UDK 512.544

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On some generalization of pronormal subgroups in locally finite group

We proved that if every subgroup of locally finite group is monoprormal, then this group is a $\overline{T}$ -group.

Key words: locally finite group, monoprormal subgroup, contranormal subgroup, Dedekind group, supersoluble group, locally nilpotent residual, $\overline{T}$ -group.

Доведено, що локально скінченна група, усі підгрупи якої є монопронормальними підгрупами, буде $\overline{T}$ -групою.

Ключові слова: локально скінченна група, монопронормальна підгрупа, контранормальна підгрупа, дедекіндова група, надрозв'язна група, локально нільпотентний резидуал, $\overline{T}$ -група.

Доказано, что локально конечная группа, все подгруппы которой являются монопронормальными подгруппами, будет $\overline{T}$ -группой.

Ключевые слова: локально конечная группа, монопронормальная подгруппа, контранормальная подгруппа, дедекиндова группа, сверхразрешимая группа, локально нильпотентный резидуал, $\overline{T}$ -группа. A subgroup H of a group G is said to be *pronormal* in G if the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$ for each element $g \in G$. Well known that normalizer of pronormal subgroup is abnormal. In the same time, every abnormal subgroup is contranormal. Thus, we naturally obtained the following generalization of pronormal subgroups.

Let G be a group and H be a subgroup of G . Let $K = \langle H, x \rangle$, $x \in G$. A subgroup H is called *monoprormal* in G if for every element $x \in G$ either $H^x = H$ or $N_K(H)^K = K$.

The groups, in which every subgroup is pronormal, have been studied by N.F. Kuzennyj and I.Ya. Subbotin [2]. They proved that if every subgroup of a group G is pronormal, then G is a $\overline{T}$ -group. Recall that a group G is said to be a T -group, if every subnormal subgroup of G is normal [1]. A group G is said to be a $\overline{T}$ -group, if every subgroup of G is a T -group [1].

In this connection it is interesting to consider the relationships between monoprormal subgroups and $\overline{T}$ -groups.

Lemma 1. *Let G be a group such that every subgroup of G is monoprormal in G .*

- (i) *If H is a subgroup of G , then every subgroup of H is monoprormal in H .*
- (ii) *If H is a normal subgroup of G , then every subgroup of factor-group G/H is monoprormal in G/H .*

Lemma 2. *Let G be a group such that every subgroup of G is monoprormal in G . If H is nilpotent subgroup of G then H is a Dedekind group.*

Proof. Let H be a nilpotent subgroup of G and $x, y \in H$. Put $K = \langle x, y \rangle$. Since a subgroup K is nilpotent, every subgroup of K is not contranormal, so $N_K(H)^K \neq K$. Thus $\langle x \rangle^y = \langle x \rangle$, which means that any cyclic subgroup of H is normal in H . Hence any subgroup of H is normal in H . So H is a Dedekind group.

Corollary 1. *Let G be a group such that every subgroup of G is monoprormal in G . If H is a locally nilpotent subgroup of G then H is Dedekind.*

Proof. Let H be a locally nilpotent subgroup of G and $x, y \in H$. Put $K = \langle x, y \rangle$. Since H is a locally nilpotent subgroup, K is nilpotent. By Lemma 2 $\langle x \rangle^y = \langle x \rangle$ for every element $y \in H$. It follows that H is a Dedekind group.

Lemma 3. *Let G be a group such that every subgroup of G is monoprormal in G . Let H be a subgroup of G and A be a normal abelian subgroup of H . Then every subgroup of A is G -invariant.*

Proof. Let H be a subgroup of G and A be a normal abelian subgroup of H . Let $a \in A$. If $A = \langle a \rangle$ then the proof is obvious. Let $\langle a \rangle$ be a proper subgroup of A (i.e. $\langle a \rangle \neq A$). Put $K = \langle A, y \rangle$, where $y \in H \setminus A$. Obviously in this case $A \leq N_K(\langle a \rangle)$ and factor-group K/A is abelian. It follows that $K \geq N_K(\langle a \rangle)$. Thus $N_K(\langle a \rangle) = K$. So every proper cyclic subgroup $\langle a \rangle$ is H -invariant and then G -invariant, since H is arbitrary subgroup of G . Hence every subgroup of A is G -invariant.

Corollary 2. *Let G be a group such that every subgroup of G is monoprormal in G . Let H be a nilpotent subgroup of G and A be a minimal abelian H -invariant subgroup of H . Then A is a cyclic subgroup of prime order.*

Lemma 4. *Let G be a finite group such that every subgroup of G is monoprormal in G . Then G is supersoluble.*

Proof. Let S_p be a Sylow p -subgroup of G , where p is a least prime number dividing the order of group G . Put $L = N_G(S_p)$. By Corollary 2 S_p has a chief series, whose factors are cyclic of prime order. Denote the chief factor of subgroup S_p as A/B . The order of factor-group $L/C_L(A/B)$ divides $p-1$. But p is least and we have $C_L(A/B) = N_G(S_p) = L$. So chief factor A/B is central. Hence $N_G(S_p) = S_p \times S_{p'}$. If $p > 2$ then S_p is abelian and $S_p \leq \zeta(N_G(S_p))$. Then by Burnside's Theorem S_p has a normal complement in G . If $p = 2$ then by Lemma 2 S_2 is Dedekind and by Theorem 1 from [3] S_2 also has a normal complement in G . So in both cases there is a normal subgroup H such that $G = H\lambda S_p$. Further, using similar arguments we choose Sylow q -subgroup S_q in H , where q is a least prime number dividing the order of H . So S_q has a normal complement K in H , i.e. $H = K\lambda S_q$. Continuing this process we obtain Sylow series such that all its factors are cyclic of prime order. It follows that group G is supersoluble.

Corollary 3. *Let G be a locally finite group such that every subgroup of G is monoprormal in G . Then G is a locally supersoluble.*

Proof. Let H be a finitely generated subgroup of G . Since G is a locally finite, H is finite. Note that every subgroup of H is monoprormal in H . By Lemma 4 a subgroup H is supersoluble. Thus every finitely generated subgroup of G is supersoluble. Hence G is locally supersoluble.

Theorem 1. [4] *Let G be a finite group and A be an abelian normal subgroup of G . If the factor-group $G/C_G(A)$ is nilpotent then $A = Z \times E$, where all chief factors of Z are central and all chief factors of E are eccentric.*

In this case we say that a subgroup A has a Z -decomposition in G .

Lemma 5. *Let G be a locally finite group such that every subgroup of G is monoprormal in G . Then all G -chief factors of locally nilpotent residual L is eccentric.*

Proof. Suppose that locally nilpotent residual L has a central factors. Let A be an abelian normal subgroup of locally nilpotent residual L . Denote by $\mathfrak{S}$ the set of all finite subgroups of G . Let $H \in \mathfrak{S}$ and suppose that $\mathfrak{H}$ is a set of all finite H -invariant subgroups of A . Let $B \in \mathfrak{H}$. By Theorem 1 a subgroup B has a Z -decomposition: $B = Z_B \times E_B$, where all chief factors of Z_B are central and all chief factors of E_B are eccentric. Note that if $B \leq C$, where $C \in \mathfrak{S}$, then $Z_B \geq Z_C$, $E_B \leq E_C$. Put $E = \bigcup_{H \in \mathfrak{S}} E_H$. So E is a normal subgroup of G . All chief factors of E are eccentric and all chief factors of the factor-group A/E are central. Thus $A/E \leq \zeta_\infty(G/E)$. The factor-group G/A is locally nilpotent, then G/E is also locally nilpotent. A subgroup E is a proper subgroup of A , i.e. $A \neq E$. On the other hand G/E is nilpotent, hence $A \leq E$. So we obtain a contradiction. Thus L doesn't has central factors.

Theorem 2. *Let G be a locally finite group such that every subgroup of G is monoprormal in G . Then G is a $\bar{T}$ -group.*

Proof. Since G is locally finite group such that every subgroup of G is monoprormal in G , by Corollary 3 G is locally supersoluble. This means that G has a Sylow descending series (no necessarily finite)

$$G = G_0 \geq G_1 \geq G_2 \geq \dots,$$

where G/G_1 is a Sylow p_1 -subgroup, G_1/G_2 is a Sylow p_2 -subgroup etc., in addition $p_1 < p_2 < \dots$. In particular, the factor-group G/G_1 can be a 2-group. Let L be a locally nilpotent residual of group G . Since G/G_1 is a locally nilpotent 2-group then $G_1 \geq L$, and then $2 \notin \Pi(L)$. Thus Sylow p -subgroups of L are abelian.

Since G is supersoluble, the derived subgroup $[G, G]$ is locally nilpotent. So L is also locally nilpotent because $[G, G]$ contains L . Since $2 \notin \Pi(L)$, L is abelian.

The factor-group G/L is locally nilpotent, then by Corollary 1 it is Dedekind. A subgroup L is abelian and normal in G , then by Lemma 3 every subgroup of L is G -invariant.

By [1] it remains to prove that $\Pi(L) \cap \Pi(G/L) = \emptyset$. Assume that it is not true. Let there is a prime number p , such that $p \in \Pi(L) \cap \Pi(G/L)$. Note that $p \in \Pi(L)$ and $2 \notin \Pi(L)$, so $p \neq 2$. Since L is abelian, $L = S_p \times S_{p'}$. Put $H = G/S_{p'}$ and $K = L/H$. Then factor-group L/H is a p -subgroup and H/K is nilpotent. Let P be a Sylow p -subgroup of H/K (i.e. $P \in \text{Syl}_p(H/K)$). Note that P contains K and $\zeta(P/K) \neq \langle 1 \rangle$ because H/K is nilpotent. By Lemma 5 all G -chief factors of nilpotent residual L are eccentric. Put $Z = \langle zK \rangle \leq \zeta(P/K)$. Since factor-group P/K is central, $[H, Z] \leq K$. By Theorem 1 and last inclusion $Z = C \times K$, where C is a some subgroup. On the other hand, by Lemma 3 every subgroup of Z is G -invariant. Pick the element $x \in \Omega_1(K)$, where $\Omega_1(K)$ is the lower layer of K , i.e. $\Omega_1(K) = \{x \in K \mid |x| = p\}$. So there is an integer k such that k is relatively prime to p and $x^g = x^k$, $g \in G$. Pick an element c of C . Then $c^g = c$. So $(xc)^g = x^g c^g = x^k c \neq (xc)^s$ for some s . So we obtain a contradiction. It shows that Z contains an element which is not G -invariant. Hence G is a $\bar{T}$ -group.

Note that Theorem 2 can't be generalized to the case of arbitrary periodic groups. The following example illustrates this.

Example 1. Let G be a Tarski monster group, i.e. G is infinite group which proper non-identity subgroups are finite cyclic of prime order p . Note that A. Yu. Ol'shanskii proved the existence of such groups for any prime $p > 10^{75}$ [5, §28]. In G every element generates cyclic subgroup of prime order p and every two non-commuting elements generate the whole group G . If $y \in \langle x \rangle$, then $\langle x \rangle^y = \langle x \rangle$. If $y \notin \langle x \rangle$, then $\langle x, y \rangle = G$ and $\langle x \rangle = N_G(\langle x \rangle)$. Thus $N_G(\langle x \rangle)^G = G$. So G is the periodic group such that every subgroup of G is monoprormal in G , but G isn't a $\bar{T}$ -group.

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Надійшла до редакції 27.04.2015