\
=\ & c\mathcal{M}_\mu\left[\varepsilon + \int_0^{\frac{\rho}{\varepsilon}} \frac{t^{n+1+2\nu}}{(1+t^2)^{\frac{n+3}{2}+\nu}}\mu(\varepsilon t)\,dt\right] \\
\le\ & c\mathcal{M}_\mu\left[\varepsilon + \int_0^\infty \mu(\varepsilon t)\frac{t^{n+1+2\nu}}{(1+t^2)^{\frac{n+3}{2}+\nu}}dt\right] \\
\le\ & c\mathcal{M}_\mu\left[\varepsilon + \int_0^\infty \frac{\mu(\varepsilon t)}{1+t^2}dt\right].
\end{aligned}$$

(we use this to find last inequality:

$$\frac{t^{n+1+2\nu}}{(1+t^2)^{\frac{n+3}{2}+\nu}} = \frac{(t^2)^{\frac{n+1+2\nu}{2}}}{(1+t^2)^{\frac{n+3}{2}+\nu}} \le \frac{(1+t^2)^{\frac{n+1+2\nu}{2}}}{(1+t^2)^{\frac{n+3}{2}+\nu}} = \frac{1}{1+t^2}.)$$

Corollary 1. *Let the function $\mu(r),(0\le r\le\rho<1)$ be continuous on $[0,\rho]$, positive on $(0,\rho]$ and $\mu(0)=0$. Let, also, $\mu(t)\ge st, 0\le t\le\rho$ for some $s>0$ and $\mu(t)=\mu(\rho)$ for $\rho\le t<\infty$. If there exist a locally bounded function $\omega(t)>0$ such that*

$$\mu(\varepsilon t)\le\mu(\varepsilon)\,\omega(t) \ \text{ and } \int_0^\infty \frac{\omega(t)}{1+t^2}dt<\infty, (0<\varepsilon<\rho, 0<t<\infty), \tag{2.24}$$

then there exist $S>0$ not depending on $\varepsilon\in(0,\rho)$, such that

$$\int_{|x|\le\rho} \|T^x\varphi(t)-\varphi(t)\|_{p,\nu}\, p_\nu(x,\varepsilon)\, x^{2\nu}dx \le S\mu(\varepsilon), \text{for all } \varepsilon\in(0,\rho). \tag{2.25}$$

Proof. By taking into account (2.24) in (2.23) and using the condition

$$\mu(\varepsilon)\ge s\varepsilon, (0\le\varepsilon\le\rho),$$

we obtain inequality (2.25).

Now, let us give the examples of the functions that satisfy all the conditions of Corollary 1.

Example 1. Let $0 < \gamma < 1$ and $0 < \beta < \infty$. Then the function

$$\mu(r) = \left\{ \begin{array}{ccc} 0 & if & r = 0 \\ r^{\gamma} \left|\ln r\right|^{\beta} & if & 0 < r < \rho \\ \rho^{\gamma} \left|\ln \rho\right|^{\beta} & if & r \geq \rho \end{array} \right\}$$

satisfies all the conditions of Corollary 1 for

$$\omega(t) = t^{\gamma} \left(1 + \frac{|\ln t|}{|\ln \rho|}\right)^{\beta}.$$

Indeed, for $0 < \varepsilon < \rho$ and $0 < t < \infty$ we have

$$\begin{aligned} \mu(\varepsilon t) &= \left\{ \begin{array}{ccc} 0 & if & t = 0 \\ \varepsilon^{\gamma} t^{\gamma} \left|\ln \varepsilon + \ln t\right|^{\beta} & if & 0 < t < \frac{\rho}{\varepsilon} \\ \rho^{\gamma} \left|\ln \rho\right|^{\beta} & if & t > \frac{\rho}{\varepsilon} \end{array} \right\} \\ &\leq \left\{ \begin{array}{ccc} 0 & if & t = 0 \\ \mu(\varepsilon) t^{\gamma} \left(1 + \frac{|\ln t|}{|\ln \varepsilon|}\right)^{\beta} & if & 0 < t < \frac{\rho}{\varepsilon} \\ \rho^{\gamma} \left|\ln \rho\right|^{\beta} & if & t > \frac{\rho}{\varepsilon} \end{array} \right\} \\ &\leq \mu(\varepsilon)\, \omega(t) \quad , \quad (0 < \varepsilon < \rho, t > 0), \end{aligned}$$

where $\omega(t) = t^{\gamma} \left(1 + \frac{|\ln t|}{|\ln \rho|}\right)^{\beta}$.

Example 2. Let $0 < \gamma < 1$. Then the function

$$\mu(r) = \left\{ \begin{array}{cc} r^{\gamma}, & if\ 0 \leq r \leq \rho < 1 \\ \rho^{\gamma}, & if\ r \geq \rho \end{array} \right\}$$

satisfies all the conditions of Corollary 1 with $\omega(t) = t^{\gamma}$.

3. Main Results

Theorem 1. *Let the function $\mu(r), (0 < r < \infty)$ satisfies all the conditions of the Corollary 1. Further, let the function $\varphi \in L_{p,\nu}\left(\mathbb{R}^n_+\right), (1 \leq p < \infty)$ has the μ-smoothness property (2.20) in $L_{p,\nu}$-sense. If the operator $\mathbb{D}^{\alpha}_{\varepsilon}$ is defined by* (2.18) *and the parameter $l \in \mathbb{N}$ satisfies the condition $l > \frac{\alpha}{2} + 1$, then*

$$\left\|\mathbb{D}^{\alpha}_{\varepsilon} \mathcal{F}^{\alpha}_{\nu} \varphi - \varphi\right\|_{p,\nu} = O\left(\mu(\varepsilon)\right) \ \ as\ \varepsilon \to 0^{+}, \tag{3.1}$$

where $\mathcal{F}^{\alpha}_{\nu}\varphi, (\alpha > 0)$ is the generalized Flett potentials.

Proof. Using Lemmas 2 and 3-(i) we have

$$\begin{aligned}
&\left\|\mathbb{D}_{\varepsilon}^{\alpha}\mathcal{F}_{\nu}^{\alpha}\varphi-\varphi\right\|_{p,\nu} \\
= &\left\|\int_{0}^{\infty}K_{\alpha}^{(l)}(\eta)\,e^{-\varepsilon\eta}\left(V_{\varepsilon\eta}\varphi\right)(t)\,d\eta-\int_{0}^{\infty}K_{\alpha}^{(l)}(\eta)\,\varphi(t)\,d\eta\right\|_{p,\nu} \\
\leq &\int_{0}^{\infty}\left|K_{\alpha}^{(l)}(\eta)\right|\left\|\left(V_{\varepsilon\eta}\varphi\right)(t)-\varphi(t)\right\|_{p,\nu}d\eta. \qquad (3.2)
\end{aligned}$$

Since $\int_{\mathbb{R}_{+}^{n}}p_{\nu}(y,\eta)\,dy=1$ for all $\eta>0$ ([21, p.217]) it follows that

$$\begin{aligned}
\left\|\left(V_{\varepsilon\eta}\varphi\right)(t)-\varphi(t)\right\|_{p,\nu} &= \left\|\int_{\mathbb{R}_{+}^{n}}p_{\nu}(y,\varepsilon\eta)\left[T^{y}\varphi(t)-\varphi(t)\right]y^{2\nu}dy\right\|_{p,\nu} \\
&\leq \int_{|y|\leq\rho}p_{\nu}(y,\varepsilon\eta)\left\|T^{y}\varphi(t)-\varphi(t)\right\|_{p,\nu}y^{2\nu}dy \\
&\quad+\int_{|y|>\rho}p_{\nu}(y,\varepsilon\eta)\left\|T^{y}\varphi(t)-\varphi(t)\right\|_{p,\nu}y^{2\nu}dy \\
&\equiv i_{1}+i_{2}.
\end{aligned}$$

By making use of (2.25) we have

$$i_{1}\leq S\mu(\varepsilon\eta)$$

where S does not depend on ε and η.

Now we estimate i_2.

$$\begin{aligned}
i_{2} &\leq 2\left\|\varphi\right\|_{p,\nu}\int_{|y|>\rho}p_{\nu}(y,\varepsilon\eta)\,y^{2\nu}dy \\
&\quad (\text{ we use formula (2.3)}) \\
&= 2\left\|\varphi\right\|_{p,\nu}a_{n,\nu}\int_{|y|>\rho}\frac{\varepsilon\eta}{\left((\varepsilon\eta)^{2}+|y|^{2}\right)^{\frac{n+1}{2}+\nu}}y^{2\nu}dy \\
(\text{we set } y &= r\theta,\ \rho<r<\infty,\ \theta\in S^{n-1},dy=r^{n-1}drd\sigma(\theta)) \\
&= c_{1}\varepsilon\eta\int_{\rho}^{\infty}\frac{r^{n-1}}{\left((\varepsilon\eta)^{2}+r^{2}\right)^{\frac{n+1}{2}+\nu}}r^{2\nu}dr \\
&\leq c_{1}\varepsilon\eta\int_{\rho}^{\infty}\frac{r^{n-1}}{r^{n+1}}dr=c_{2}\varepsilon\eta,
\end{aligned}$$

where $c_{2}=c_{2}(\rho,\eta)$ dose not depend on ε and η. Therefore, we have the following estimate:

$$\left\|\left(V_{\varepsilon\eta}\varphi\right)(t)-\varphi(t)\right\|_{p,\nu}\leq S\mu(\varepsilon\eta)+c_{2}\varepsilon\eta. \qquad (3.3)$$

Now, we consider the last estimation (3.3) in (3.2)

$$\begin{aligned}
\left\|\left(\mathbb{D}_{\varepsilon}^{\alpha}\mathcal{F}_{\nu}^{\alpha}\varphi\right)(t)-\varphi(t)\right\|_{p,\nu} &\leq c_3\int_0^{\infty}\left|K_{\alpha}^{(l)}(\eta)\right|\left(\mu(\varepsilon\eta)+\varepsilon\eta\right)d\eta \\
&\leq c_3\int_0^{\infty}\left|K_{\alpha}^{(l)}(\eta)\right|\left(\mu(\varepsilon)\,\omega(\eta)+\varepsilon\eta\right)d\eta \\
\text{(using the condition } \mu(\varepsilon) &\geq s\varepsilon,\ \varepsilon\in(0,\rho)\text{)} \\
&\leq c_4\mu(\varepsilon)\int_0^{\infty}\left|K_{\alpha}^{(l)}(\eta)\right|\left(\omega(\eta)+\eta\right)d\eta. \qquad (3.4)
\end{aligned}$$

Lemma 3-(ii) and the formula (2.24) yield that

$$\begin{aligned}
\int_0^{\infty}\left|K_{\alpha}^{(l)}(\eta)\right|\omega(\eta)\,d\eta &= \int_0^{1}\left|K_{\alpha}^{(l)}(\eta)\right|\omega(\eta)\,d\eta+\int_1^{\infty}\left|K_{\alpha}^{(l)}(\eta)\right|\omega(\eta)\,d\eta \\
&\leq c_5+\int_1^{\infty}\frac{\omega(\eta)}{1+\eta^2}\left(1+\eta^2\right)\left|K_{\alpha}^{(l)}(\eta)\right|d\eta \\
\text{(we use } K_{\alpha}^{(l)}(\eta) &= O(\eta^{\alpha-l-1}) \text{ as } \eta\to\infty \text{ and } l>\alpha+1\text{)} \\
&\leq c_5+c_6\int_1^{\infty}\frac{\omega(\eta)}{1+\eta^2}d\eta\equiv c_7<\infty.
\end{aligned}$$

Since $K_{\alpha}^{(l)}(\eta)=O(\eta^{\alpha-l-1})$, $\eta\to\infty$ and $l>\alpha+1$ we have

$$\begin{aligned}
\int_0^{\infty}\left|K_{\alpha}^{(l)}(\eta)\right|\eta d\eta &= \int_0^{1}\left|K_{\alpha}^{(l)}(\eta)\right|\eta d\eta+\int_1^{\infty}\left|K_{\alpha}^{(l)}(\eta)\right|\eta d\eta \\
&\leq c_8+\int_1^{\infty}\left|K_{\alpha}^{(l)}(\eta)\right|\eta d\eta\leq c_9.
\end{aligned}$$

Finally, the summation of these estimations in (3.4) yields

$$\left\|\left(\mathbb{D}_{\varepsilon}^{\alpha}\mathcal{F}_{\nu}^{\alpha}\varphi\right)(t)-\varphi(t)\right\|_{p,\nu}\leq c\mu(\varepsilon)\ \text{ as } \varepsilon\to 0^{+},$$

where c being independent of ε.
The proof of the theorem is completed.

Corollary 2. *Let $\mu(t)=t^{\gamma}, 0<\gamma<1, t\in[0,\rho)$ and $t\in\mathbb{R}_{+}^{n}$ be a μ-smoothness point of $\varphi\in L_{p,\nu}$. Then*

$$\left\|\left(\mathbb{D}_{\varepsilon}^{\alpha}\mathcal{F}_{\nu}^{\alpha}\varphi\right)(t)-\varphi(t)\right\|_{p,\nu}=O\left(\varepsilon^{\gamma}\right)\ \textit{ as } \varepsilon\to 0^{+}.$$

References

1. *Aliev, I.A., Bayrakci, S.*: On inversion of B-elliptic potentials by the method of Balakrishnan-Rubin. Fractional Calculus and Applied Analysis, Vol.1, No:4, (1998) 365-384.

2. *Aliev I.A.* On the Bochner-Riesz summability and restoration of μ-smooth functions by means of their Fourier transforms, Fractional Calculus and Applied Analysis, **2**, No.3 (1999) 265-277.
3. *Aliev I.A, Bayrakci S.*: On inversion of Bessel potentials associated with the Laplace-Bessel differential operator,Acta Math. Hungar. 95 (1-2) (2002) 125-145.
4. *Aliev I.A, Rubin B.*: Wavelet-like transforms for admissible semi-groups; Inversion formulas for potentials and Radon transforms;Journal of Fourier Analysis and Applications, Volume 11, No 3 (2005) 333-352.
5. *Aliev I.A, Rubin B, Uyhan S, Sezer S.*: Composite Wavelet Transforms: Applications and Perspectives, Radon Transforms, Geometry, and Wavelets; Book Series: Contemporary Mathematics, Amer. Math. Soc., Providence, RI, Vol.: 464, 2008, pp. 1-25.
6. *Aliev I.A, Eryiğit M.*: On a rate of convergence of truncated hypersingular integrals associated to Riesz and Bessel potentials,J. Math. Anal. Appl., 406 (2013) 352-359.
7. *Aliev,I. A., Çobanoğlu, S.*: The rate of convergence of truncated hypersingular integrals generated by the Poisson and metaharmonic semigroups,Integral Transforms and Special Functions, (2014) Vol. 25, No. 12, 943–954.
8. *Eryigit, M., Çobanoğlu, S.*: On the rate of L_p-convergence of Balakrishnan-Rubin-type hypersingular integrals associated to Gauss-Weierstrass semigroup,Turk. J. Math., vol. 41, num. 6, p. 1376-1384, (2017).
9. *Sezer Evcan, S., Eryigit, M., Çobanoğlu, S.*: A Balakrishnan-Rubin type Hypersingular Integral Operators and Inversion of Flett Potentials, Hacet. J. Math. Stat. DOI : 10.15672 (accepted).
10. *Flett, T. M.*: Temperatures, Bessel potentials and Lipschitz spaces.Proceedings of the London Mathematical Society, 3 (3), 385-451, 1971.
11. *Gadjiev, A.D., Aral A., Aliev I.A.*: On Behavior Of The Riesz And Generalized Riesz Potentials As Order Tends To Zero,Mathematical Inequalities and Applications, vol.10, pp.875-888, 2007.
12. *Guliev, V.S.*, Sobolev's theorem for Riesz B-potentials. Dokl. Rus. Akad. Nauk ,358, No:4, (1998) 450-451.
13. *Guliev, V.S.*: On maximal function and fractional integral associated with the Bessel differential operator.Mathematical Inequalities and Applications, 6, (2003) 317-330.
14. Kipriyanov, I.A.: Singular elliptic boundary problems, Mirovozrenie Zhizn Moskow, Fizmatlit, Nauka,(1997).
15. *Kurokawa, T.*: On the Riesz and Bessel kernels as approximations of the identity,Sci, Rep. Kagoshima Univ. No:30(1981), 31-45.
16. *Levitan, B. M.: Expansion in Fourier series and integrals with Bessel functions.Uspekhi Matematicheskikh Nauk, 6(2):102–143, 1951.*
17. *Lizorkin P.I.*: The functions of Hirshman type and relations between the spaces $B_p^r(E_n)$ and $L_p^r(\mathbb{R}^n)$, Mat. Sb. 63(4), (1964) 505-535 (Russian).
18. *Lofstorm J., Peetre J.*: Approximation theorems connected with generalized translation,Math. Ann. 181,255-268, (1969).
19. *Rubin, B.*: A method of characterization and inversion of Bessel and Riesz potentials,Sov. Math. (Iz. VUZ) 30 (5) (1986) 78-89.

20. *Rubin, B.*: Inversion of potentials on $\mathbb{R}^n$ with the aid of Gauss-Weierstrass integrals,Math. Notes, 41 (1-2) (1987), 22-27. English translation from Math. Zametki 41 (1) (1987) 34-42.
21. *Rubin B.*: Fractional integrals and potentials. Pitman Monographs and Surveys in Pure and Applied Mathematics. Longman, Harlow, (1996).
22. *Samko S.G.*: Spaces $L^{\alpha}_{p,r}(\mathbb{R}_n)$ and hypersingular integrals,Studia Math., 61, No:3, (1977), 193-230.
23. *Samko S.G, Kilbas A.A, Marichev O.I.*: Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Sci. Publ., London, New York, (1993).
24. *Samko S.G.*: Hypersingular Integrals and Their Applications, in: Ser.: Analytical Methods and Special Functions, Taylor and Francis, London, (2002).
25. *Stein, E., Weiss, G.*: On the theory of harmonic functions of several variables, I; The theory of H^p spaces,Acta Math, 103(1-2), (1960) 25-62.
26. *Stein, E.*: Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton New Jersey, (1970).
27. *Sezer S.*: An Inversion Formula For The Generalized Flett Potentials,International Journal of Mathematics and Computation, vol.2, pp.115-125, (2009).
28. *Sezer S, Aliev I.A.*: On the Gauss Weierstrass Summability of Multiple Trigonometric Series at μ-smoothness points, Acta Mathematica Sinica, English series, Vol.27, No:4, (2011) 741-746.
29. *Uyhan, S.*: On Poisson semigroup generated by the generalized B-Translation,Dynamical Systems and Applications, Proceeding, 679-686, (2004).

Received: 27.01.2020. *Accepted:* 02.05.2020