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Kolmogorov inequalities for norms of Marchaud-type fractional derivatives of multivariate functions

Нехай $C(\mathbb{R}^n)$ – простори обмежених неперервних функцій $x : \mathbb{R}^n \to \mathbb{R}$ з нормами

$$\|x\|_C = \|x\|_{C(\mathbb{R}^n)} := \sup\{|x(t)| : \ t \in \mathbb{R}^n\}.$$

Точки $t \in \mathbb{R}^n$ будемо зображати у вигляді $t = (\underline{t}, \overline{t}) = \underline{t} + \overline{t}$, where $\underline{t} = (t_1, ..., t_k, 0, ..., 0)$, $\overline{t} = (0, ..., 0, t_{k+1}, ..., t_n)$. Для заданих модулів неперервності ω_1, ω_2 означимо простори

$$H^{k,\omega_1} = \{x \in C(\mathbb{R}^n) : \|x\|_{H^{k,\omega_1}} = \sup_{\underline{t} \neq \theta} \frac{\|x(\cdot + \underline{t}) - x(\cdot)\|_C}{\omega_1(|\underline{t}|)} < \infty\},$$

$$H^{n-k,\omega_2} = \{x \in C(\mathbb{R}^n) : \|x\|_{H^{n-k,\omega_2}} = \sup_{\overline{t} \neq \theta} \frac{\|x(\cdot + \overline{t}) - x(\cdot)\|_C}{\omega_2(|\overline{t}|)} < \infty\}.$$

Для функцій $x \in H^{k,\omega_1} \cap H^{n-k,\omega_2}$ одержано нові точні нерівності типу Колмогорова для C-норм змішаних частинних похідних типу Маршо, що означаються в такий спосіб

$$(D_\varepsilon^\alpha)x(u) = \frac{\alpha_1 \alpha_2}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} \int\limits_{\mathbb{R}^n_+} \frac{x(u) - x(u - \varepsilon \underline{t}) - x(u - \varepsilon \overline{t}) + x(u - \varepsilon t)}{|\underline{t}|^{k+\alpha_1} |\overline{t}|^{n-k+\alpha_2}} \, dt,$$

де $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$, $\varepsilon_i = \pm$, $i = 1, ..., n$, $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 \in (0, 1)$.

Наведено деякі застосування одержаних нерівностей, зокрема, до задачі наближення необмеженого оператора дробового диференціювання обмеженими.

Ключові слова: нерівність Колмогорова, дробова похідна, модуль неперервності, норма

We obtain new sharp Kolmogorov type inequalities, estimating the norm of mixed Marchaud type derivative of multivariate function through the C-norm of function itself and its norms in Hölder spaces.

Key words: Kolmogorov inequality, fractional derivative, modulus of continuity, norm

MSC2010: Pri 41A10, 41A17, 41A35, 41A65, Sec 26A33, 26D10, 35A23, 41A29

1. Introduction. Statements of the problems. Main results

Sharp Kolmogorov-type inequalities are of great importance for many branches of mathematics and its applications. Many inequalities of this type for norms of integer derivatives of univariate functions are known (see [16, 17, 18, 2, 3, 7, 20, 22]). But in the case of functions of many variables very few results were obtained (see [19, 12, 25, 6, 4, 10, 11]). Some known sharp Kolmogorov-type inequalities for fractional derivatives can be found in [10, 11, 5, 15, 1, 21].

Let $\mathbb{R}^n$ be the Euclidian space of points $t = (t_1, ..., t_n)$, $|t| = \sqrt{t_1^2 + ... + t_n^2}$ and $\mathbb{R}^n_+ = \{t \in \mathbb{R}^n : \ t_i \geq 0, \ \forall i = 1, ..., n\}$.

Introduce the space $\mathbb{R}^n$ in the form $\mathbb{R}^k \oplus \mathbb{R}^{n-k}$. The point t we will present in the form $t = (\underline{t}, \overline{t}) = \underline{t} + \overline{t}$, where $\underline{t} = (t_1, ..., t_k, 0, ..., 0)$, $\overline{t} = (0, ..., 0, t_{k+1}, ..., t_n)$.

By $C(\mathbb{R}^n)$ denote the space of all bounded and continuous functions $x : \mathbb{R}^n \to \mathbb{R}$ with the norm

$$\|x\|_C = \|x\|_{C(\mathbb{R}^n)} := \sup\{|x(t)| : \ t \in \mathbb{R}^n\}.$$

Let $x \in C(\mathbb{R}^n)$, $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 \in (0, 1)$, $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$, $\varepsilon_i = \pm$, $i = 1, ..., n$. For $t \in \mathbb{R}^n$ set $\varepsilon t = (\varepsilon_1 t_1, ..., \varepsilon_n t_n)$.

We define the mixed difference of the spaces $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ by equality

$$\Delta_t^{k,n-k} x(u) = x(u) - x(u + \underline{t}) - x(u + \overline{t}) + x(u + t)$$

The mixed Marchaud derivative of order α is denoted as

$$(D^\alpha_\varepsilon) x(u) = A_\alpha \int\limits_{\mathbb{R}^n_+} \frac{\Delta_{-\varepsilon t}^{k,n-k} x(u)}{|\underline{t}|^{k+\alpha_1} |\overline{t}|^{n-k+\alpha_2}} \, dt,$$

where $A_\alpha = A_{\alpha_1} A_{\alpha_2}$, $A_{\alpha_i} = \dfrac{\alpha_i}{\Gamma(1 - \alpha_i)}$, $i = 1, 2$.

The different definitions of fractional derivatives of one- and multivariate functions can be found in [23].

For $k \in \mathbb{N}$ and $\beta \in (0, 1)$ let $H^{k,\beta}$ and $H^{n-k,\beta}$ be the spaces:

$$H^{k,\beta} = \{x \in C(\mathbb{R}^n) : \|x\|_{k,\beta} = \sup_{t \neq \theta} \frac{\|x(\cdot + \underline{t}) - x(\cdot)\|_C}{|\underline{t}|^\beta} < \infty\},$$

$$H^{n-k,\beta} = \{x \in C(\mathbb{R}^n) : \|x\|_{n-k,\beta} = \sup_{t \neq \theta} \frac{\|x(\cdot + \overline{t}) - x(\cdot)\|_C}{|\overline{t}|^\beta} < \infty\},$$

where $\theta = (0, ..., 0) \in \mathbb{R}^n$.

Theorem A. (see [8]). Let $\beta_1, \beta_2 \in (0, 1)$ $x \in H^{k,\beta_1} \cap H^{n-k,\beta_2}$ and $\dfrac{\alpha_1}{\beta_1} + \dfrac{\alpha_2}{\beta_2} < 1$. The following sharp inequality holds

$$\|(D^\alpha_\varepsilon) x(\cdot)\|_C \leq \frac{\sigma_k \sigma_{n-k}}{2^n} A_\alpha \frac{2^{2 - \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}}}{\alpha_1 \alpha_2 \left(1 - \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}\right)} \|x\|_C^{1 - \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}} \|x\|_{k,\beta_1}^{\frac{\alpha_1}{\beta_1}} \|x\|_{n-k,\beta_2}^{\frac{\alpha_2}{\beta_2}}, \qquad (1.1)$$

where σ_n is the surface area of the unite ball in $\mathbb{R}^n$.

Note that for one-variable functions inequalities of such type were obtained in [5] and for functions of two variables (1.1) was proved in [10].

Let $\omega(t)$ be a modulus of continuity, i.e. continuous, non-decreasing, sub-additive function defined on $[0,+\infty)$ and such that $\omega(0)=0$. By $H^\omega = H^\omega(\mathbb{R})$ denote the space of functions $x\in C(\mathbb{R})$, for which

$$\|x\|_{H^\omega} = \sup_{\substack{t_1,t_2\in\mathbb{R}\\ t_1\neq t_2}} \frac{|x(t_1)-x(t_2)|}{\omega(|t_1-t_2|)} < \infty.$$

If $\omega(t)=t^\beta$, $\beta\in(0,1]$, then we write H^β instead of H^ω.

Let ω_1, ω_2 be given moduli of continuity. We will consider the following spaces:

$$H^{k,\omega_1} = \{x\in C(\mathbb{R}^n) : \|x\|_{H^{k,\omega_1}} = \sup_{\underline{t}\neq\theta} \frac{\|x(\cdot+\underline{t})-x(\cdot)\|_C}{\omega_1(|\underline{t}|)} < \infty\},$$

$$H^{n-k,\omega_2} = \{x\in C(\mathbb{R}^n) : \|x\|_{H^{n-k,\omega_2}} = \sup_{\overline{t}\neq\theta} \frac{\|x(\cdot+\overline{t})-x(\cdot)\|_C}{\omega_2(|\overline{t}|)} < \infty\},$$

If $\omega_1(\underline{t}) = |\underline{t}|^{\beta_1}$, $\omega_2(\overline{t}) = |\overline{t}|^{\beta_2}$, $\beta_1,\beta_2\in(0,1]$, then we write H^{k,β_1} and H^{n-k,β_2} instead of H^{k,ω_1} and H^{n-k,ω_2}.

In what follows, for $\alpha_1,\alpha_2\in(0,1)$ and for given moduli of continuity ω_1, ω_2, we will need the following condition:

$$\int\limits_{\mathbb{R}^n_+} \frac{\min\{1,\omega_1(|\underline{t}|),\omega_2(|\overline{t}|)\}}{(|\underline{t}|^{k+\alpha_1}(|\overline{t}|^{n-k+\alpha_2}} \, dt < \infty. \tag{1.2}$$

The main result of this paper is contained in the following

Theorem 1. *Let moduli of continuity ω_1, ω_2 and numbers $\alpha_1,\alpha_2\in(0,1)$ be such that condition (1.2) is satisfied. Then for any function $x\in H^{k,\omega_1}\cap H^{n-k,\omega_2}$ and any vector varepsilon of sign distribution, the following sharp inequality holds:*

$$\|(D^\alpha_\varepsilon)x\|_C \le 2A_\alpha \int\limits_{\mathbb{R}^n_+} \frac{\min\{2\|x\|_C, \|x\|_{H^{k,\omega_1}}\cdot\omega_1(|\underline{t}|), \|x\|_{H^{n-k,\omega_2}}\cdot\omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}} \, dt. \tag{1.3}$$

Remark 1. If $\omega_1(|\underline{t}|) = |\underline{t}|^{\beta_1}$, $\omega_2(|\overline{t}|) = |\overline{t}|^{\beta_2}$, $\beta_1,\beta_2\in(0,1]$, we obtain the result of Theorem A.

Remark 2. Theorem 1 is the multivariate analog of the result from [5].

Remark 3. The problems, analogous to Theorem A and Theorem 1 were considered in [9].

For given moduli of continuity ω_1, ω_2 by UH^{k,ω_1} and UH^{n-k,ω_2} denote the unit balls in the spaces H^{k,ω_1} and H^{n-k,ω_2} respectively. Set $UH^{\omega_1,\omega_2} = UH^{k,\omega_1} \cap UH^{n-k,\omega_2}$.

For $\beta_j \in (0,1]$, $j = 1,2$ by UH^{k,β_1} and UH^{n-k,β_2} denote the unit balls in the spaces H^{k,β_1} and H^{n-k,β_2} respectively and let $UH^{\beta_1,\beta_2} = UH^{k,\beta_1} \cap UH^{n-k,\beta_2}$.

Consider the function

$$\Omega(\delta, UH^{\omega_1,\omega_2}) = \sup_{\substack{x \in UH^{\omega_1,\omega_2} \\ \|x\|_C \leq \delta}} \|D_\varepsilon^\alpha x\|_C, \quad \delta \geq 0 \tag{1.4}$$

The function (1.4) is called the modulus of continuity of the operator D_ε^α on the set UH^{ω_1,ω_2} .

Theorem 1 implies the following statement

Corollary 1. *Under conditions of Theorem 1 for any $\delta > 0$*

$$\Omega(\delta, UH^{\omega_1,\omega_2}) = 2A_\alpha \int\limits_{\mathbb{R}_+^n} \frac{\min\{2\delta, \omega_1(|\underline{t}|), \omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1} |\overline{t}|^{n-k+\alpha_2}}\, dt. \tag{1.5}$$

In particular, if $\beta_1, \beta_2 \in (0,1]$ and $\alpha_1, \alpha_2 \in (0,1)$ are such that $\dfrac{\alpha_1}{\beta_1} + \dfrac{\alpha_2}{\beta_2} < 1$, then

$$\Omega(\delta, UH^{\beta_1,\beta_2}) = \frac{\sigma_k \sigma_{n-k}}{2^n} A_\alpha \frac{2^{2-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}}}{\alpha_1\alpha_2\left(1 - \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}\right)} \delta^{1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}}.$$

The problem of finding of the modulus of continuity for a given operator on a given set is closely related to the problem about approximation of an unbounded operator by bounded ones.

We now consider the general statement of this problem.

Let X and Y be the Banach spaces, let $\mathcal{L}(X,Y)$ be the space of linear bounded operators $S : X \to Y$, and let $A : X \to Y$ be an operator (not necessarily linear) with the domain $D_A \subset X$. Let also $Q \subset D_A$ be some class of elements.

For $N > 0$, the quantity

$$E_N(A,Q) = \inf_{\substack{S \in \mathcal{L}(X,Y) \\ \|S\| \leq N}} \sup_{x \in Q} \|Ax - Sx\|_Y \tag{1.6}$$

is called the best approximation of the operator A on the set Q by linear operators $S : X \to Y$ such that $\|S\| = \|S\|_{X \to Y} \leq N$.

The problem is to compute the quantity (1.6) and to find the extremal operator, i.e. the operator delivering the infimum on the right-hand side of (1.6).

This problem appeared in Stechkin's investigations in 1965. The statement of this problem, the first important results and the solution of this problem for low order differentiation operators were presented in [24]. For a survey of further research on this problem see [2, 3].

The function

$$\Omega(\delta, Q) := \sup_{\substack{x \in Q \\ \|x\|_X \le \delta}} \|Ax\|_Y, \quad \delta \ge 0,$$

is called the modulus of continuity of the operator A on the set Q.

Note that this definition generalizes the above presented definition of the modulus of continuity of the operator D^α_ε on the set UH^{ω_1,ω_2}.

It is easily seen that the problem of computation of the function $\Omega(\delta, Q)$ for a given operator is the abstract version of the problem about the Kolmogorov inequality.

S.B. Stechkin [24] proved that

$$E_N(A, Q) \ge \sup_{\delta \ge 0}\{\Omega(\delta, Q) - N\delta\}. \tag{1.7}$$

Namely, the inequality (1.7) shows the relation between the Stechkin problem and Kolmogorov type inequalities.

Theorem 2. *Let the strictly increasing moduli of continuity ω_1, ω_2 and the numbers $\alpha_1, \alpha_2 \in (0,1)$ be such that the condition (1.2) is satisfied. Given $N > 0$, let $h^N = (h^N_1, h^N_2) \in \mathbb{R}^2_+$ be such that*

$$\omega_1(h^N_1) = \omega_2(h^N_2) \qquad \text{and} \qquad \frac{4A_\alpha}{\alpha_1\alpha_2}\sigma_k\sigma_{n-k}(h^N_1)^{-\alpha_1}(h^N_2)^{-\alpha_2} = N. \tag{1.8}$$

Let

$$G(h^N) := \{u = (u_1, \ldots, u_n) \in \mathbb{R}^n \ : \ |\underline{u}| \ge h^N_1, |\overline{u}| \ge h^N_2\}.$$

Then

$$E_N\left(D^\alpha_\varepsilon, UH^{\omega_1,\omega_2}\right) = 2A_\alpha \int\limits_{\mathbb{R}^n_+ \setminus G(h^N)} \frac{\min\{\omega_1(|\underline{t}|), \omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt.$$

In addition, the operator

$$B_{h^N}x(u) = A_\alpha \int\limits_{G(h^N)} \frac{\Delta_{-\varepsilon t}x(u)}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt$$

is the extremal operator.

Note that in the case $\omega_1(|\underline{t}|) = |\underline{t}|^{\beta_1}$, $\omega_2(|\overline{t}|) = |\overline{t}|^{\beta_2}$ $\beta_1, \beta_2 \in (0,1]$ applying Theorem 2, we immediately obtain the following statement.

Corollary 2. *Suppose that $\beta_1, \beta_2 \in (0,1]$ and $\alpha_1, \alpha_2 \in (0,1)$, satisfy the condition $\dfrac{\alpha_1}{\beta_1} + \dfrac{\alpha_2}{\beta_2} < 1$. Then for any $N > 0$,*

$$E_N\left(D^\alpha_\varepsilon, UH^{\beta_1,\beta_2}\right) =$$

$$= 2^{1 + 2\cdot\frac{1 - \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}}{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}}} \cdot A_\alpha^{\frac{1}{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}}} \cdot \frac{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}}{1 - \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}} \cdot \left(\frac{\sigma_k\sigma_{n-k}}{\alpha_1\alpha_2}\right)^{1 - \frac{1 - \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}}{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}}} \cdot N^{-\frac{1 - \frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}}{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}}}.$$

The inequalities for intermediate derivatives are also closely related to the Kolmogorov problem about necessary and sufficient conditions of the existence of a function, for which given numbers are the upper bounds of absolute values of its derivatives of corresponding order (see [17], [18]). For some known results in this direction see, for example, [26, 27, 13, 14] and [7].

We consider the Kolmogorov type problem in the following setting. It is required to find the necessary and sufficient conditions on the numbers $M_0, M_\alpha, M_{\omega_1}, M_{\omega_2}$ for existence of the function $x \in H^{k,\omega_1} \cap H^{n-k,\omega_2}$ such that

$$\|x\|_C = M_0,\ \ \|D^\alpha_\varepsilon x\|_C = M_\alpha,\ \ \|x\|_{H^{k,\omega_1}} = M_{\omega_1}, \|x\|_{H^{n-k,\omega_2}} = M_{\omega_2}.$$

Theorem 3. *Let moduli of continuity ω_1, ω_2 and numbers $\alpha_1, \alpha_2 \in (0,1)$, $j = 1, ..., m$, be such that (1.2) holds, and let numbers $M_0, M_\alpha, M_{\omega_1}, M_{\omega_2}$ be given. There exists a function $x \in H^{k,\omega_1} \cap H^{n-k,\omega_2}$ such that*

$$\|x\|_C = M_0,\ \ \|D^\alpha_\varepsilon x\|_C = M_\alpha,\ \ \|x\|_{H^{k,\omega_1}} = M_{\omega_1}, \|x\|_{H^{n-k,\omega_2}} = M_{\omega_2}.$$

if and only if the inequality

$$M_\alpha \le 2A_\alpha \int\limits_{\mathbb{R}^n_+} \frac{\min\{2M_0, M_{\omega_1}\omega_1(|\underline{t}|), M_{\omega_2}\omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt \qquad (1.9)$$

holds.

Corollary 3. *Suppose that $\beta_1, \beta_2 \in (0,1]$ and $\alpha_1, \alpha_2 \in (0,1)$ satisfy the condition $\dfrac{\alpha_1}{\beta_1} + \dfrac{\alpha_2}{\beta_2} < 1$, and let numbers $M_0, M_\alpha, M_{\beta_1}, M_{\beta_2}$ be given. There exists a function $x \in H^{k,\beta_1} \cap H^{n-k,\beta_2}$ such that*

$$\|x\|_C = M_0,\ \ \|D^\alpha_\varepsilon x\|_C = M_\alpha,\ \ \|x\|_{k,\beta_1} = M_{\beta_1},\ \|x\|_{n-k,\beta_2} = M_{\beta_2},$$

if and only if the inequality

$$M_\alpha \le \frac{\sigma_k \sigma_{n-k}}{2^n} A_\alpha \cdot \frac{2^{2-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}}}{\alpha_1\alpha_2\left(1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}\right)} \cdot M_0^{1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}} \cdot M_{\beta_1}^{\frac{\alpha_1}{\beta_1}} M_{\beta_2}^{\frac{\alpha_2}{\beta_2}}$$

holds.

2. Proofs

Proof of Theorem 1. We prove the theorem in the case $\varepsilon = (-, ..., -)$ only, since for any other ε one can use analogous arguments. Taking into account the definition of the fractional derivative we have

$$|D_\varepsilon^\alpha x(u)| \le A_\alpha \int\limits_{\mathbb{R}_+^n} \frac{\|\Delta_t^{k,n-k} x(\cdot)\|_C}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt, \quad \forall u \in \mathbb{R}^n. \tag{2.1}$$

To estimate the norm $\|\Delta_t^{k,n-k}x\|_C$ we will use the following inequalities

$$\|\Delta_t^{k,n-k}x\|_C \le 4\|x\|_C,$$

$$\|\Delta_t^{k,n-k}x\|_C \le 2\omega_1(|\underline{t}|)\|x\|_{H^{k,\omega_1}},$$

$$\|\Delta_t^{k,n-k}x\|_C \le 2\omega_2(|\overline{t}|)\|x\|_{H^{n-k,\omega_2}}.$$

Combining these estimates we obtain

$$\|\Delta_t^{k,n-k}x\|_C \le 2\min\{2\|x\|_C, \|x\|_{H^{k,\omega_1}}\omega_1(|\underline{t}|), \|x\|_{H^{n-k,\omega_2}}\omega_2(|\overline{t}|)\}.$$

Applying the last estimate to the right-hand side of (2.1) we obtain that $\forall u \in \mathbb{R}^m$

$$|D_\varepsilon^\alpha x(u)| \le 2A_\alpha \int\limits_{\mathbb{R}_+^n} \frac{\min\{2\|x\|_C, \|x\|_{H^{k,\omega_1}}\omega_1(|\underline{t}|), \|x\|_{H^{n-k,\omega_2}}\omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt. \tag{2.2}$$

Let us show that for every function $x \in H^{k,\omega_1} \cap H^{n-k,\omega_2}$, its fractional derivative $D_\varepsilon^\alpha x(u)$ depends on u continuously.

For $\xi > 0$ let

$$\omega(x,\xi) := \sup_{\substack{|t|<\xi \\ t\in\mathbb{R}^n}} \|x(\cdot) - x(\cdot + t)\|_C.$$

Applying the inequality (2.2) to the difference $x(u) - x(u+\delta)$, $\delta \in \mathbb{R}^n$, we obtain

$$|D_\varepsilon^\alpha x(u) - D_\varepsilon^\alpha x(u+\delta)|$$

$$\le 2A_\alpha \int\limits_{\mathbb{R}_+^n} \frac{\min\{2\omega(x,|\delta|), 2\|x\|_{H^{k,\omega_1}}\omega_1(|\underline{t}|), 2\|x\|_{H^{n-k,\omega_2}}\omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt.$$

Note that the function

$$\frac{\min\{2\omega(x,|\delta|), 2\|x\|_{H^{k,\omega_1}}\omega_1(|\underline{t}|), 2\|x\|_{H^{n-k,\omega_2}}\omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}$$

uniformly converges to zero (as $|\delta| \to 0$) on any set of points $(t_1, ..., t_n) \in \prod\limits_{j=1}^n [\sigma_j, \infty)$, $\sigma_j > 0$, $j = 1, ..., n$, and the integral

$$\int\limits_{\mathbb{R}_+^n} \frac{\min\{2\omega(x,|\delta|), 2\|x\|_{H^{k,\omega_1}}\omega_1(|\underline{t}|), 2\|x\|_{H^{n-k,\omega_2}}\omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt$$

uniformly converges on any bounded set of values of the parameter δ.

Therefore

$$|D_\varepsilon^\alpha x(u) - D_\varepsilon^\alpha x(u+\delta)| \to 0, \quad |\delta| \to 0,$$

which proves the continuity of $D_\varepsilon^\alpha x(u)$ for all $u \in \mathbb{R}^n$.

Thus, we obtain from (2.2):

$$\|D_\varepsilon^\alpha x\|_C \le 2A_\alpha \int\limits_{\mathbb{R}_+^n} \frac{\min\{2\|x\|_C, \|x\|_{H^{k,\omega_1}}\omega_1(|\underline{t}|), \|x\|_{H^{n-k,\omega_2}}\omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt,$$

and inequality (1.3) is proved.

Construct now a function , which turns (1.3) into equality. To this end, we will use the methods from [8]. For $\delta > 0$ and $u \in \mathbb{R}_+^n$ define $f_\delta(u)$ for $u \in \mathbb{R}_+^n$ as follows:

$$\begin{aligned}
&f_\delta(u) = \frac{\delta}{2},\ if\ \omega_1(|\underline{u}|) \ge \delta,\ \omega_2(|\overline{u}|) \ge \delta,\\
&f_\delta(u) = \omega_1(|\underline{u}|) - \omega_2(|\overline{u}|) + \frac{\delta}{2},\ if\ \omega_1(|\underline{u}|) \le \omega_2(|\overline{u}|) \le \delta,\\
&f_\delta(u) = \omega_2(|\overline{u}|) - \omega_1(|\underline{u}|) + \frac{\delta}{2},\ if\ \omega_2(|\overline{u}|) \le \omega_1(|\underline{u}|) \le \delta,\\
&f_\delta(u) = \omega_1(|\underline{u}|) - \frac{\delta}{2},\ if\ \omega_1(|\underline{u}|) \le \delta \le \omega_2(|\overline{u}|),\\
&f_\delta(u) = \omega_2(|\overline{u}|) - \frac{\delta}{2},\ if\ \omega_2(|\overline{u}|) \le \delta \le \omega_1(|\underline{u}|),
\end{aligned}$$

and then extend it to the whole space $\mathbb{R}^n$ evenly with respect to each variable

It is easy to see that

$$\|f_\delta\|_C = \frac{\delta}{2},\ \ \|f_\delta\|_{H^{k,\omega_1}} = 1,\ \ \|f_\delta\|_{H^{n-k,\omega_2}} = 1.$$

In addition for $t_1, t_2, ..., t_n > 0$

$$\Delta_t^{k,n-k} f_\delta(0) = \begin{cases} 2\omega_1(|\underline{u}|),\ if\ \omega_1(|\underline{u}|) \le \min\{\omega_2(|\overline{u}|), \delta\},\\ 2\omega_2(|\overline{u}|),\ if\ \omega_2(|\overline{u}|) \le \min\{\omega_1(|\underline{u}|), \delta\},\\ 2\delta,\ if\ \omega_1(|\underline{u}|) \ge \delta, \omega_2(|\overline{u}|) \ge \delta. \end{cases}$$

Now for $\varepsilon = (-, ..., -)$ we have:

$$\|D_\varepsilon^\alpha f_\delta\|_C \ge |D_\varepsilon^\alpha f_\delta(0)| = 2A_\alpha \int\limits_{\mathbb{R}_+^n} \frac{\min\{\delta, \omega_1(|\underline{t}|), \omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt, \tag{2.3}$$

Combining (1.3) (for the function f) with (2.3), we see that

$$\|D_\varepsilon^\alpha f_\delta\|_C = 2A_\alpha \int\limits_{\mathbb{R}_+^n} \frac{\min\{2\|f\|_C, \|f\|_{H^{k,\omega_1}}\omega_1(|\underline{t}|), \|f\|_{H^{n-k,\omega_2}}\omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt, \tag{2.4}$$

i.e. relation (1.3) turns into equality.

The proof is complete.

Proof of Corollary 1. It follows from equality (1.3) that for any $\delta > 0$,

$$\Omega\left(\delta, UH^{\omega_1,\omega_2}\right) \leq 2A_\alpha \int\limits_{\mathbb{R}^n_+} \frac{\min\{2\delta, \omega_1(|\underline{t}|), \omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt.$$

For the function f, constructed in the proof of Theorem 1,

$$\|f\|_C \leq \delta, \quad f \in UH^{\omega_1,\omega_2}.$$

Using (2.4), we obtain

$$\Omega\left(\delta, UH^{\omega_1,\omega_2}\right) \geq \|D^\alpha_\varepsilon f_\delta\|_C = 2A_\alpha \int\limits_{\mathbb{R}^n_+} \frac{\min\{2\delta, \omega_1(|\underline{t}|), \omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt.$$

The corollary is proved.

Proof of Theorem 2. As in the proof of Theorem 1 suppose that $\varepsilon = (-, ..., -)$. Remind that for a given $N > 0$, the vector $h^N = (h_1^N, h_2^N) \in \mathbb{R}^2_+$ is defined by the following conditions:

$$\omega_1(h_1^N) = \omega_2(h_2^N), \qquad \frac{4A_\alpha}{\alpha_1\alpha_2}\sigma_k\sigma_{n-k}(h_1^N)^{-\alpha_1}(h_2^N)^{-\alpha_2} = N,$$

and

$$G(h^N) := \{u = (\underline{u}, \overline{u}) \in \mathbb{R}^n \ : \ |\underline{u}| \geq h_1^N, |\overline{u}| \geq h_2^N\}.$$

Define the operator B_{h^N} as follows

$$B_{h^N} x(u) = A_\alpha \int\limits_{G(h^N)} \frac{\Delta_t^{k,n-k} x(u)}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt.$$

Show that B_{h^N} is the bounded operator from $C(\mathbb{R}^n)$ to $C(\mathbb{R}^n)$, and moreover $\|B_{h^N}\| \leq N$. Indeed for all $x \in C(\mathbb{R}^n)$,

$$\|B_{h^N} x\|_C \leq 4A_\alpha \int\limits_{G(h^N)} \frac{\|x\|_C}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt =$$

$$= 4A_\alpha \|x\|_C \int\limits_{G(h^N)} \frac{dt}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}} =$$

$$= 4A_\alpha \|x\|_C \int\limits_{|\underline{t}| \geq h_1^N} \frac{d\underline{t}}{|\underline{t}|^{k+\alpha_1}} \int\limits_{|\overline{t}| \geq h_2^N} \frac{d\overline{t}}{|\overline{t}|^{n-k+\alpha_2}} =$$

$$= 4A_\alpha \|x\|_C \int\limits_{S^{k-1}} dx' \int\limits_{h_1^N}^{\infty} \frac{d\rho}{\rho^{1+\alpha_1}} \int\limits_{S^{n-k-1}} dx' \int\limits_{h_2^N}^{\infty} \frac{d\rho}{\rho^{1+\alpha_2}} =$$

$$= 4A_\alpha \|x\|_C \frac{\sigma_{k-1}\sigma_{n-k-1}}{\alpha_1\alpha_2} (h_1^N)^{-\alpha_1}(h_2^N)^{-\alpha_2} = N\|x\|_C.$$

For any $x \in H^{k,\omega_1} \cap H^{n-k,\omega_2}$, estimate the deviation $\|D_\varepsilon^\alpha x - B_{h^N} x\|_C$. We have

$$\|D_\varepsilon^\alpha x - B_{h^N} x\|_C \le \|A_\alpha \int\limits_{\mathbb{R}_+^n \setminus G(h^N)} \Delta_t^{k,n-k} x(u) |\underline{t}|^{k+\alpha_1} |\overline{t}|^{n-k+\alpha_2}\, dt\|_C \le$$

$$\le 2A_\alpha \int\limits_{\mathbb{R}_+^n \setminus G(h^N)} \frac{\min\{\omega_1(|\underline{t}|), \omega_2(|\underline{t}|)\}}{|\underline{t}|^{k+\alpha_1} |\overline{t}|^{n-k+\alpha_2}}\, dt.$$

We have obtained the estimation of the value $E_N\left(D_\varepsilon^\alpha, UH^{\omega_1,\omega_2}\right)$ from above.

Let us estimate this value from below. From (1.7) we have

$$E_N\left(D_\varepsilon^\alpha, UH^{\omega_1,\omega_2}\right) \ge \sup_{\delta>0} \left\{\Omega\left(\delta, UH^{\omega_1,\omega_2}\right) - N\delta\right\}. \tag{2.5}$$

Using Corollary 1 and condition (1.8) we obtain

$$E_N\left(D_\varepsilon^\alpha, UH^{\omega_1,\omega_2}\right) \ge \sup_{\delta>0} \left\{ 2A_\alpha \int\limits_{\mathbb{R}_+^n} \frac{\min\{2\delta, \omega_1(|\underline{t}|), \omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1} |\overline{t}|^{n-k+\alpha_2}}\, dt \right.$$

$$\left. -4A_\alpha \delta \int\limits_{G(h^N)} \frac{dt}{|\underline{t}|^{k+\alpha_1} |\overline{t}|^{n-k+\alpha_2}} \right\}. \tag{2.6}$$

Set

$$\delta_N = \omega_1(h_1^N) = \omega_2(h_2^N).$$

Note that for $t \in G(h^N)$,

$$\min\{2\delta, \omega_1(|\underline{t}|), \omega_2(|\overline{t}|)\} = 2\delta_N$$

and for $t \in \mathbb{R}_+^n \setminus G(h^N)$,

$$\min\{2\delta, \omega_1(|\underline{t}|), \omega_2(|\overline{t}|)\} = \min\{\omega_1(|\underline{t}|), \omega_2(|\overline{t}|)\}.$$

From (2.6) we derive

$$E_N\left(D_\varepsilon^\alpha, UH^{\omega_1,\omega_2}\right) \ge 2A_\alpha \int\limits_{\mathbb{R}_+^n} \frac{\min\{2\delta_N, \omega_1(|\underline{t}|), \omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1} |\overline{t}|^{n-k+\alpha_2}}\, dt -$$

$$-2A_\alpha \int\limits_{G(h^N)} \frac{\min\{2\delta_N, \omega_1(|\underline{t}|), \omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt =$$

$$= 2A_\alpha \int\limits_{\mathbb{R}^n_+\setminus G(h^N)} \frac{\min\{\omega_1(|\underline{t}|), \omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt. \qquad (2.7)$$

We have obtained the required estimation from below. Therefore we have proved Theorem 2.

Proof of Corollary 2. Let $\omega_1(|\underline{t}|) = |\underline{t}|^{\beta_1}$ and $\omega_2(|\overline{t}|) = |\overline{t}|^{\beta_2}$ in (2.7). To compute the integral

$$I = \int\limits_{\mathbb{R}^n_+\setminus G(h^N)} \frac{\min\{|\underline{t}|^{\beta_1}, |\overline{t}|^{\beta_2}\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt$$

let us devide $\mathbb{R}^n_+ \setminus G(h^N)$ into two parts:

$$\mathbb{R}^n_+ \setminus G(h^N) = D_1 \cup D_2,$$

where

$$D_1 = \{t \in \mathbb{R}^n_+ \setminus G(h^N) : \min\{|\underline{t}|^{\beta_1}, |\overline{t}|^{\beta_2}\} = |\underline{t}|^{\beta_1}\},$$

$$D_2 = \{t \in \mathbb{R}^n_+ \setminus G(h^N) : \min\{|\underline{t}|^{\beta_1}, |\overline{t}|^{\beta_2}\} = |\overline{t}|^{\beta_2}\}.$$

Note, that $\mathrm{mes}(D_1 \cap D_2) = 0$, so

$$I = I_1 + I_2,$$

where

$$I_j = \int\limits_{D_j} \frac{\min\{|\underline{t}|^{\beta_1}, |\overline{t}|^{\beta_2}\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\, dt, \quad j = 1, 2.$$

Let us compute integral I_1:

$$I_1 = \int\limits_{|\underline{t}|\le h_1^N} \frac{|\underline{t}|^{\beta_1}}{|\underline{t}|^{k+\alpha_1}}\, d\underline{t} \int\limits_{|\overline{t}|\ge|\underline{t}|^{\frac{\beta_1}{\beta_2}}} \frac{d\overline{t}}{|\overline{t}|^{n-k+\alpha_2}} =$$

$$= \sigma_{n-k} \int\limits_{|\underline{t}|\le h_1^N} |\underline{t}|^{\beta_1-k-\alpha_1}\, d\underline{t} \cdot \int\limits_{|\underline{t}|}^{\infty} \frac{d\rho}{\rho^{1+\alpha_2}} = \frac{\sigma_{n-k}}{\alpha_2} \int\limits_{|\underline{t}|\le h_1^N} |\underline{t}|^{\beta_1-k-\alpha_1-\frac{\alpha_2\beta_1}{\beta_2}}\, d\underline{t} =$$

$$= \frac{\sigma_{n-k}\sigma_k}{\alpha_2} \int\limits_0^{h_1^N} \rho^{\beta_1-\alpha_1-\frac{\alpha_2\beta_1}{\beta_2}-1}\, d\rho = \frac{\sigma_{n-k-1}\sigma_{k-1}}{\alpha_2\beta_1\left(1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}\right)} (h_1^N)^{\beta_1\left(1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}\right)}.$$

Analogously, we obtain

$$I_2 = \frac{\sigma_{n-k}\sigma_k}{\alpha_1\beta_2\left(1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}\right)}(h_2^N)^{\beta_2\left(1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}\right)}.$$

Using the conditions (1.8) we have

$$I = I_1 + I_2 = \frac{\sigma_{n-k}\sigma_k}{\left(1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}\right)}\frac{\left(\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}\right)}{\alpha_1\alpha_2}(h_1^N)^{\beta_1\left(1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}\right)}. \tag{2.8}$$

and

$$(h_1^N)^{\beta_1} = \left(\frac{N\alpha_1\alpha_2}{4A_\alpha\sigma_{n-k}\sigma_k}\right)^{-\frac{1}{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}}} \tag{2.9}$$

Combining (2.8, 2.9) and (2.7)we obtain

$$E_N\left(D_\varepsilon^\alpha, UH^{\beta_1,\beta_2}\right) =$$

$$= 2^{1+2\cdot\frac{1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}}{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}}} \cdot A_\alpha^{2-\frac{1}{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}}} \cdot \frac{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}}{1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}} \cdot \left(\frac{\sigma_{n-k-1}\sigma_{k-1}}{\alpha_1\alpha_2}\right)^{1-\frac{1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}}{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}}} \cdot N^{-\frac{1-\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}}{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}}}.$$

Corollary 2 is prooved.

Proof of Theorem 3. Let us consider the case $\varepsilon = (-, ..., -)$. For $\delta > 0$ and modulus of continuity ω_1, ω_2 by $f(\cdot;\delta;\omega_1,\omega_2)$ denote the function f constructed in the proof of Theorem 1. Suppose the inequality (1.9) holds true and select $0 < L_0 < M_0$ such that

$$M_\alpha = 2A_\alpha \int\limits_{\mathbb{R}_+^n} \frac{\min\{2L_0, M_{\omega_1}\omega_1(|\underline{t}|), M_{\omega_2}\omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\,dt.$$

For the function $f(\cdot;L_0;M_{\omega_1}\omega_1,M_{\omega_2}\omega_2)$, we have

$$\|f(\cdot;L_0;M_{\omega_1}\omega_1,M_{\omega_2}\omega_2)\|_C \leq L_0 \leq M_0.$$

In addition, it is easy to verify that

$$\|f(\cdot;L_0;M_{\omega_1}\omega_1,M_{\omega_2}\omega_2)\|_{H^{k,\omega_1}} = M_{\omega_1},$$

$$\|f(\cdot;L_0;M_{\omega_1}\omega_1,M_{\omega_2}\omega_2)\|_{H^{n-k,\omega_2}} = M_{\omega_2}.$$

As in the proof of Theorem 1, we obtain

$$\|D_\varepsilon^\alpha f(\cdot;L_0,M_{\omega_1}\omega_1,M_{\omega_2}\omega_2)\|_C = 2A_\alpha \int\limits_{\mathbb{R}_+^n} \frac{\min\{2L_0, M_{\omega_1}\omega_1(|\underline{t}|), M_{\omega_2}\omega_2(|\overline{t}|)\}}{|\underline{t}|^{k+\alpha_1}|\overline{t}|^{n-k+\alpha_2}}\,dt = M_\alpha.$$

Now construct the function

$$x(u) = f(u; L_0, M_{\omega_1}\omega_1, M_{\omega_2}\omega_2) + M_0 - \|f(\cdot; L_0, M_{\omega_1}\omega_1, M_{\omega_2}\omega_2)\|_C$$

It is obvious that $x \in UH^{\omega_1,\omega_2}$, and also

$$\|x\|_C = M_0, \quad \|D^\alpha_\varepsilon x\|_C = M_\alpha, \quad \|x\|_{H^{k,\omega_1}} = M_{\omega_1}, \quad \|x\|_{H^{n-k,\omega_2}} = M_{\omega_2}.$$

Theorem 3 is proved.

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