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The uniqueness of the best non-symmetric L_1-approximant with a weight by $A_{\alpha,\beta}$-subspace

Питання єдиності елемента найкращого L_1-наближення та одностороннього наближення достатньо широко досліджене. Проте задачі несиметричного наближення розглядалися не так інтенсивно. В.Ф. Бабенком було встановлено зв'язок між несиметричними та односторонніми наближеннями. У статті вивчаються питання єдиності елемента найкращого несиметричного L_1-наближення з вагою для неперервних на компакті K функцій зі значеннями у КВ-просторі X скінченновимірним підпростором простору $C(K, X)$ неперервних на K функцій зі значеннями в КВ-просторі X з нормою $||\cdot||_X$.

У просторі $C(K, X)$ введена несиметрична L_1-норма з вагою виду $||f||_{1;\theta;\alpha,\beta} = \int\limits_Q \theta(x)||\alpha \cdot f_+(x) + \beta \cdot f_-(x)||_X d\mu(x)$, де вага θ є μ-вимірною дійсною функцією, заданою на K, яка задовольняє умову $0 < inf\{\theta(x) : x \in K\} \leq sup\{\theta(x) : x \in K\} < \infty$.

Уведено поняття $A_{\alpha,\beta}$-підпростору та показано, що якщо X є (α, β)-гладким КВ-простором і кожна неперервна на компакті K функція зі значеннями у КВ-просторі X має єдиний елемент найкращого (α, β)-наближення з вагою θ елементами скінченно вимірного підпростору H для будь-якої ваги θ, що задовольняє умову $0 < inf\{\theta(x) : x \in K\} \leq sup\{\theta(x) : x \in K\} < \infty$, то підпростір H є $A_{\alpha,\beta}$-підпростором простору неперервних на K функцій зі значеннями у X. Обернене твердження має місце за умов, що КВ-простір X є строго нормованим зі строго монотонною нормою. Об'єднуючи обидва твердження, як наслідок отримано , що, якщо КВ-простір X є строго нормованим зі строго монотонною нормою та (α, β)-гладким, то кожна неперервна на K функція зі значенями у X буде мати єдиний елемент найкращого (α, β)-наближення з вагою скінченно вимірним підпростором H не залежно від ваги θ тоді і тільки тоді, коли H є $A_{\alpha,\beta}$-підпростором простору $C(K, X)$.

У випадку звичайного наближення, тобто, коли $\alpha = \beta = 1$, так звані, А-підпростори вивчалися багатьма авторами. Увів це поняття Ганс Штраус при дослідженні єдиності елемента найкращого L_1-наближення для дійснозначних функцій, неперервних на відрізку. Результати даної статті узагальнюють результати А.Кроо на випадок несиметричного наближення .

Ключові слова: (α, β)-наближення, КВ-простір, L_1-норма, вага, неперервні функції.

The questions of the uniqueness of the best non-symmetric L_1-approximant with weight in the finite dimensional subspace and the connection of such tasks with $A_{\alpha,\beta}$-subspaces were considered in this article. This result generalizes the known result of Kroo on the case of non-symmetric approximation.

Key words: (α, β)-approximation, KB-space, L_1-norm, weight, continuous functions
MSC2010: Pri 41A52, Sec 41A65, 46B40

Let X be a partially ordered set and its order is consistent with algebraic operations. The following definitions are given in [5] .

Let $E \subset X$ be a non-empty set. The element $y \in X$ is called supremum (infimum) of the set E and is denoted by $\sup E$ ($\inf E$) if the following conditions hold:

1) $x \leq y$ ($x \geq y$) $\forall x \in E$;

2) for any element $z \in X$ such that $x \leq z$ ($x \geq z$), it follows that $y \leq z$ ($y \geq z$).

The supremum of the set E is denoted by $x_1 \vee x_2 \vee ... \vee x_n$ and the infimum of the set E is denoted by $x_1 \wedge x_2 \wedge ... \wedge x_n$ if the set E consists of elements $x_1, x_2, ..., x_n$.

Suppose in the space X for any two elements $x, y \in X$ there exists their supremum $x \vee y$; then the element $x_+ = x \vee 0$ is called the positive part of the element $x \in X$, the element $x_- = (-x) \vee 0$ is its negative part, and the element $|x| = x_+ + x_-$ is the module of the element x. Two elements $x, y \in X$ are called disjunctive and are denoted by $x \Delta y$ if $|x| \wedge |y| = 0$.

Let an order of a partially ordered vector space X is consistent with algebraic operations and for any two elements $x, y \in X$ there exists their supremum $x \vee y$. Then a space X is called a KN-lineal if in X the monotone norm is defined, i. e., $|x| \leq |y| \Rightarrow ||x||_X \leq ||y||_X$.

A KN-lineal is called a KN-space (or $K_\sigma N$-space) if for any (or any numbered) non-empty set bounded above or below there exists the its upper or lower bound respectively.

Let X is a KN-linear and X^* is a space of linear continuous in the usual sense functionals on X, then X^* is a complete KN-space in the usual sense.

A $K_\sigma N$-space is called a KB-space if its norm satisfies two conditions:

1) $||x_n||_X \to 0$ if $x_n \downarrow 0$;

2) $||x_n||_X \to +\infty$ if $x_n \uparrow +\infty$ $(x_n \geq 0)$.

Let K be a compact subset of $\mathbb{R}^m$ such that $K = \overline{IntK}$, μ be the Lebesgue measure in $\mathbb{R}^m$ and $\mu(intK) > 0$.

Let X be a KB-space with the norm $|| \cdot ||_X$.

By $C(K, X)$ denote the space of continuous functions $f : K \to X$ and by Θ denote the set of all μ-measurable real functions θ on K such that $0 < inf\{\theta(x) : x \in K\} \leq sup\{\theta(x) : x \in K\} < \infty$.

For any $x \in K$ and positive numbers α, β put

$$|f(x)|_{\alpha,\beta} = \alpha \cdot f_+(x) + \beta \cdot f_-(x),$$

$$||f(x)||_{X;\alpha,\beta} = ||\alpha \cdot f_+(x) + \beta \cdot f_-(x)||_X,$$

where $f_\pm(x) = (\pm f(x)) \vee 0$.

Suppose the space $C(K, X)$ is supplied with the non-symmetric L_1-norm with a weight $\theta \in \Theta$:

$$||f||_{1;\theta;\alpha,\beta} = \int\limits_K \theta(x)||f(x)||_{X;\alpha,\beta} d\mu(x),$$

and denote by $C_\theta(K,X)$ the space $C(K,X)$ endowed with the above norm.

For $f \in C_\theta(K,X)$, $H \subset C_\theta(K,X)$ and $\theta \in \Theta$ the quantity

$$E(f,H)_{1;\theta;\alpha,\beta} = \inf_{g\in H} \|f-g\|_{1;\theta;\alpha,\beta} \tag{1}$$

is called the best (α,β)-approximation of a function f by a set H in the metric L_1 with a weight θ. The function $g^* \in H$ is the best (α,β)-approximant of a function f by elements of a set H in the metric L_1 with a weight θ if g^* realizes the greatest lower bound in the equality (1). By Z_f denote the set of zeros for a function f, and $N_f = K \setminus Z_f$.

For $f,g \in C_\theta(K,X)$ put

$$\tau_\pm^{(\alpha,\beta)}(f,g)_{1;\theta} = \lim_{t\to\pm 0} \frac{||f+tg||_{1;\theta;\alpha,\beta} - ||f||_{1;\theta;\alpha,\beta}}{t}$$

and for $x \in K$ put

$$\tau_\pm^{(\alpha,\beta)}(f(x),g(x))_X = \lim_{t\to\pm 0} \frac{||(f+tg)(x)||_{X;\alpha,\beta} - ||f(x)||_{X;\alpha,\beta}}{t}.$$

For $\alpha=\beta=1$ such functionals were considered in [2] and [1].

A normalized space X is called strictly convex if for any $x,y\in X$ such that $||x+y|| = ||x||+||y||$, it follows that there is $\lambda \in \mathbb{R}$ such that $y = \lambda\cdot x$.

A KB-space X is the space with a strictly monotone norm if $|x|<|y| \Rightarrow ||x||_X < ||y||_X$.

Let H be a subspace of the space $C(K,X)$. We set

$$H' = \{h \in C(K,X): \quad \exists g_h \in H \quad \forall x\in K \quad h(x) = \pm g_h(x)\}.$$

Such classes were considered in [1]. Originally such sets were introduced by Hans Strauss [3] for $X=\mathbb{R}$, $K=[a,b]$.

The following theorem (see [4]) is needed for the sequel.

Theorem 1. *Let X be a KB-space, H be a subspace of space $C_\theta(K,X)$, $\theta\in\Theta$. Then in order that each function $f\in C_\theta(K,X)$ has at most one the best (α,β)-approximant with a weight $\theta\in\Theta$ by elements from H it is necessary that for any function $h\in H'\setminus\{0\}$ there exists a function $g_0\in H$ such that*

$$\int\limits_{N_h} \theta(x)\tau_-^{(\alpha,\beta)}(h(x),g_0(x))_X d\mu(x) > \int\limits_{Z_h} \theta(x)||g_0(x)||_{X;\beta,\alpha} d\mu(x).$$

If the KB-space X is strictly convex with a strictly monotone norm then this condition is sufficient.

Following Strauss [3] and Kroo [1] we consider next property of subspace H.

The finite dimensional subspace $H \subset C(K,X)$ is called an $A_{\alpha,\beta}$-subspace (or is said to satisfy the $A_{\alpha,\beta}$-property) if for any $h \in H' \setminus \{0\}$ there exists a $g \in H$ such that

(i) $g(x) = 0$ a.e. on Z_h;

(ii) $\tau_-^{(\alpha,\beta)}(h(x), g(x))_X \geq 0$ a.e. on N_h and this inequality is strict on a subset of N_h of positive measure.

This Theorem was proved in [4].

Theorem 2. *Let X be a strictly convex KB-space with a strictly monotone norm and assume that H is an $A_{\alpha,\beta}$-subspace of $C(K,X)$. Then each function $f \in C_\theta(K,X)$ has at most one best (α,β)-approximant with a weight $\theta \in \Theta$ by elements from H for every $\theta \in \Theta$.*

The converse is true if the space X is (α,β)-smooth. Recall that the space X is (α,β)-smooth if at every point of its unit sphere there exists a unique tangent functional $\tau_X^{(\alpha,\beta)}(f(x), g(x)) = \tau_-^{(\alpha,\beta)}(f(x), g(x))_X = \tau_+^{(\alpha,\beta)}(f(x), g(x))_X$. In this case $\tau_X^{(\alpha,\beta)}(f,\cdot)$ is a linear functional.

In this paper, following Strauss and Kroo we show that the converse of Theorem 2.

First we prove the lemma.

Lemma 1. *Let X be a (α,β)-smooth KB-space and let H be a finite dimensional subspace $C(K,X)$. For given $h \in H' \setminus \{0\}$ we set $\tilde{H}_h = \{g \in H : g = 0$ a.e. on $Z_h\}$. Suppose each function $f \in C_\theta(K,X)$ has unique best (α,β)-approximant with a weight $\theta \in \Theta$ by elements from H for every $\theta \in \Theta$; then for any $h \in H' \setminus \{0\}$ there exists a $g_0 \in \tilde{H}_h$ such that*

$$\int\limits_{N_h} \theta(x)\tau_X^{(\alpha,\beta)}(h(x), g_0(x))d\mu(x) \neq 0 \tag{2}$$

for any $\theta \in \Theta$.

Proof. Suppose each function $f \in C_\theta(K,X)$ has unique best (α,β)-approximant with a weight $\theta \in \Theta$ by elements from H for every $\theta \in \Theta$. It follows by Theorem 1 for any $h \in H' \setminus \{0\}$ there exists a $g \in H$ such that

$$\int\limits_{N_h} \theta(x)\tau_X^{(\alpha,\beta)}(h(x), g(x))d\mu(x) > \int\limits_{Z_h} \theta(x)||g(x)||_{X;\beta,\alpha}d\mu(x), \tag{3}$$

for any $\theta \in \Theta$.

Assume the converse. Then there exists a $h \in H' \setminus \{0\}$ such that for any $g \in \tilde{H}_h$ we can find $\theta \in \Theta$ satisfying

$$\int\limits_{N_h} \theta(x)\tau_X^{(\alpha,\beta)}(h(x), g(x))d\mu(x) = 0. \tag{4}$$

Let $g_1, g_2, ..., g_k$ be a basis in $\tilde{H}_h$. Since X is a (α,β)-smooth KB-space, it follows that the functional $\tau_X^{(\alpha,\beta)}(f,g)$ is linear of g for any fixed $f \in X$, $f \neq 0$. Then for any $\bar{b} = (b_1 b_2, ..., b_k) \in \mathbb{R}^k$ there exists a $\theta_0 \in \Theta$ such that

$$0 = \int\limits_{N_h} \theta_0(x)\tau_X^{(\alpha,\beta)}(h, \sum_{i=1}^{k} b_i g_i)(x)d\mu(x) = \sum_{i=1}^{k} b_i \int\limits_{N_h} \theta_0(x)\tau_X^{(\alpha,\beta)}(h, g_i)(x)d\mu(x). \qquad (5)$$

Following Kroo [1] we consider the set

$$A_0 = \{ \left(\int\limits_{N_h} \theta(x)\tau_X^{(\alpha,\beta)}(h, g_i)(x)d\mu(x) \right)_{i=1}^{k} : \theta \in \Theta \}.$$

A_0 is a convex subset of $\mathbb{R}^k$ and A_0 has nonempty intersection with any hyperplane $H(\bar{b}) = \{\bar{a} \in \mathbb{R}^k : \langle \bar{a}, \bar{b} \rangle = 0\}$, $\bar{b} \in \mathbb{R}^k$, $\langle \bar{a}, \bar{b} \rangle$ denotes the inner product in $\mathbb{R}^k$.

Let A_0 be an r-dimensional convex subset of $\mathbb{R}^k$. Obviously, $\bar{0}$ is the limit point of A_0. Let us show that $\bar{0} \in A_0$. If $r = 0$, this is trivial. Now let be $1 \leq r \leq k$ then A_0 consists r linearly independent vectors, i.e., there exists $\theta_1, ..., \theta_r \in \Theta$ that

$$\bar{e}_j = \left(\int\limits_{N_h} \theta_j(x)\tau_X^{(\alpha,\beta)}(h(x), g_i(x))d\mu(x) \right)_{i=1}^{k}, \qquad 1 \leq j \leq r,$$

are linearly independent.

Show that A_0 is an open subset $F_r = \text{span}\{\bar{e}_{11}, ..., \bar{e}_r\}$. Give an arbitrary point $\bar{c} = \left(\int\limits_{N_h} \theta_{\bar{c}}(x)\tau_X^{(\alpha,\beta)}(h(x), g_i(x))d\mu(x) \right)_{i=1}^{k} \in A_0$ ($\theta_c \in \Theta$) and as well as in [1] indicate ball with center at this point which belongs to A_0.

Choosing a sufficiently small $h > 0$, we obtain $\theta_{\bar{c}} \pm h\theta_j \in \Theta, 1 \leq j \leq r$ and $\bar{d}_j^{\pm} = \left(\int\limits_{N_h} (\theta_{\bar{c}} \pm h\theta_j)(x)\tau_X^{(\alpha,\beta)}(h(x), g_i(x))d\mu(x) \right)_{i=1}^{k} = \bar{c} \pm h\bar{e}_j \in A_0, 1 \leq j \leq r.$

Further convexity of A_0 implies that for any $\alpha_j^{\pm} \geq 0 (1 \leq j \leq r)$ such that $\sum\limits_{j=1}^{r} \alpha_j^+ + \sum\limits_{j=1}^{r} \alpha_j^- = 1$ we have $\sum\limits_{j=1}^{r} \alpha_j^+ \bar{d}_j^+ + \sum\limits_{j=1}^{r} \alpha_j^- \bar{d}_j^- \in A_0$. Then

$$\sum_{j=1}^{r} \alpha_j^+ \bar{d}_j^+ + \sum_{j=1}^{r} \alpha_j^- \bar{d}_j^- = \bar{c} + h \sum_{j=1}^{r} (\alpha_j^+ - \alpha_j^-)\bar{e}_j \in A_0.$$

Since $\bar{e}_j, 1 \leq j \leq r$ are linearly independent then we indicated the r-dimensional ball with center at the point $\bar{c}$ from A_0 and hence, A_0 is the open subset of F_r.

Assuming that $\bar{0} \notin A_0$, we obtain that $\bar{0} \in \text{Bd}A_0$ and exists a hyperplane $\tilde{F}$ in F_r supporting A_0 at $\bar{0}$. But A_0 has nonempty intersection with any hyperplane $H(\bar{b}) =$

$\{\bar{a} \in \mathbb{R}^k : \langle \bar{a}, \bar{b} \rangle = 0\}$ and, hence, with $\tilde{F}$. This contradicts the fact that A_0 is open in F_r. It follows that $\bar{0} \in A_0$ and exists a weight $\tilde{\theta} \in \Theta$ such that

$$\int\limits_{N_h} \tilde{\theta}(x) \tau_X^{(\alpha,\beta)}(h(x), g_i(x)) d\mu(x) = 0, \quad 1 \le i \le k.$$

Since $\tau_X^{(\alpha,\beta)}(h, \cdot)(h \neq 0)$ is the linear functional then

$$\int\limits_{N_h} \tilde{\theta}(x) \tau_X^{(\alpha,\beta)}(h, g)(x) d\mu(x) = 0, \quad g \in \tilde{H}_h. \tag{6}$$

Let be $\dim H = n$ and $g_1, ..., g_n$ is the basis in H such that the elements $g_1, ..., g_k$ are the basis in $\tilde{H}_h$ and set $H_0 = \mathrm{span}\{g_{k+1}, ..., g_n\}$. Consider the functionals

$$\eta_1(g) = \int\limits_{Z_h} ||g(x)||_{X;\beta,\alpha} d\mu(x), \quad \eta_2(g) = \sup_{x \in K} ||g(x)||_{X;\alpha,\beta}, \quad g \in H_0.$$

These functionals are the norms in H_0. Since the norms are equivalent in the finite-dimensional space then exists $\xi > 0$ such that $\eta_2(g) \le \xi \eta_1(g)$ for every $g \in H_0$. Let the weight $\theta^* \in \Theta$ be defined as follows $\theta^*(x) = \tilde{\theta}(x)$ for $x \in N_h$ and $\theta^*(x) = \xi \sup\limits_{x \in K} \tilde{\theta}(x) \mu(K)$ for $x \in Z_h$.

Then, since the functional $\tau_X^{(\alpha,\beta)}(h, g)$ is linear, for any $g = \tilde{g}_1 + \tilde{g}_2 \in H$, where $\tilde{g}_1 \in \tilde{H}_h, \tilde{g}_2 \in H_0$, we obtain that

$$\int\limits_{N_h} \theta^*(x) \tau_X^{(\alpha,\beta)}(h, g)(x) d\mu(x) = \int\limits_{N_h} \tilde{\theta}(x) \tau_X^{(\alpha,\beta)}(h, \tilde{g}_1 + \tilde{g}_2)(x) d\mu(x) =$$

$$= \int\limits_{N_h} \tilde{\theta}(x) \tau_X^{(\alpha,\beta)}(h, \tilde{g}_2)(x) d\mu(x) \le \sup_{x \in K} \tilde{\theta}(x) \mu(K) \eta_2(\tilde{g}_2) \le$$

$$\le \xi \sup_{x \in K} \tilde{\theta}(x) \mu(K) \eta_1(\tilde{g}_2) = \int\limits_{Z_h} \theta^*(x) ||\tilde{g}_2(x)||_{X;\beta,\alpha} d\mu(x) = \int\limits_{Z_h} \theta^*(x) ||g(x)||_{X;\beta,\alpha} d\mu(x).$$

It contradicts (3).

The lemma is proved.

Theorem 3. *Let X be a (α, β)-smooth KB-space, let H be a finite dimensional subspace $C(K, X)$ and assume that each function $f \in C_\theta(K, X)$ has unique best (α, β)-approximant with a weight $\theta \in \Theta$ by elements from H for every $\theta \in \Theta$. Then H is the $A_{\alpha,\beta}$-subspace of $C(K, X)$.*

Proof. By Lemma 1, for any $h \in H' \setminus \{0\}$ there exists $g_0 \in \tilde{H}_h$ such that $\int\limits_{N_h} \theta(x)\tau_X^{(\alpha,\beta)}(h(x), g_0(x))d\mu(x) \neq 0$ for all $\theta \in \Theta$.

It means that or $\tau_X^{(\alpha,\beta)}(h(x), g_0(x)) \geq 0$ a.e. on N_h, or $\tau_X^{(\alpha,\beta)}(h(x), g_0(x)) \leq 0$ a.e. on N_h. Indeed, assuming the contrary, that each set

$$S_- = \{x \in N_h : \tau_X^{(\alpha,\beta)}(h(x), g_0(x)) < 0\}, \quad S_+ = \{x \in N_h : \tau_X^{(\alpha,\beta)}(h(x), g_0(x)) > 0\}$$

has positive measure, then

$$\int\limits_{S_-} \tau_X^{(\alpha,\beta)}(h(x), g_0(x))d\mu(x) < 0,$$

$$\int\limits_{S_+} \tau_X^{(\alpha,\beta)}(h(x), g_0(x))d\mu(x) > 0.$$

For a sufficiently small $\epsilon > 0$ we take the weight functions $\theta_1(x) = 1$ for $x \in S_-$ and $\theta_1(x) = \epsilon$ for $x \in K \setminus S_-$, $\theta_2(x) = 1$ for $x \in S_+$ and $\theta_2(x) = \epsilon$ for $x \in K \setminus S_+$, for which

$$\int\limits_{N_h} \theta_1(x)\tau_X^{(\alpha,\beta)}(h(x), g_0(x))d\mu(x) < 0,$$

$$\int\limits_{N_h} \theta_2(x)\tau_X^{(\alpha,\beta)}(h(x), g_0(x))d\mu(x) > 0.$$

Then there exists $\gamma \in (0;1)$ such that

$$\int\limits_{N_h} (\gamma\theta_1(x) + (1-\gamma)\theta_2(x))\tau_X^{(\alpha,\beta)}(h(x), g_0(x))d\mu(x) = 0,$$

where $\gamma\theta_1(x) + (1-\gamma)\theta_2(x) \in \Theta$.

This contradicts the lemma.

Since $\tau_X^{(\alpha,\beta)}$ is linear functional we can assume that $\tau_X^{(\alpha,\beta)}(h(x), g_0(x)) \geq 0$ a.e. at N_h.

Thus, we indicate the element $g_0 \in \tilde{H}_h$ such that $g_0 = 0$ a.e. on Z_h and $\tau_X^{(\alpha,\beta)}(h(x), g_0(x)) \geq 0$ a.e. at N_h. Moreover, by Lemma 1, $\tau_X^{(\alpha,\beta)}(h(x), g_0(x)) > 0$ on a subset of N_h of the positive measure. Hence, set H is the $A_{\alpha,\beta}$-subspace of $C(K, X)$.

The theorem is proved.

Combining the theorems 2 and 3, we get the corollary.

Corollary 1. *Let X be a strictly convex (α, β)-smooth KB-space with a strictly monotone norm and let H be a finite dimensional subspace $C(K, X)$. Then each function $f \in C_\theta(K, X)$ has unique best (α, β)-approximant with a weight $\theta \in \Theta$ by elements from H for every $\theta \in \Theta$ iff H is the $A_{\alpha,\beta}$-subspace of $C(K, X)$.*

These results have been proven in [1] in the case $\alpha = \beta = 1$. In the article we use Kroo proof methods of [1].

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