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ON SOME SMOOTH APPROXIMATIONS ON THICK PERIODIC MULTI-STRUCTURES

Досліджуються питання апроксимації L^2 - обмежених функцій, означених на густих сингулярних з'єднаннях. Показано, що довільну функцію з означеного класу можна наблизити в просторі L^2 гладкою функцією, за її значеннями на боковій поверхні швидко осцилюючого сингулярного з'єднання. Отримано оцінки таких наближень та досліджено їх асимптотичну поведінку.

Ключові слова: густе сингулярне з'єднання, апроксимаційні властивості, сингулярні міри, збіжність у змінних просторах.

Изучается проблема аппроксимации L^2 - ограниченных функций, заданных на густых сингулярных соединениях. Показано, что любая функция из указанного класса может быть аппроксимирована в норме пространства L^2 гладкой функцией, которая задана только на боковой поверхности быстро осциллирующего сингулярного соединения. Получены оценки таких приближений и изучено их асимптотическое поведение.

Ключевые слова: густое сингулярное соединение, аппроксимационные свойства, сингулярные меры, сходимость в переменных пространствах.

In this paper we study the approximation properties of measurable and square-integrable functions. In particular we show that any L^2 - bounded functions can be approximated in L^2 -norm by smooth functions defined on a highly oscillating boundary of thick multi-structures in R^n . We derive the norm estimates for the approximating functions and study their asymptotic behavior.

Key words: thick multi-structure, approximating properties, singular measures, convergence in variable spaces.

Introduction

The main object of our consideration in this paper is a class of functions defined on a domain $\Omega_\varepsilon \subset R^n$, whose boundary $\partial\Omega_\varepsilon$ contains the very highly oscillating part with respect to a small parameter ε , as $\varepsilon \rightarrow 0$.

We say that Ω_ε is a thick multi-structure in R^n , if Ω_ε consists of some fixed domain Ω^+ and a large number of cylinders with axes parallel to Ox_n and ε -periodically distributed along some manifold Σ on the boundary of Ω^+ (see Fig.1). This manifold is called the joint zone and the domain Ω^+ is called the junction's body. Here ε is a small positive parameter,

which characterizes the distance between the neighboring cylinders and their thickness. So, each attached cylinder has a small cross section of the size ε and its limiting dimension (as $\varepsilon \rightarrow 0$) is equal to 1. In view of this, such cylinders will be called thin domains.

Thick junctions have a special character of the connectedness: there are points in a thick junction which are at a short distance of order $O(\varepsilon)$, but the length of all curves, which connect these points in the junction, is order $O(1)$.

As a result, there are no extension operators $P: W^{1,p}(\Omega_\varepsilon) \rightarrow W_{loc}^{1,p}(R^n)$ that would be uniformly bounded in ε . At the same time thick multi-structures (or thick junctions) are prototypes of widely used engineering constructions as well as many other physical and biological systems with very distinct characteristic scales (microscopic radiators, ferrite-filled rod radiators, and others). As a rule, the computational calculation of the solutions of any problems on Ω_ε is very complicated due to singularities of the thick junctions. As a result, asymptotic analysis is one of the main approaches to study majority of problems in such domains.

Fig. 1. Thick multi-structure Ω_ε

In view of this we consider the approximation problem of L^2 -bounded functions by sequences of smooth functions defined on lateral side of thin cylinders and their bases which form a quickly oscillating counterpart of thick junctions. We derive the norm estimates for the approximating functions and study their asymptotic behaviour.

Notation and Preliminaries

Let $a > 0$, $d > 0$ and $c > 0$ be given positive constants. Let $B = (0, a)^{n-1}$ and C be bounded open smooth domain in R^{n-1} ($n \geq 2$). Moreover we assume that $C \subset\subset (0, 1)^{n-1}$, that is the set C belongs to the $n-1$ -dimensional cube $(0, 1)^{n-1}$ together with its boundary ∂C . Throughout this paper the parameter ε varies in a strictly decreasing sequence of

positive numbers which converges to zero, i.e. we can suppose that $\varepsilon = a/N$, where N is a large positive integer. When we write $\varepsilon > 0$, we consider only the elements of this sequence.

Let us denote the elements of R^n by $x = (x_1, x_2, \dots, x_n) = (x', x_n)$ and introduce the following sets: $\theta_\varepsilon = \{x' = (k_1, k_2, \dots, k_{n-1}) \in N^{n-1} : \varepsilon C + \varepsilon \cdot \subset \subset B\}$,

$$\begin{aligned}\Omega &= B \times (-d, c), \quad G_\varepsilon = \{(x', x_n) : x' \in \varepsilon C + \varepsilon \cdot, -d < x_n \leq 0\}, \\ \Sigma &= B \times \{0\}, \quad \Omega^+ = B \times (0, c), \quad \Omega^- = B \times (-d, 0), \\ \Gamma_0 &= B \times \{-d\}, \quad \Omega_\varepsilon^- = \Omega_\varepsilon \cap \Omega^-\end{aligned}\quad (1)$$

Then in view of our previous description, the set

$$\Omega_\varepsilon = (B \times (0, c)) \cup \left(\bigcup_{\cdot \in \theta_\varepsilon} (\varepsilon C + \varepsilon \cdot) \times (-d, 0] \right) = \Omega^+ \cup \left(\bigcup_{\cdot \in \theta_\varepsilon} G_\varepsilon \right),$$

is a thick multi-structure in R^n , which consists of the junction's body Ω^+ and a large number N^{n-1} of the thin cylinders G_ε with axis Ox_n and ε -periodically distributed on the basis Σ of Ω^+ (see Fig. 1 for 3-d example). Here, each cylinder G_ε is obtained with ε -homothety in the first $(n-1)$ variables. It is easy to see that $\Omega_\varepsilon = \Omega^+ \cup \left(\bigcup_{\cdot \in \theta_\varepsilon} G_\varepsilon \right)$.

Denote by Γ_ε the union of the lower bases

$$\Gamma_\varepsilon = \{(x', x_n) : x' \in \varepsilon \cdot C + \varepsilon \cdot, x_n = -d\}$$

of the thin cylinders G_ε when $\cdot \in \theta_\varepsilon$ (i.e. $\Gamma_\varepsilon = \Gamma_0 \cap \partial \Omega_\varepsilon$, and by S_ε the union of their boundaries along the axis Ox_n :

$$S_\varepsilon = \{(x', x_n) : x' \in \varepsilon \cdot \partial C + \varepsilon \cdot, -d < x_n < 0\}.$$

We make also use of the idea of classical smoothing. Let $K(x)$ be a positive compactly supported smooth function ($K \in C_0^\infty$) such that $\int_{R^n} K(x) dx = 1$ and $K(x) = K(-x)$ for all $x \in R^n$. For an arbitrary measurable function $\varphi \in L_{loc}^1(R^n)$ we define an ordinary smoothing operator by the equality

$$(\varphi)_h(x) = h^{-n} \int_{R^n} K\left(\frac{x-y}{h}\right) \varphi(y) dy = \int_{R^n} K(y) \varphi(x+hy) dy. \quad (2)$$

On smooth Γ_ε -approximation of $L^2(\Gamma_0)$ -functions

Let $u^0 \in L^2(\Gamma_0)$ be a given function such that $\|u^0\|_{L^2(\Gamma_0)} \leq \dots$ with some constant $\dots > 0$. Our aim is to construct a sequence of smooth functions $\{u_\varepsilon \in C^\infty(\Gamma_\varepsilon)\}_{\varepsilon > 0}$ satisfying the following conditions:

$$\|u_\varepsilon\|_{L^2(\Gamma_\varepsilon)} \leq \left(\sqrt{|C|}\right)^{-1}, \quad \forall \varepsilon > 0, \quad (3)$$

$$\tilde{u}_\varepsilon \xrightarrow{w} u^0 \text{ in } L^2(\Gamma_0) \text{ as } \varepsilon \rightarrow 0. \quad (4)$$

Here $|C|$ is the $(n-1)$ -dimensional Lebesgue measure of the set C , and $\tilde{u}_\varepsilon$ is the trivial extension by zero of a function $u_\varepsilon \in L^2(\Gamma_\varepsilon)$ to the set Γ_0 . It is clear that in this case $\tilde{u} \in L^2(\Gamma_0)$.

Remark 1. Note that for every $\varepsilon > 0$ the restriction $u^0|_{\Gamma_\varepsilon}$ is an element of $L^2(\Gamma_\varepsilon)$ whereas the functions u_ε in (3)–(4) must be elements of $C^\infty(\Gamma_\varepsilon)$. In view of this we call the sequence $\{u_\varepsilon \in C^\infty(\Gamma_\varepsilon)\}_{\varepsilon > 0}$ with properties (3)–(4) a smooth Γ_ε -approximation of $u^0 \in L^2(\Gamma_0)$.

In order to construct the sequence $\{u_\varepsilon \in C^\infty(\Gamma_\varepsilon)\}_{\varepsilon > 0}$, we make use the idea of classical smoothing. Let $K_u \in C_0^\infty(R^{n-1})$ be nonnegative function such that

$$\int_{R^{n-1}} K_u(x') dx' = 1 \text{ and } K_u(x') = K_u(-x') \quad \forall x' \in R^{n-1}.$$

Let $\tilde{u}^0$ be the trivial extension by zero of function $u^0 \in L^2(\Gamma_0)$ to the set $O = \{(x', x_n) \in R^n : x' \in R^{n-1}, x_n = -\beta_0\}$. For every fixed $\varepsilon > 0$ we define the following smoothing operator

$$\begin{aligned} (u^0)_\varepsilon &= (K_u^\varepsilon u^0)(x') = \varepsilon^{-n+1} \int_O K_u\left(\frac{x' - y'}{\varepsilon}\right) \tilde{u}^0(y', y_n) dH^{n-1} = \\ &= \varepsilon^{-n+1} \int_{R^{n-1}} K_u\left(\frac{x' - y'}{\varepsilon}\right) \tilde{u}^0(y') dy'. \end{aligned} \quad (5)$$

We begin with the following result:

Proposition 1. Let $u^0 \in L^2(\Gamma_0)$ be a given function such that $\|u^0\|_{L^2(\Gamma_0)} \leq \dots$. Then for every $\delta > 0$ there exists $\varepsilon(\delta) = a/N$, where $N \in \dots$, such that

$$\|u_\varepsilon\|_{L^2(\Gamma_\varepsilon)} \leq \left(\sqrt{|C|}\right)^{-1}. \quad (6)$$

provided

$$u_\varepsilon := |C|^{-1} (u^0)_\delta|_{\Gamma_\varepsilon}. \quad (7)$$

Proof. First of all we note that $u_\varepsilon \in C^\infty(\Gamma_\varepsilon)$ by the properties of smoothing operator (5). Following the definition of the smoothing operator and using the Cauchy-Bunyakovskii inequality, we have

$$\begin{aligned}
 \|(u^0)_\delta\|_{L^2(R^{n-1})} &= \int_{R^{n-1}} \left(\delta^{-n+1} \int_{R^{n-1}} K_u \left(\frac{x' - y'}{\delta} \right) \tilde{u}^0(y') dy' \right)^2 dx' \leq \\
 &\leq \int_{R^{n-1}} \left(\delta^{-n+1} \int_{R^{n-1}} K_u \left(\frac{x' - y'}{\delta} \right) dy' \right) \times \\
 &\times \left(\delta^{-n+1} \int_{R^{n-1}} K_u \left(\frac{x' - y'}{\delta} \right) (\tilde{u}^0(y'))^2 dy' \right) dx' = \\
 &= \int_{R^{n-1}} \left(\delta^{-n+1} \int_{R^{n-1}} K_u \left(\frac{x' - y'}{\delta} \right) (\tilde{u}^0(y'))^2 dy' \right) dx' \\
 &= \int_{R^{n-1}} \left(\delta^{-n+1} \int_{R^{n-1}} K_u \left(\frac{x' - y'}{\delta} \right) dx' \right) (\tilde{u}^0(y'))^2 dy' = \\
 &= \int_{R^{n-1}} (\tilde{u}^0(y'))^2 dy' = \int_{\Gamma_0} (u^0(y'))^2 dy' = \|u^0\|_{L^2(\Gamma_0)}^2.
 \end{aligned} \tag{8}$$

Due to this estimate, we come to the inequality

$$\begin{aligned}
 \|(u^0)_\delta\|_{L^2(\Gamma_\varepsilon)} &= \|\chi_{\Gamma_\varepsilon}(u^0)_\delta\|_{L^2(\Gamma_\varepsilon)} \\
 &\leq \|\chi_{\Gamma_0}(u^0)_\delta\|_{L^2(R^{n-1})} \leq \|u^0\|_{L^2(\Gamma_0)} \leq
 \end{aligned} \tag{9}$$

which holds true for every $\varepsilon > 0$. Here $\chi_{\Gamma_\varepsilon} \in L^\infty(\Gamma_0)$ is the characteristic function of the zone Γ_ε , i.e.,

$$\chi_{\Gamma_\varepsilon}(x') = \begin{cases} 1, & x' \in \Gamma_\varepsilon, \\ 0, & \text{otherwise} \end{cases} \quad \forall x' \in \Gamma_0.$$

Note that

$$\chi_{\Gamma_\varepsilon} \xrightarrow{w} |C| \text{ in } L^\infty(\Gamma_0) \text{ as } \varepsilon \rightarrow 0. \tag{10}$$

Indeed, in view of $\varepsilon[0,1)^{n-1}$ -periodicity of this characteristic function and the mean value property, we have

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_0} \varphi(x') \chi_{\Gamma_\varepsilon}(x') dH^{n-1} &= \lim_{\varepsilon \rightarrow 0} \int_B \varphi(x') \chi_{\cup_{\varepsilon \in \theta_\varepsilon} (\varepsilon C + \varepsilon)}(x') dx' = \\
 &= \int_B \varphi(x') dx' \left(\int_{[0,1)^{n-1}} \chi_C(x') dx' \right) = \\
 &= |C| \int_B \varphi(x') dx' = |C| \int_{\Gamma_0} \varphi(x') dH^{n-1} \quad \forall \varphi \in C_0^\infty(\Gamma_0).
 \end{aligned}$$

Thus, the property (10) holds true.

Further for a given $\delta > 0$ we define a parameter $\varepsilon = a/N$ so that the following inequality would be valid

$$\frac{1}{|\Gamma_\varepsilon|} \int_{\Gamma_\varepsilon} (u^0)_\delta^2 dx' \leq \frac{1}{|\Gamma_0|} \int_{\Gamma_0} (u^0)_\delta^2 dx'.$$

Then from (9) we deduce

$$\left\| |C|^{-1} (u^0)_\delta \right\|_{L^2(\Gamma_\varepsilon)}^2 \leq \frac{|\Gamma_\varepsilon|}{|\Gamma_0|} \left\| |C|^{-1} (u^0)_\delta \right\|_{L^2(\Gamma_0)}^2 \leq \frac{|\Gamma_\varepsilon|}{|\Gamma_0|} |C|^{-2}. \quad (11)$$

Since $|\Gamma_0| = |B| = a^{n-1}$ and

$$|\Gamma_\varepsilon| = \sum_{\varepsilon \in \theta_\varepsilon} \int_{\Gamma_\varepsilon} dx' = \sum_{k \in \theta_\varepsilon} \varepsilon^{n-1} |C| = \left(\frac{a}{\varepsilon} \right)^{n-1} \varepsilon^{n-1} |C| = a^{n-1} |C|,$$

it follows from (11) that

$$\|u_\varepsilon\|_{L^2(\Gamma_\varepsilon)} = \left\| |C|^{-1} (u^0)_\delta \right\|_{L^2(\Gamma_\varepsilon)} \leq \left(\sqrt{|C|} \right)^{-1}. \quad (12)$$

Thus for the chosen value of ε inequality (6) is valid. This concludes the proof.

The next step to prove the property of strong convergence (4).

Proposition 2. Assume the functions $u_\varepsilon \in C^\infty(\Gamma_\varepsilon)$ are defined by (7). Then the property (4) holds true.

Proof. To begin with, we note that

$$(u^0)_\delta \rightarrow u^0 \text{ strongly in } L^2(\Gamma_0) \text{ as } \delta \rightarrow 0 \quad (13)$$

by properties of the classical smoothing. Hence in view of (10), (13), and the fact that the value of ε depends on a given parameter δ , we have

$$\tilde{u}_\varepsilon(\delta) = |C|^{-1} \chi_{\Gamma_\varepsilon(\delta)} (u^0)_\delta \xrightarrow{w} |C|^{-1} |C| u^0 = u^0 \text{ as } \delta \rightarrow 0$$

as a limit of the product of the weak and strong convergent sequences in $L^2(\Gamma_0)$. This implies the required property (4). The proof is complete.

Some auxiliary results

We begin this section with the description of the $(n-1)$ -dimensional surface S_ε in the terms of singular measures in R^n which do not satisfy the regularity property with respect to the corresponding Lebesgue measure.

Let μ_0 be a periodic finite positive Borel measure in R^{n-1} . Let $\square = [0,1)^{n-1}$ be the cell or torus of periodicity for μ_0 . We assume that Borel measure μ_0 is the probability measure in R^{n-1} , concentrated and uniformly distributed on the set ∂C , so $\int_\square d\mu_0 = 1$.

Remark 2. By definition we have $\mu_0(\square \setminus \partial C) = 0$. Therefore any functions, taking the same values on the set ∂C , coincide as elements of $L^2(\square, d\mu_0)$. Here the Lebesgue space

$L^2(\square, d\mu_0)$ with respect to the measure μ_0 is defined in a usual way with the corresponding norm

$$\|f\|_{L^2(\square, d\mu_0)}^2 = \int_{\square} |f(x)|^2 d\mu_0$$

(we adopt the standard notation $L^2(\square)$ when μ_0 is the Lebesgue measure).

We set $\square_n = \square \times [0,1] = [0,1]^n$ and consider the following measure $d\mu = d\mu_0 \times dx_n$ in $\square_n$. It is easy to see that this measure concentrated on the set $\partial C \times [0,1]$, and for any smooth function g we have

$$\int_{\square_n} g d\mu = \int_0^1 \int_{\square} g dx_n d\mu_0 = \left[H^{n-1}(\partial C \times [0,1]) \right]^{-1} \int_{\partial C \times [0,1]} g dH^{n-1}.$$

However, as follows from the properties of the Hausdorff measure, we have $H^{n-1}(\partial C \times [0,1]) = H^{n-2}(\partial C)$ (see [1]). In what follows, we use the notation $|\partial C|_H = H^{n-2}(\partial C)$ for $(n-2)$ -dimensional Hausdorff measure of the set ∂C . Then the previous relation can be rewritten in the form

$$\int_{\square_n} g d\mu = \int_0^1 \int_{\square} g dx_n d\mu_0 = |\partial C|_H^{-1} \int_{\partial C \times (0,1)} g dH^{n-1}. \quad (14)$$

In order to clarify relation (14), we consider the planar thick multi-structure $\Omega_\varepsilon \subset R^2$. Then $n=2$ and the set C is some part of the segment $(0,1)$. For instance, let $C = \{x_1 \in (0,1) : |x_1 - 1/2| < h/2\}$, where $h \in (0,1)$ is a fixed value. In this case $|\partial C|_H = 2$ and the 1-periodic measure μ_0 in R^1 can be defined as

$$\mu_0(B) = \frac{1}{|\partial C|_H} (\partial_{M_1} + \partial_{M_2})(B) = \frac{1}{2} (\partial_{M_1} + \partial_{M_2})(B)$$

for any Borel set $B \subset [0,1]$, where $M_i = \frac{1}{2} + \left(i - \frac{3}{2}\right)h$, $i=1,2$. Here δ_{M_i} is the Dirac measures

located at the points M_i . Thus, the multiplier $|\partial C|_H^{-1}$ in (14) is equal to $1/2$.

Let Λ be any Borel set of R^n . We introduce the scaling measure μ_ε by the rule $\mu_\varepsilon(\Lambda) = \varepsilon \mu(\varepsilon^{-1}\Lambda)$. This measure has a period ε , and moreover, since $\mu_\varepsilon(\varepsilon \square_n) = \varepsilon \mu_0(\varepsilon \square)$, it follows that

$$\mu_\varepsilon(\varepsilon \square_n) = \varepsilon^n \int_0^\varepsilon \int_{\varepsilon \square} d\mu_0(x'/\varepsilon) d(x_n/\varepsilon) = \varepsilon^n \int_0^1 \int_{\square} d\mu_0 dx_n = \varepsilon^n.$$

As a result, the measure μ_ε weakly converges to Lebesgue measure in R^n as $\varepsilon \rightarrow 0$ (in symbols $d\mu_\varepsilon \xrightarrow{w} dx$), that is,

$$\lim_{\varepsilon \rightarrow 0} \int_{R^n} \varphi d\mu_\varepsilon = \int_{R^n} \varphi dx \text{ for all functions } \varphi \in C_0^\infty(R^n).$$

Now, we give the convergence formalism of the sequence of boundary controls $\{\hat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)\}_{\varepsilon>0}$. First of all we recall that the sequence of Borel measures $\{d\mu_\varepsilon\}$ weakly converges to Lebesgue measure dx . Let $\{\hat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)\}_{\varepsilon>0}$ be any bounded sequence, i.e.

$$\limsup_{\varepsilon \rightarrow 0} \int_{\Omega^-} \hat{p}_\varepsilon^2 d\mu_\varepsilon < +\infty.$$

Definition 1. [3]. We say that a bounded sequence $\{\hat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)\}_{\varepsilon>0}$ is weakly convergent if there exists an element $p^* \in L^2(\Omega^-)$ such that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega^-} \varphi \hat{p}_\varepsilon d\mu_\varepsilon = \int_{\Omega^-} \varphi p^* dx \quad \text{for any function } \varphi \in C_0^\infty(R^n)$$

(in symbols we will use the following notation $\hat{p}_\varepsilon \xrightarrow{w} p^*$ in $L^2(\Omega^-, d\mu_\varepsilon)$).

Further we make use the observation

$$\begin{aligned} \varepsilon \int_{S_\varepsilon} p_\varepsilon \varphi dH^{n-1} &= \varepsilon \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\partial C^+_{\cdot j})} \int_{-\beta(x')}^0 p_\varepsilon \varphi dH^{n-2} dx_n \\ &= \varepsilon |\partial C|_H \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square^+_{\cdot j})} \int_{-\beta(x')}^0 \hat{p}_\varepsilon \varphi \varepsilon^{n-2} d\mu_0(x'/\varepsilon) dx_n \\ &= |\partial C|_H \sum_{j=1}^{N^{n-1}} \int_{\varepsilon(\square^+_{\cdot j})} \int_{-\beta(x')}^0 \hat{p}_\varepsilon \varphi \varepsilon^n d\mu_0(x'/\varepsilon) d(x_n/\varepsilon) \\ &= |\partial C|_H \sum_{j=1}^{N^{n-1}} \int_{\left\{ (x', x_n) : \begin{array}{l} x' \in \varepsilon(\square^+_{\cdot j}) \\ -\beta(x') < x_n < 0 \end{array} \right\}} \hat{p}_\varepsilon \varphi d\mu_\varepsilon \\ &= |\partial C|_H \int_{\Omega^-} \hat{p}_\varepsilon \varphi d\mu_\varepsilon \quad \forall \varphi \in C_0^\infty(R^n). \end{aligned} \quad (15)$$

Here a function $\hat{p}_\varepsilon$ is defined as follows: $\hat{p}_\varepsilon = p_\varepsilon$ on $S_\varepsilon \equiv \text{supp } p(\mu_\varepsilon)$, and $\hat{p}_\varepsilon = 0$ in $\Omega^- \setminus S_\varepsilon$. Hence, $\hat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)$. It is clear that any functions, taking the same values on the set S_ε , coincide as elements of $L^2(\Omega^-, d\mu_\varepsilon)$. In view of this, any function $\hat{p}_\varepsilon \in L^2(\Omega^-, d\mu_\varepsilon)$ with the property

$$|\partial C|_H \int_{\Omega^-} \hat{p}_\varepsilon \varphi d\mu_\varepsilon = \varepsilon \int_{S_\varepsilon} p_\varepsilon \varphi dH^{n-1} \quad \forall \varphi \in C_0^\infty(R^n)$$

will be called a prototype of the function $p_\varepsilon \in L^2(S_\varepsilon)$. We note also that the integral $\int_{\Omega^-} \hat{p}_\varepsilon \varphi d\mu_\varepsilon$ is well defined for every function $\varphi \in C_0^\infty(R^n; \Gamma_\varepsilon)$.

Indeed, since the set Ω^- is bounded, and $\hat{p}_\varepsilon d\mu_\varepsilon$ is a Radon measure, it follows that $\int_{\Omega^-} \hat{p}_\varepsilon \varphi d\mu_\varepsilon$ is a linear continuous functional on $C_0^\infty(R^n; \Gamma_\varepsilon)$.

In a similar manner to (15), we can derive the following relation

$$\begin{aligned} \varepsilon \|p_\varepsilon\|_{L^2(S_\varepsilon)}^2 &= \varepsilon \int_{S_\varepsilon} p_\varepsilon^2 dH^{n-1} \\ &= |\partial C|_H \int_{\Omega^-} \hat{p}_\varepsilon^2 d\mu_\varepsilon = |\partial C|_H \|\hat{p}_\varepsilon\|_{L^2(\Omega^-, d\mu_\varepsilon)}^2. \end{aligned} \quad (16)$$

On smooth S_ε -approximation of $L^2(\Omega^-)$ -functions

Let $p^0 \in L^2(\Omega^-)$ be a given function such that $\|p^0\|_{L^2(\Omega^-)}^2 \leq \dots$ with some constant $\dots > 0$. Our aim is to construct a sequence of smooth functions $\{p_\varepsilon \in C^\infty(S_\varepsilon)\}_{\varepsilon>0}$ satisfying the following conditions:

$$\|p_\varepsilon\|_{L^2(S_\varepsilon)}^2 \leq \sqrt{\frac{|\partial C|_H}{\varepsilon}}, \quad \forall \varepsilon > 0, \quad (17)$$

$$\tilde{p}_\varepsilon \xrightarrow{w} p^0 \text{ in } L^2(\Omega^-, d\mu_\varepsilon) \text{ as } \varepsilon \rightarrow 0. \quad (18)$$

Remark 3. Note that we can not use the restriction of this function to the construction of the sequence $\{p_\varepsilon\}_{\varepsilon>0}$ because a trace of an L^2 -function on the $(n-1)$ -dimensional surface S_ε is not defined. In view of this we call the sequence $\{p_\varepsilon \in C^\infty(\Gamma_\varepsilon)\}_{\varepsilon>0}$ with properties (17)–(18) a smooth S_ε -approximation of the function $p^0 \in L^2(\Gamma_0)$.

Let $K_p \in C_0^\infty(R^n)$ be nonnegative function such that

$$\int_{R^n} K_p(x) dx = 1, \text{ and } K_p(x) = K_p(-x) \quad \forall x \in R^n.$$

Let $\tilde{p}^0$ be the trivial extension by zero of the function $p^0 \in L^2(\Omega^-)$ to R^n . For every fixed $\varepsilon > 0$ we define the following smoothing operator

$$(p^0)_\varepsilon = (K_p^\varepsilon p^0)(x) = \varepsilon^{-n} \int_{R^n} K_p\left(\frac{x-y}{\varepsilon}\right) \tilde{p}^0(y) dy.$$

We begin with the following result:

Proposition 3. Let $p^0 \in L^2(\Omega^-)$ be a given function such that $\|p^0\|_{L^2(\Omega^-)}^2 \leq \dots$. Then there exists $\varepsilon_0 > 0$ such that the functions

$$p_\varepsilon = (p^0)_\varepsilon|_{S_\varepsilon} \quad (19)$$

satisfy the estimate (17) for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. Since $(p^0)_\varepsilon \in C(\overline{\Omega^-})$ it is clear that $p_\varepsilon^{sub} = (p^0)_\varepsilon|_{S_\varepsilon} \in L^2(S_\varepsilon, \cdot^{n-1})$. By analogy with (8) it is easy to show the validity of the following inequality

$$\|p^0\|_{L^2(\Omega^-)}^2 \leq \int_{\Omega^-} (p^0)^2 dx = \|p^0\|_{L^2(\Omega^-)}^2. \quad (20)$$

Thus, taking into account (20), we have to prove that the estimate (17) is valid at least for ε small enough. In order to verify this fact we will show that the value

$$\sigma = \left| \varepsilon |\partial C|_H^{-1} \|(p^0)_\varepsilon\|_{L^2(S_\varepsilon, \cdot^{n-1})}^2 - \|(p^0)_\varepsilon\|_{L^2(\Omega^-)}^2 \right|$$

can be done as small as is wished for ε small enough.

To begin with, we note that

$$\|(p^0)_\varepsilon\|_{L^2(\Omega^-)}^2 = \sum_{j=1}^{N^{n-1}} \int_{\left\{x=(x', x_n) : \begin{array}{l} x' \in \varepsilon(\square + j) \\ -\beta_0 < x_n < 0 \end{array} \right\}} (p^0)_\varepsilon^2 dx. \quad (21)$$

Here $\square = [0, 1)^{n-1}$. Further we make use the following notation:

$$L_\varepsilon = \{(x', x_n) : x' \in \varepsilon \square + \varepsilon, -\beta < x_n \leq 0\},$$

$$G_\varepsilon = \{(x', x_n) : x' \in \varepsilon C + \varepsilon, -\beta < x_n \leq 0\},$$

$$\Delta_i S_\varepsilon = \{(x', x_n) : x' \in \varepsilon \partial C + \varepsilon, -\beta + (i-1)\Delta < x_n \leq -\beta_0 + i\Delta\},$$

where $\Delta = \beta_0/J$.

Since $(p^0)_\varepsilon \in C_0^\infty(R^n)$ it means that a wild oscillation of this function is excluded on the cross sections of thin cylinders L_ε for ε small enough. Hence for a given $\eta > 0$ there exist an $\varepsilon_\eta > 0$, an integer $J \in N$, and a collection of points $\{x_{\varepsilon, i} \in \Delta_i S_\varepsilon\}$ such that

$$\left| \int_{L_\varepsilon} (p^0)_\varepsilon^2 dx - \sum_{i=1}^J (p^0)_\varepsilon^2(x_{\varepsilon, i}) L^n(\varepsilon \square) \Delta \right| < \frac{\eta}{2N^{n-1}} \quad \forall \varepsilon \in (0, \varepsilon_\eta).$$

Taking into account the chain of transformations

$$\begin{aligned} & \sum_{i=1}^J (p^0)_\varepsilon^2(x_{\varepsilon, i}) L^n(\varepsilon \square) \Delta \\ &= \sum_{i=1}^J (p^0)_\varepsilon^2(x_{\varepsilon, i}) \varepsilon^n \Delta = \sum_{i=1}^J (p^0)_\varepsilon^2(x_{\varepsilon, i}) \varepsilon^n \frac{H^{n-1}(\Delta_i S_\varepsilon)}{H^{n-2}(\varepsilon \partial C)} \end{aligned}$$

$$= \frac{\varepsilon}{H^{n-2}(\partial C)} \sum_{i=1}^J (p^0)_\varepsilon^2(x_{\varepsilon,i}) H^{n-1}(\Delta_i S_\varepsilon) \\ = \varepsilon |\partial C|_H^{-1} \sum_{i=1}^J (p^0)_\varepsilon^2(x_{\varepsilon,i}) H^{n-1}(\Delta_i S_\varepsilon)$$

and the fact that

$$\left| \sum_{i=1}^J (p^0)_\varepsilon^2(x_{\varepsilon,i}) H^{n-1}(\Delta_i S_\varepsilon) - \int_{S_\varepsilon} (p^0)_\varepsilon^2 dH^{n-1} \right| < \frac{\eta |\partial C|_H}{2\varepsilon N^{n-1}}$$

for J large enough, we deduce: for a given $\eta > 0$ there exists an $\varepsilon_\eta > 0$ such that

$$\left| \int_{L_\varepsilon} (p^0)_\varepsilon^2 dx - \varepsilon |\partial C|_H^{-1} \int_{S_\varepsilon} (p^0)_\varepsilon^2 dH^{n-1} \right| \\ \leq \left| \int_{L_\varepsilon} (p^0)_\varepsilon^2 dx - \varepsilon |\partial C|_H^{-1} \sum_{i=1}^J (p^0)_\varepsilon^2(x_{\varepsilon,i}) H^{n-1}(\Delta_i S_\varepsilon) \right| \\ + \left| \varepsilon |\partial C|_H^{-1} \sum_{i=1}^J (p^0)_\varepsilon^2(x_{\varepsilon,i}) H^{n-1}(\Delta_i S_\varepsilon) - \int_{S_\varepsilon} (p^0)_\varepsilon^2 dH^{n-1} \right| < \frac{\eta}{N^{n-1}} \quad (22)$$

for all $\varepsilon \in (0, \varepsilon_\eta)$.

As a result, combining this estimate with the representation (21), we come to the conclusion:

$$\left| \left\| (p^0)_\varepsilon \right\|_{L^2(\Omega^-)}^2 - \varepsilon |\partial C|_H^{-1} \left\| (p^0)_\varepsilon \right\|_{L^2(S_\varepsilon; H^{n-1})}^2 \right| \\ = \left| \sum_{j=1}^{N^{n-1}} \left(\int_{L_\varepsilon^j} (p^0)_\varepsilon^2 dx - \varepsilon |\partial C|_H^{-1} \int_{S_\varepsilon^j} (p^0)_\varepsilon^2 dH^{n-1} \right) \right| < \eta. \quad (23)$$

Since relation (20) turns into the equality only in the case when p^0 is a constant, it follows that choosing an appropriate value $\eta > 0$ we can find an $\varepsilon_\eta > 0$ such that for every $\varepsilon \in (0, \varepsilon_\eta)$ the inequality

$$\varepsilon |\partial C|_H^{-1} \left\| (p^0)_\varepsilon \right\|_{L^2(S_\varepsilon; H^{n-1})}^2 \leq \left\| p^0 \right\|_{L^2(\Omega^-)}^2 \leq \eta^2$$

holds true. Hence the estimate (17) is valid for $\varepsilon_\eta > 0$ small enough. $\square$

It remains to substantiate the convergence property (18).

Proposition 4. Assume the functions $p_\varepsilon \in C^\infty(S_\varepsilon)$ are defined by (19). Then the property (18) holds true.

Proof. Let $\{\varsigma_\varepsilon\}$ be any sequence of smooth functions such that $\varsigma_\varepsilon \rightarrow p$ strongly in $L^2(\Omega^-)$ as $\varepsilon \rightarrow 0$. Let us show that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega^-} \varsigma_\varepsilon \varphi d\mu_\varepsilon = \int_{\Omega^-} p \varphi dx \quad \forall \varphi \in C_0^\infty(R^n). \quad (24)$$

Indeed, let us partition Ω^- into cubes with edges ε and denote these cubes by the symbols $\varepsilon \square_n^j$. Then

$$\begin{aligned} \int_{\Omega^-} \varsigma_\varepsilon \varphi d\mu_\varepsilon &= \sum_j \int_{\varepsilon \square_n^j} \varsigma_\varepsilon(x) \varphi(x) d\mu_\varepsilon \\ &+ \sum_j \int_{\Omega^- \cap \varepsilon \square_n^j} \varsigma_\varepsilon(x) \varphi(x) d\mu_\varepsilon = \sum_j \varsigma_\varepsilon(x_j) \varphi(x_j) \int_{\varepsilon \square_n^j} d\mu_\varepsilon \\ &+ \sum_j \int_{\Omega^- \cap \varepsilon \square_n^j} \varsigma_\varepsilon(x) \varphi(x) d\mu_\varepsilon, \end{aligned}$$

where x_j is a Lebesgue point of φ in the cube $\varepsilon \square_n^j$ and the second sum is calculated over the set of boundary cubes. By definition of the measure μ_ε we have $\int_{\varepsilon \square_n^j} d\mu_\varepsilon = \varepsilon^n \int_{\square_n} d\mu = \varepsilon^n$.

Moreover, $\left| \sum_j \int_{\Omega^- \cap \varepsilon \square_n^j} \varsigma_\varepsilon(x) \varphi(x) d\mu_\varepsilon \right| \leq \sup_{x \in \Omega^-} |\varsigma_\varepsilon(x) \varphi(x)| \varepsilon^n \cdot D(\varepsilon)$, where $D(\varepsilon)$ is a quantity

of boundary cubes, and $\varepsilon^n D(\varepsilon) \rightarrow 0$ by Jordan's measurability of the set $\partial\Omega^-$. Thus, summarizing the above cited facts, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega^-} \varsigma_\varepsilon \varphi d\mu_\varepsilon &= \lim_{\varepsilon \rightarrow 0} \left[\sum_j \varsigma_\varepsilon(x_j) \varphi(x_j) \varepsilon^n - \int_{\Omega^-} \varsigma_\varepsilon \varphi dx \right] + \\ &+ \lim_{\varepsilon \rightarrow 0} \int_{\Omega^-} \varsigma_\varepsilon \varphi dx = \lim_{\varepsilon \rightarrow 0} \left[\sum_j \varsigma_\varepsilon(x_j) \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon \square_n^j} \varsigma_\varepsilon \varphi dx \right] \varepsilon^n + \\ &+ \lim_{\varepsilon \rightarrow 0} \sum_j \int_{\Omega^- \cap \varepsilon \square_n^j} \varsigma_\varepsilon(x) \varphi(x) dx + \int_{\Omega^-} p \varphi dx = \\ &= \lim_{\varepsilon \rightarrow 0} \sum_j \left[\varsigma_\varepsilon(x_j) \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon \square_n^j} \varsigma_\varepsilon \varphi dx \right] \varepsilon^n + \int_{\Omega^-} p \varphi dx. \end{aligned}$$

Let us suppose that $\lim_{\varepsilon \rightarrow 0} \sum_j \left| \varsigma_\varepsilon(x_j) \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon \square_n^j} \varsigma_\varepsilon \varphi dx \right| \varepsilon^n > 0$.

Then there exist a constant $C^* > 0$ and a value $\varepsilon^* > 0$ such that

$$\left| \varsigma_\varepsilon(x_j) \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon \square_n^j} \varsigma_\varepsilon \varphi dx \right| \geq C^* \quad \forall \varepsilon \in (0, \varepsilon^*)$$

(possibly except, for a finite number of indices j for every fixed ε). Hence extremely wild oscillations are present in the sequence $\{\varsigma_\varepsilon \varphi\}$. However (see [1]; [2]), if we have very rapid

fluctuations in the functions $\{\varsigma_\varepsilon \varphi\}$, then the convergence $\varsigma_\varepsilon \varphi \rightarrow p\varphi$ almost everywhere in Ω^- is excluded. This fact immediately reflects the failure of the strong convergence in $L^2(\Omega^-)$ as $\varepsilon \rightarrow 0$ (see Valadier's Theorem [2]). So, our supposition was wrong and we get

$$\lim_{\varepsilon \rightarrow 0} \sum_j \left| \varsigma_\varepsilon(x_j) \varphi(x_j) - \varepsilon^{-n} \int_{\varepsilon \square_j^n} \varsigma_\varepsilon \varphi dx \right| \varepsilon^n = 0.$$

As a result, we come to the relation (24).

To conclude the proof it remains to use the main property of classical smoothing

$$(p^0)_\varepsilon \rightarrow p^0 \text{ strongly in } L^2(\Omega^-) \text{ as } \varepsilon \rightarrow 0$$

and apply the relation (24). $\square$

As the direct consequence of Proposition 3, which we feel to be interesting per se, we present one observation concerning L^2 -functions defined on thick junctions.

Corollary 1. *Let ε be a thick multi-structure in R^n , which consists of some domain Ω^+ and a large number of thin cylinders G_ε with a small cross section of the size εC and ε -periodically distributed along some manifold Σ on the boundary of Ω^+ (see Fig. 1 for 3-d example). Let $\Omega^- = \Sigma \times (a, b)$ be a domain which is filled up by the thin cylinders in the limit passage as $\varepsilon \rightarrow 0$. Let $f \in L^2_{loc}(R^n)$ a given function. Then*

$$\|f\|_{L^2(\Omega^-)}^2 \geq \frac{\varepsilon}{H^{n-2}(\partial C)} \|(f)_\varepsilon\|_{L^2(S_\varepsilon; H^{n-1})}^2$$

for ε small enough, where $(f)_\varepsilon$ denotes the direct smoothing of the function $\chi_{\Omega^-} f$, i.e.

$$(f)_\varepsilon = \varepsilon^{-n} \int_{R^n} K\left(\frac{x-y}{\varepsilon}\right) \chi_{\Omega^-}(y) f(y) dy.$$

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