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Sharp inequalities of various metrics on the classes of functions with given comparison function

Abstract. For any $q > p > 0$, $\omega > 0$, $d \geq 2\omega$, we obtain the following sharp inequality of various metrics

$$\|x\|_{L_q(I_d)} \leq \frac{\|\varphi + c\|_{L_q(I_{2\omega})}}{\|\varphi + c\|_{L_p(I_{2\omega})}} \|x\|_{L_p(I_d)}$$

on the set $S_\varphi(\omega)$ of d -periodic functions x having zeros with given the sine-shaped 2ω -periodic comparison function φ , where $c \in [-\|\varphi\|_\infty, \|\varphi\|_\infty]$ is such that

$$\|x_\pm\|_{L_p(I_d)} = \|(\varphi + c)_\pm\|_{L_p(I_{2\omega})}.$$

In particular, we obtain such type inequalities on the Sobolev sets of periodic functions and on the spaces of trigonometric polynomials and polynomial splines with given quotient of the norms $\|x_+\|_{L_p(I_d)} / \|x_-\|_{L_p(I_d)}$.

Key words: Inequality of various metrics, a class of functions with given comparison function, Sobolev class of functions, polynomial, spline.

Анотація. Для довільних $q > p > 0$, $\omega > 0$, $d \geq 2\omega$, отримана точна нерівність різних метрик

$$\|x\|_{L_q(I_d)} \leq \frac{\|\varphi + c\|_{L_q(I_{2\omega})}}{\|\varphi + c\|_{L_p(I_{2\omega})}} \|x\|_{L_p(I_d)} \quad (1)$$

на класах $S_\varphi(\omega)$ d -періодичних функцій x , що мають нулі, із заданою синусоподібною 2ω -періодичною функцією порівняння φ , де $c \in [-\|\varphi\|_\infty, \|\varphi\|_\infty]$ задовольняє умову

$$\|x_\pm\|_{L_p(I_d)} = \|(\varphi + c)_\pm\|_{L_p(I_{2\omega})}.$$

Як наслідок доведена точна нерівність

$$\|x\|_{L_q(I_{2\pi})} \leq \frac{\|\varphi_r + c\|_{L_q(I_{2\pi})}}{\|\varphi_r + c\|_{L_p(I_{2\pi})}^\alpha} \|x\|_{L_p(I_{2\pi})}^\alpha \cdot \|x^{(r)}\|_\infty^{1-\alpha}, \quad \alpha = (r + 1/q)(r + 1/p)^{-1}$$

на соболевських класах диференційовних 2π -періодичних функцій із заданим відношенням L_p -норм додатної і від'ємної частин функції, де φ_r – ідеальний сплайн Ейлера порядку r . Крім того, з нерівності (1) виведена нерівність типу Нікольського

$$\|T\|_{L_q(I_{2\pi})} \leq \left(\frac{n}{m}\right)^{\frac{1}{p} - \frac{1}{q}} \frac{\|\sin(\cdot) + c\|_{L_q(I_{2\pi})}}{\|\sin(\cdot) + c\|_{L_p(I_{2\pi})}} \cdot \|T\|_{L_p(I_{2\pi})},$$

на просторах тригонометричних поліномів T порядку $\leq n$ з періодом $2\pi/m$, $m \leq n$, і заданим відношенням L_p -норм додатної і від'ємної частин, і аналогічна нерівність типу Нікольського

$$\|s\|_{L_q(I_{2\pi})} \leq \left(\frac{n}{m}\right)^{\frac{1}{p} - \frac{1}{q}} \frac{\|\varphi_r + c\|_{L_q(I_{2\pi})}}{\|\varphi_r + c\|_{L_p(I_{2\pi})}} \cdot \|s\|_{L_p(I_{2\pi})},$$

на просторах поліноміальних сплайнів s порядку r мінімального дефекту з вузлами в точках $k\pi/n$, $k \in \mathbf{Z}$, та періодом $2\pi/m$, $m \leq n$, і заданим відношенням L_p -норм додатної і від'ємної частин

Ключові слова: Нерівність різних метрик, клас функцій із заданою функцією порівняння, соболевський клас, поліном, сплайн.

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1. Introduction. Let $G \subset \mathbf{R}$. We will consider the spaces $L_p(G)$, $0 < p \leq \infty$, of all measurable functions $x : G \rightarrow \mathbf{R}$ such that $\|x\|_p = \|x\|_{L_p(G)} < \infty$, where

$$\|x\|_p := \left(\int_G |x(t)|^p dt \right)^{1/p}, \quad \text{if } 0 < p < \infty,$$

$$\|x\|_\infty := \operatorname{vrai\,sup}_{t \in G} |x(t)|.$$

Let $d > 0$ and I_d denote the circle which is realized as the interval $[0, d]$ with coincident endpoints.

For $r \in \mathbf{N}$, $G = \mathbf{R}$ or $G = I_d$, denote by $L_\infty^r(G)$ the space of all functions $x \in L_\infty(G)$ for which $x^{(r-1)}$ is locally absolutely continuous and $x^{(r)} \in L_\infty(G)$.

SHARP INEQUALITIES OF VARIOUS METRICS

Let $\varphi_r(t)$, $r \in \mathbf{N}$, be the shift of the r^{th} 2π -periodic integral with zero mean value on a period of the function $\varphi_0(t) = \text{sgn} \sin t$ satisfying $\varphi_r(0) = 0$ and $\varphi_r'(0) > 0$.

Теорема А [1]. *Let $r \in \mathbf{N}$, $q > p > 0$. For any function $x \in L_\infty^r(I_{2\pi})$ having zeros the inequality*

$$\|x\|_q \leq \sup_{c \in [0, K_r]} \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|_p^\alpha} \|x\|_p^\alpha \|x^{(r)}\|_\infty^{1-\alpha}, \quad (1.1)$$

holds true, where $\alpha = \frac{r+1/q}{r+1/p}$, $K_r := \|\varphi_r\|_\infty$.

In addition, the inequality

$$\|x_\pm\|_q \leq \frac{\|(\varphi_r + c)_\pm\|_q}{\|(\varphi_r + c)_\pm\|_p^\alpha} \|x_\pm\|_p^\alpha \|x^{(r)}\|_\infty^{1-\alpha} \quad (1.2)$$

is proved in [1] for any function $x \in L_\infty^r(I_{2\pi})$ having zeros, where $c \in [-K_r, K_r]$ satisfying

$$\frac{\|x_+\|_p}{\|x_-\|_p} = \frac{\|(\varphi_r + c)_+\|_p}{\|(\varphi_r + c)_-\|_p}.$$

In this paper we generalize the inequalities (1.1) and (1.2) on the classes of the functions with given comparison function.

We need the following definitions.

A function $f \in L_\infty^1(\mathbf{R})$ is called a comparison function for $x \in L_\infty^1(\mathbf{R})$ if there exists a constant $c \in \mathbf{R}$ satisfying

$$\min_{t \in \mathbf{R}} f(t) + c \leq x(t) \leq \max_{t \in \mathbf{R}} f(t) + c, \quad t \in \mathbf{R},$$

and from $x(\xi) = f(\eta) + c$, $\xi, \eta \in \mathbf{R}$, the inequality $|x'(\xi)| \leq |f'(\eta)|$ follows (if corresponding derivatives exist).

Let $\omega > 0$. By definition, S -function is a 2ω -periodic function $\varphi \in L_\infty^1(I_{2\omega})$ that has the following properties: vanishes at 0, is odd about 0, is even about $\omega/2$, is positive and concave on $(0, \omega)$, and strictly increasing on $[0, \omega/2]$.

For 2ω -periodic S -function φ denote by $S_\varphi(\omega)$ the class of functions $x \in L_\infty^1(\mathbf{R})$ for which φ is the comparison function. Note that the classes $S_\varphi(\omega)$ were considered in [2], [3]. Examples of such classes $S_\varphi(\omega)$ are the Sobolev classes $\{x \in L_\infty^r(I_d) : \|x^{(r)}\|_\infty \leq 1\}$, the bounded subsets of the space T_n of all trigonometric polynomials of order at most n , and the same subsets of the space $S_{n,r}$ of polynomial splines of order r having defect 1 with knots at the points $k\pi/n, k \in \mathbf{Z}$.

In this paper we prove the inequality of various metrics on the classes $S_\varphi(\omega)$ of d -periodic functions with given quotient of the norms $\|x_+\|_{L_p(I_d)}/\|x_-\|_{L_p(I_d)}$ (Theorem 1). In particular, we obtain such type inequalities for a function $x \in L_\infty^r(I_{2\pi})$ (Theorem 2) and for functions in spaces T_n and $S_{n,r}$ (Theorem 3 and Theorem 4) with given quotient of the L_p -norms positive and negative parts of a function.

2. The inequalities of various metrics on the classes $S_\varphi(\omega)$.

Theorem 1. *Let $q, p > 0$, $q > p$ and φ is 2ω -periodic S -function. If for d -periodic function $x \in S_\varphi(\omega)$ having zeros there exists the number $c \in [-\|\varphi\|_\infty, \|\varphi\|_\infty]$ such that*

$$\|x_\pm\|_{L_p(I_d)} = \|(\varphi + c)_\pm\|_{L_p(I_{2\omega})}, \quad (2.1)$$

then the inequality

$$\|x\|_{L_q(I_d)} \leq \frac{\|\varphi + c\|_{L_q(I_{2\omega})}}{\|\varphi + c\|_{L_p(I_{2\omega})}} \|x\|_{L_p(I_d)} \quad (2.2)$$

holds true.

For fix $c \in [-\|\varphi\|_\infty, \|\varphi\|_\infty]$ the inequality (2.2) is the best possible on the set of functions $x \in S_\varphi(\omega)$ having zeros and satisfying the condition (2.1). The inequality (2.2) becomes equality for the function $x(t) = \varphi(t) + c$.

For proving Theorem 1 we require the following lemmas.

Lemma 1. *Under the conditions of Theorem 1*

$$\|x_\pm\|_\infty \leq \|(\varphi + c)_\pm\|_\infty, \quad (2.3)$$

and

$$d \geq 2\omega. \quad (2.4)$$

Proof. Fix a function $x \in S_\varphi(\omega)$ and a number $c \in [-\|\varphi\|_\infty, \|\varphi\|_\infty]$ satisfying the conditions of Theorem 1. Let us suppose that the statement (2.3) is false. Since φ is comparison function for function x , then $\|x_+\|_\infty + \|x_-\|_\infty \leq \|(\varphi + c)_+\|_\infty + \|(\varphi + c)_-\|_\infty$. So exactly one of the inequalities (2.3) is false. Let for example

$$\|x_+\|_\infty \leq \|(\varphi + c)_+\|_\infty, \quad \|x_-\|_\infty > \|(\varphi + c)_-\|_\infty.$$

Then there is an $a > 0$ satisfying

$$\|(x + a)_+\|_\infty \leq \|(\varphi + c)_+\|_\infty, \quad \|(x + a)_-\|_\infty = \|(\varphi + c)_-\|_\infty. \quad (2.5)$$

It is clear that $x + a \in S_\varphi(\omega)$. Let m be a point of maximum of the spline $\varphi + c$ and let $t_1(t_2)$ be closest to the left (right) of m zero of the spline. In view of the second relation (2.5) there is an shift $x(\cdot + \tau)$ of the function x such that $x(m + \tau) + a = \varphi(m) + c$. Since $\varphi + c$ is the comparison function for function x , then

$$x(t + \tau) + a \leq \varphi(t) + c < 0, \quad t \in (t_1, t_2).$$

By virtue of the inequality $a > 0$, this implies the estimate

$$\|x_-\|_{L_p(I_d)} > \|(x + a)_-\|_{L_p(I_d)} \geq \|(\varphi + c)_-\|_{L_p(I_{2\omega})},$$

that contradicting (2.1). The inequality (2.3) is proved. Since $x \in S_\varphi(\omega)$, the relation (2.4) follows immediately from (2.1) and (2.3).

Lemma 1 is proved.

For a function $f \in L_1(I_d)$ denote by $r(f, t)$, $t \in [0, d]$, the decreasing rearrangement (see, for example, [4, §1.3]) of the restriction of the function $|f|$ on $[0, d]$. Set $r(f, t) = 0$ for $t > d$.

Lemma 2. *Under the conditions of Theorem 1*

$$\int_0^\xi r^p(\bar{x}_\pm, t) dt \leq \int_0^\xi r^p(\bar{\varphi}_\pm, t) dt, \quad \xi > 0, \quad (2.6)$$

where \bar{x} is the restriction of the function x on I_d , and $\bar{\varphi}$ is the restriction of the function $\varphi + c$ on $I_{2\omega}$. In particular,

$$\|x_\pm\|_{L_q(I_d)} \leq \|(\varphi + c)_\pm\|_{L_q(I_{2\omega})}. \quad (2.7)$$

Proof. For proving (2.6) note that by virtue of (2.3) for any $y_\pm \in (0, \|\bar{x}_\pm\|_\infty)$ there are the points

$$t_i^\pm \in I_d, \quad i = 1, 2, \dots, m, \quad m \geq 2, \quad y_j^\pm \in I_{2\omega}, \quad j = 1, 2,$$

such that

$$y_\pm = \bar{x}_\pm(t_i^\pm) = \bar{\varphi}_\pm(y_j^\pm).$$

Since $\varphi + c$ is comparison function for function x , then

$$|\bar{x}'_\pm(t_i^\pm)| \leq |\bar{\varphi}'_\pm(y_j^\pm)|.$$

Show that if the points $\theta_1^\pm \in [0, d]$ and $\theta_2^\pm \in [0, 2\omega]$ are chosen so that

$$y_\pm = r(\bar{x}_\pm, \theta_1^\pm) = r(\bar{\varphi}_\pm, \theta_2^\pm),$$

then

$$|r'(\bar{x}_\pm, \theta_1^\pm)| \leq |r'(\bar{\varphi}_\pm, \theta_2^\pm)|.$$

Indeed, this immediately follows from the theorem on the derivative of the rearrangement (see, for example, [4, the statement 1.3.2]). By this theorem

$$|r'(\bar{x}_\pm, \theta_1^\pm)| = \left[\sum_{i=1}^m |\bar{x}'_\pm(t_i)|^{-1} \right]^{-1} \leq \left[\sum_{j=1}^2 |\bar{\varphi}'_\pm(y_j^\pm)|^{-1} \right]^{-1} = |r'(\bar{\varphi}_\pm, \theta_2^\pm)|.$$

Moreover, (2.3) implies the relation

$$r(\bar{x}_\pm, 0) = \|\bar{x}_\pm\|_\infty \leq \|\bar{\varphi}_\pm\|_\infty = r(\bar{\varphi}_\pm, 0).$$

Therefore, the difference $\Delta^\pm(t) := r(\bar{x}_\pm, t) - r(\bar{\varphi}_\pm, t)$ changes sign on $[0, \infty)$ at most once (from minus to plus). The same is true for the difference $\Delta_p^\pm(t) := r^p(\bar{x}_\pm, t) - r^p(\bar{\varphi}_\pm, t)$.

Set $I_\pm(\xi) := \int_0^\xi \Delta_p^\pm(t) dt$. Then, $I_\pm(0) = 0$ and since the rearrangement is L_p -norm-preserving, by (2.1) and (2.4) we have

$$I(d) = \|\bar{x}_\pm\|_{L_p(I_d)} - \|\bar{\varphi}_\pm\|_{L_p(I_{2\omega})} = 0.$$

Besides, $I'_\pm(\xi) = \Delta_p^\pm(\xi)$ changes sign on $[0, \infty)$ at most once (from minus to plus). Therefore, $I(\xi) \leq 0$, $\xi > 0$. This is equivalent to (2.6). From (2.6) by virtue of the Hardy-Littlewood-Polya theorem (see, for example, [4], theorem 1.3.1) follows the inequality (2.7).

Lemma 2 is proved.

Proof of Theorem 1. Fix any d -periodic function $x \in S_\varphi(\omega)$ and a number $c \in [-\|\varphi\|_\infty \|\varphi\|_\infty]$ satisfying the conditions of Theorem 1. Then by virtue of Lemma 1, the inequality (2.7) holds true. The relations (2.1) and (2.7) immediately implies the inequality (2.2). Its accuracy is obvious.

Theorem 1 is proved.

3. The inequality of various metric for the functions $x \in L_\infty^r(I_{2\pi})$. Recall that $\varphi_r(t)$, $r \in \mathbf{N}$, is the shift of the r^{th} 2π -periodic integral with zero mean value on a period of the function $\varphi_0(t) = \text{sgn} \sin t$ satisfying $\varphi_r(0) = 0$, and $K_r := \|\varphi_r\|_\infty$ is the Favard constant. For $\lambda > 0$ set $\varphi_{\lambda,r}(t) := \lambda^{-r} \varphi_r(\lambda t)$. Obviously the spline $\varphi_{\lambda,r}(t)$ is $2\pi/\lambda$ -periodic S -function.

For $r \in \mathbf{N}$; $p > 0$, $f_p \in [0, \infty]$ set

$$f_p L_\infty^r(I_{2\pi}) := \left\{ x \in L_\infty^r(I_{2\pi}) : \frac{\|x_+\|_p}{\|x_-\|_p} = f_p \right\}.$$

Obviously, for given p, f_p there is unique $c \in [-K_r, K_r]$ such that

$$\varphi_r + c \in f_p L_\infty^r(I_{2\pi}). \quad (3.1)$$

Theorem 2. Let $r \in \mathbf{N}$; $p, q > 0$, $q > p$, $f_p \in [0, \infty]$. Then for any function $x \in f_p L_\infty^r(I_{2\pi})$ having zeros we have

$$\|x\|_q \leq \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|_p^\alpha} \|x\|_p^\alpha \cdot \|x^{(r)}\|_\infty^{1-\alpha}, \quad (3.2)$$

where $\alpha = \frac{r+1/q}{r+1/p}$ and c satisfies (3.1).

The inequality (3.2) is sharp on the class $f_p L_\infty^r(I_{2\pi})$ having zeros and becomes equality for the function $x(t) = \varphi_r(t) + c$.

Proof. Fix a function $x \in f_p L_\infty^r(I_{2\pi})$ having zeros. In view of homogeneity of the inequality (3.2) we can assume that

$$\|x^{(r)}\|_\infty = 1. \quad (3.3)$$

Choose $\lambda > 0$ satisfying

$$\|x\|_p = \|\varphi_{\lambda,r} + \lambda^{-r} c\|_{L_p(I_{2\pi/\lambda})}. \quad (3.4)$$

Then (3.1) and (3.4) in view of the definition of the class $f_p L_\infty^r(I_{2\pi})$ implies that

$$\|x_\pm\|_p = \|(\varphi_{\lambda,r} + \lambda^{-r} c)_\pm\|_{L_p(I_{2\pi/\lambda})}, \quad (3.5)$$

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Show that

$$\|x_{\pm}\|_{\infty} \leq \|(\varphi_{\lambda,r} + \lambda^{-r}c)_{\pm}\|_{\infty}. \quad (3.6)$$

Let us suppose that the inequality (3.6) is false. Then $\|x\|_{\infty} = \|\varphi_{\omega,r} + \omega^{-r}c\|_{\infty}$ for some $\omega \in (0, \lambda)$. Let for example

$$\|x_{+}\|_{\infty} = \|(\varphi_{\omega,r} + \omega^{-r}c)_{+}\|_{\infty}, \quad \|x_{-}\|_{\infty} \leq \|(\varphi_{\omega,r} + \omega^{-r}c)_{-}\|_{\infty}. \quad (3.7)$$

Passing, if necessary, to the shift of the function x , we can assume that

$$\|x_{+}\|_{\infty} = x(m) = (\varphi_{\omega,r} + \omega^{-r}c)(m),$$

where m is the maximum point of the spline $\varphi_{\omega,r}$. By virtue of (3.7) and (3.3) the conditions of the Kolmogorov theorem [5] are satisfied. By this theorem the spline $\varphi_{\omega,r}$ is comparison function for function x . Therefore,

$$x_{+}(t) \geq (\varphi_{\omega,r} + \omega^{-r}c)_{+}(t), \quad t \in (m - \pi/\omega, m + \pi/\omega)$$

and

$$\|x_{+}\|_p \geq \|(\varphi_{\omega,r} + \omega^{-r}c)_{+}\|_{L_p(I_{2\pi/\omega})} > \|(\varphi_{\lambda,r} + \lambda^{-r}c)_{+}\|_{L_p(I_{2\pi/\lambda})},$$

which contradicts (3.5). The inequality (3.6) is proved. By virtue of (3.6) and (3.3) the conditions of the Kolmogorov theorem [5] are satisfied. By this theorem the spline $\varphi_{\lambda,r}$ is comparison function for function x . So $x \in S_{\varphi}(\frac{\pi}{\lambda})$. Then (3.5) in view of Lemma 2 implies that

$$\|x_{\pm}\|_q \leq \|(\varphi_{\lambda,r} + \lambda^{-r}c)_{\pm}\|_{L_q(I_{2\pi/\lambda})}. \quad (3.8)$$

Combining (3.4) and (3.8), and also taking into account the obvious equality

$$\|(\varphi_{\lambda,r} + \lambda^{-r}c)_{\pm}\|_{L_p(I_{2\pi/\lambda})} = \lambda^{-(r+1/p)} \|(\varphi_r + c)_{\pm}\|_p, \quad p > 0,$$

we get

$$\frac{\|x\|_q}{\|x\|_p^{\alpha}} \leq \frac{\|\varphi_{\lambda,r} + \lambda^{-r}c\|_{L_q(I_{2\pi/\lambda})}}{\|\varphi_{\lambda,r} + \lambda^{-r}c\|_{L_p(I_{2\pi/\lambda})}^{\alpha}} = \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|_p^{\alpha}}.$$

This estimates implies the inequality (3.2) by virtue of (3.3). Its accuracy is obvious.

Theorem 2 is proved.

Let

$$E_0^{\pm}(x)_p := \inf_{c \in \mathbf{R}} \{ \|x - c\|_p : \forall t \quad \pm(x(t) - c)_{\pm} \geq 0 \}$$

be the best one-sided approximations by constants of the function f in $L_p(I_{2\pi})$.

Corollary 1. *Let $r \in \mathbf{N}$; $p, q > 0$, $q > p$, $\alpha = \frac{r+1/q}{r+1/p}$, and let the number $\bar{c} \in [0, K_r]$ implements the upper bound*

$$\sup_{c \in [0, K_r]} \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|_p^{\alpha}}.$$

Then, for any function $x \in L_\infty^r(I_{2\pi})$, having zeros, the inequality

$$\|x\|_q \leq \frac{\|\varphi_r + \bar{c}\|_q}{\|\varphi_r + \bar{c}\|_p^\alpha} \|x\|_p^\alpha \cdot \|x^{(r)}\|_\infty^{1-\alpha}$$

holds true and, for any function $x \in L_\infty^r(I_{2\pi})$, we have

$$E_0^\pm(x)_q \leq \frac{\|\varphi_r + K_r\|_q}{\|\varphi_r + K_r\|_p^\alpha} E_0^\pm(x)_p^\alpha \cdot \|x^{(r)}\|_\infty^{1-\alpha}.$$

Both inequalities are sharp on the respective classes and turn into equalities for the splines $\varphi_r(t) + \bar{c}$ and φ_r respectively.

Remark. Theorem 2 and Corollary 1 are proved in [1].

4. The inequality of various metric for trigonometric polynomials. Let us recall that T_n is the space of all trigonometric polynomials of degree at most n . For $p > 0$, $f_p \in [0, \infty]$ set

$$f_p T_n := \left\{ T \in T_n : \frac{\|T_+\|_p}{\|T_-\|_p} = f_p \right\}.$$

Theorem 3. Let $n, m \in \mathbf{N}$, $m \leq n$; $p, q > 0$, $q > p$, $f_p \in [0, \infty]$. For any trigonometric polynomial $T \in f_p T_n$ with minimal period $2\pi/m$, having zeros, the inequality

$$\|T\|_q \leq \left(\frac{n}{m}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{\|\sin(\cdot) + c\|_q}{\|\sin(\cdot) + c\|_p} \cdot \|T\|_p, \quad (4.1)$$

holds true, where $c \in [-1, 1]$ is the constant satisfying

$$\sin(\cdot) + c \in f_p T_n. \quad (4.2)$$

The inequality (4.1) is sharp for $m = 1$ in the sense

$$\sup_{n \in \mathbf{N}} \sup_{T \in f_p T_n} \frac{\|T\|_q}{n^{1/p-1/q} \|T\|_p} = \frac{\|\sin(\cdot) + c\|_q}{\|\sin(\cdot) + c\|_p}.$$

Proof. Fix a polynomial $T \in f_p T_n$ with minimal period $\frac{2\pi}{m}$ having zeros. Set $\varphi(t) := \sin nt$, $\psi(t) := \varphi(t) + c$, $t \in \mathbf{R}$. In view of homogeneity of the inequality (4.1) we can assume that

$$\|T\|_{L_p(I_{2\pi/m})} = \|\psi\|_{L_p(I_{2\pi/n})}. \quad (4.3)$$

Then, by virtue of (4.2) and the definition of the class $f_p T_n$, we have

$$\|T_\pm\|_{L_p(I_{2\pi/m})} = \|\psi_\pm\|_{L_p(I_{2\pi/n})}. \quad (4.4)$$

Show that

$$\|T_\pm\|_\infty \leq \|\psi_\pm\|_\infty. \quad (4.5)$$

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Let us suppose that the inequality (4.5) is false. Then, there is $\gamma \in (0, 1)$ such that $\|\gamma T_{\pm}\|_{\infty} \leq \|\psi_{\pm}\|_{\infty}$, moreover, one of these inequalities turn into equality. Let, for example,

$$\|\gamma T_+\|_{\infty} \leq \|\psi_+\|_{\infty}, \quad \|\gamma T_-\|_{\infty} = \|\psi_-\|_{\infty}.$$

Then, the polynomial ψ is the comparison function for the polynomial γT (see the proof of the theorem 8.1.1 in [6]). Let m be a minimum point of the function ψ and let $t_1(t_2)$ be closest to the left (right) of m zero of the function ψ . Passing, if necessary, to the shift of the polynomial γT , we can assume that

$$\|\gamma T_-\|_{\infty} = -\gamma T(m).$$

Since the polynomial ψ is the comparison function for the polynomial γT , then

$$\gamma T(t) \leq \psi(t) < 0, \quad t \in (t_1, t_2).$$

This implies the estimate

$$\|T_-\|_{L_p(2\pi/m)} > \|\gamma T_-\|_{L_p(2\pi/m)} \geq \|\psi_-\|_{L_p(2\pi/n)},$$

which contradicts (4.4). The inequality (4.5) is proved.

It follows from the inequality (4.5) and the proof of the theorem 8.1.1 in [6] that $\varphi(t) = \sin nt$ is the comparison function for the polynomial T . So $T \in S_{\varphi}(\frac{\pi}{n})$. Therefore, by virtue of (4.4), the polynomial T satisfies the conditions of Theorem 1, and hence, the conditions of Lemmas 1 and 2. According to the inequality (2.7) of Lemma 2, we have

$$\|T\|_{L_q(I_{2\pi/m})} \leq \|\varphi + c\|_{L_q(I_{2\pi/n})}.$$

From this inequality, due to the $2\pi/m$ -periodic of the polynomial T and $2\pi/n$ -periodic of the polynomial φ , we obtain

$$\|T\|_q \leq \left(\frac{m}{n}\right)^{1/q} \|\varphi + c\|_q. \quad (4.6)$$

Similarly, the condition (4.4) implies the equality

$$\|T\|_p = \left(\frac{m}{n}\right)^{1/p} \|\varphi + c\|_p. \quad (4.7)$$

From (4.6) and (4.7) follows the inequality (4.1). The sharpness of (4.1) is obvious.

Theorem 3 is proved.

Corollary 2. *Let $n, m \in \mathbf{N}$, $m \leq n$; $q, p > 0$, $q > p$, and let the number $\bar{c} \in [0, 1]$ implements the upper bound*

$$\sup_{c \in [0, 1]} \frac{\|\sin(\cdot) + c\|_q}{\|\sin(\cdot) + c\|_p}.$$

Then, for any trigonometric polynomial $T \in T_n$ with minimal period $2\pi/m$, having zeros, the inequality

$$\|T\|_q \leq \left(\frac{n}{m}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{\|\sin(\cdot) + \bar{c}\|_q}{\|\sin(\cdot) + \bar{c}\|_p} \cdot \|T\|_p$$

holds, and for any trigonometric polynomial $T \in T_n$ with minimal period $2\pi/m$, we have

$$E_0^\pm(T)_q \leq \left(\frac{n}{m}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{\|\sin(\cdot) + 1\|_q}{\|\sin(\cdot) + 1\|_p} \cdot E_0^\pm(T)_p.$$

Both inequalities are sharp for $m = 1$ in the sense

$$\sup_{n \in \mathbf{N}} \sup_{T \in T_n^0} \frac{\|T\|_q}{n^{1/p-1/q} \|T\|_p} = \frac{\|\sin(\cdot) + \bar{c}\|_q}{\|\sin(\cdot) + \bar{c}\|_p},$$

where T_n^0 is the set of polynomials $T \in T_n$ having zeros, and

$$\sup_{n \in \mathbf{N}} \sup_{T \in T_n} \frac{E_0^\pm(T)_q}{n^{1/p-1/q} E_0^\pm(T)_p} = \frac{\|\sin(\cdot) + 1\|_q}{\|\sin(\cdot) + 1\|_p}.$$

Remark. Theorem 3 and Corollary 2 for $m = 1$ are proved in [1].

5. The inequality of various metric for periodic polynomial splines. Let $r, n \in \mathbf{N}$. Recall that $S_{n,r}$ stands for the space of polynomial splines of order r having defect 1 with knots at the points $k\pi/n, k \in \mathbf{Z}$. It is clear that $S_{n,r} \subset L_\infty(\mathbf{R})$. For $p > 0, f_p \in [0, \infty]$ set

$$f_p S_{n,r} := \left\{ s \in S_{n,r} : \frac{\|s_+\|_p}{\|s_-\|_p} = f_p \right\}.$$

Theorem 4. Let $n, m \in \mathbf{N}, m \leq n; p, q > 0, q > p, f_p \in [0, \infty]$. For a spline $s \in f_p S_{n,r}$ with minimal period $2\pi/m$, having zeros, the inequality

$$\|s\|_q \leq \left(\frac{n}{m}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|_p} \cdot \|s\|_p \quad (5.1)$$

holds true, where $c \in [-K_r, K_r]$ is the constant satisfying

$$\varphi_{n,r} + n^{-r}c \in f_p S_{n,r}. \quad (5.2)$$

The inequality (5.1) is sharp for $m = 1$ in the sense

$$\sup_{n \in \mathbf{N}} \sup_{s \in f_p S_{n,r}} \frac{\|s\|_q}{n^{1/p-1/q} \|s\|_p} = \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|_p}.$$

Proof. Fix a spline $s \in f_p S_{n,r}$ with minimal period $2\pi/m$ having zeros. Set $\varphi(t) := \varphi_{n,r}(t), \psi(t) := \varphi_{n,r}(t) + n^{-r}c, t \in \mathbf{R}$. In view of homogeneity of the inequality (5.1) we can assume that

$$\|s\|_{L_p(I_{2\pi/m})} = \|\psi\|_{L_p(I_{2\pi/n})}. \quad (5.3)$$

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Then, by virtue of (5.2) and the definition of the class $f_p S_{n,r}$, we have

$$\|s_{\pm}\|_{L_p(I_{2\pi/m})} = \|\psi_{\pm}\|_{L_p(I_{2\pi/n})}. \quad (5.4)$$

Show that

$$\|s_{\pm}\|_{\infty} \leq \|\psi_{\pm}\|_{\infty}. \quad (5.5)$$

Let us suppose that the inequality (5.5) is false. Then, there is $\gamma \in (0, 1)$ such that $\|\gamma s_{\pm}\|_{\infty} \leq \|\psi_{\pm}\|_{\infty}$, moreover, one of these inequalities turn into equality. Let, for example,

$$\|\gamma s_+\|_{\infty} \leq \|\psi_+\|_{\infty}, \quad \|\gamma s_-\|_{\infty} = \|\psi_-\|_{\infty}.$$

Then, $E_0(\gamma s)_{\infty} \leq E_0(\psi)_{\infty} = \|\varphi_{n,r}\|_{\infty}$ and, by the Tikhomirov inequality [7],

$$\|\gamma s^{(r)}\|_{\infty} < \|s^{(r)}\|_{\infty} \leq \frac{E_0(s)_{\infty}}{\|\varphi_{n,r}\|_{\infty}} = 1,$$

where $E_0(x)_{\infty}$ is the best uniform approximation of the function x by constants.

Thus, the spline γs satisfies the conditions of the Kolmogorov comparison theorem [5]. By this theorem the spline φ is the comparison function for the spline γs . Let m be a minimum point of the function ψ and let $t_1(t_2)$ be closest to the left (right) of m zero of the function ψ . Passing, if necessary, to the shift of the spline γs , we can assume that

$$\|\gamma s_-\|_{\infty} = -\gamma s(m).$$

Since the spline ψ is the comparison function for the spline γs , then

$$\gamma s(t) \leq \psi(t) < 0, \quad t \in (t_1, t_2).$$

This implies the estimate

$$\|s_-\|_{L_p(2\pi/m)} > \|\gamma s_-\|_{L_p(2\pi/m)} \geq \|\psi_-\|_{L_p(2\pi/n)},$$

which contradicts (5.4). The inequality (5.5) is proved.

It follows from (5.5) that $E_0(s)_{\infty} \leq E_0(\psi)_{\infty} = \|\varphi_{n,r}\|_{\infty}$ and, by the Tikhomirov inequality [7],

$$\|s^{(r)}\|_{\infty} \leq \frac{E_0(s)_{\infty}}{\|\varphi_{n,r}\|_{\infty}} = 1.$$

Thus, the spline s satisfies the conditions of the Kolmogorov comparison theorem [5]. By this theorem the spline φ is the comparison function for the spline s . So $s \in S_{\varphi}(\frac{\pi}{n})$. Therefore, by virtue of (5.4), the spline s satisfies the conditions of Theorem 1, and hence, the conditions of Lemmas 1 and 2. According to the inequality (2.7) of Lemma 2, we have

$$\|s\|_{L_q(I_{2\pi/m})} \leq \|\varphi_{n,r} + n^{-r}c\|_{L_q(I_{2\pi/n})}.$$

From this inequality, due to the $2\pi/m$ -periodic of the spline s and $2\pi/n$ -periodic of the spline $\varphi_{n,r}$, we obtain

$$\|s\|_q \leq \left(\frac{m}{n}\right)^{1/q} \|\varphi_{n,r} + n^{-r}c\|_q. \quad (5.6)$$

Similarly, the condition (5.4) implies the equality

$$\|s\|_p = \left(\frac{m}{n}\right)^{1/p} \|\varphi_{n,r} + n^{-r}c\|_p. \quad (5.7)$$

From (5.6) and (5.7) follows the inequality (5.1). The sharpness of (5.1) is obvious.

Theorem 4 is proved.

Corollary 3. *Let $n, m \in \mathbf{N}$, $m \leq n$; $q, p > 0$, $q > p$, and let the number $\bar{c} \in [0, K_r]$ implements the upper bound*

$$\sup_{c \in [0, K_r]} \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|_p}.$$

Then, for any splines $s \in S_{n,r}$ with minimal period $2\pi/m$, having zeros, the inequality

$$\|s\|_q \leq \left(\frac{n}{m}\right)^{\frac{1}{p} - \frac{1}{q}} \frac{\|\varphi_r + \bar{c}\|_q}{\|\varphi_r + \bar{c}\|_p} \cdot \|s\|_p$$

holds, and for any splines $s \in S_{n,r}$ with minimal period $2\pi/m$, we have

$$E_0^\pm(s)_q \leq \left(\frac{n}{m}\right)^{\frac{1}{p} - \frac{1}{q}} \frac{\|\varphi_r + K_r\|_q}{\|\varphi_r + K_r\|_p} \cdot E_0^\pm(s)_p.$$

Both inequalities are sharp for $m = 1$ in the sense

$$\sup_{n \in \mathbf{N}} \sup_{s \in S_{n,r}^0} \frac{\|s\|_q}{n^{1/p-1/q} \|s\|_p} = \frac{\|\varphi_r + \bar{c}\|_q}{\|\varphi_r + \bar{c}\|_p},$$

where $S_{n,r}^0$ is the set of splines $s \in S_{n,r}$ having zeros, and

$$\sup_{n \in \mathbf{N}} \sup_{s \in S_{n,r}} \frac{E_0^\pm(s)_q}{n^{1/p-1/q} E_0^\pm(s)_p} = \frac{\|\sin(\cdot) + K_r\|_q}{\|\sin(\cdot) + K_r\|_p}.$$

Remark. Theorem 4 and Corollary 3 for $m = 1$ are proved in [1].

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