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On the homology groups $H_k(\mathbb{C}\Omega_n)$, $k = 1, ..., n$

Abstract. In the paper the homology groups of the $(2n+1)$-dimensional CW-complex $\mathbb{C}\Omega_n$ are investigated. The spaces $\mathbb{C}\Omega_n$ consist of complex-valued functions and generalize the widely known in the approximation theory spaces Ω_n. The research of the homotopy properties of the spaces Ω_n has been started by V.I. Ruban who in 1985 found the n-dimensional homology group of the space Ω_n and in 1999 found all the cohomology groups of this space. The spaces $\mathbb{C}\Omega_n$ have been introduced by A.M. Pasko who in 2015 has built the structure of CW-complex on these spaces. This CW-structure is analogue of the CW-structure of the space Ω_n introduced by V.I. Ruban. In present paper in order to investigate the homology groups of the spaces $\mathbb{C}\Omega_n$ we calculate the relative homology groups $H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1})$, it turned out that the groups $H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1})$ are trivial if $1 \le k < n$ and $H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1}) = \mathbb{Z}^{C_{n+1}^{k-n}}$ if $n \le k \le 2n+1$, in particular $H_n(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1}) = \mathbb{Z}$. Further we consider the exact homology sequence of the pair $(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n)$ and prove that its inclusion operator $i_* : H_k(\mathbb{C}\Omega_n) \to H_k(\mathbb{C}\Omega_{n+1})$ is zero. Taking into account that the relative homology groups $H_k(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n)$ are zero if $1 \le k \le n$ and the inclusion operator $i_* = 0$ we have derived from the exact homology sequence of the pair $(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n)$ that the homology groups $H_k(\mathbb{C}\Omega_n), 1 \le k < n$, are trivial. The similar considerations made it possible to calculate the group $H_n(\mathbb{C}\Omega_n)$. So the homology groups $H_k(\mathbb{C}\Omega_n), n \ge 2, k = 1, ..., n$, have been found.

Key words: homology group, spline, CW-complex

Анотація. У статті піддано дослідженню гомологічні групи $2n+1$-вимірного клітинного простору $\mathbb{C}\Omega_n$. Простори $\mathbb{C}\Omega_n$ складаються з комплекснозначних функцій та узагальнюють широко відомі в теорії апроксимації топологічні простори Ω_n. Дослідження гомотопічних інваріантів топологічного простору Ω_n було започатковане В.І. Рубаном, який у 1985 р. знайшов n-вимірну групу гомологій цього простору, а в 1999 р. повністю розв'язав задачу відшукання його груп когомологій. Топологічні простори $\mathbb{C}\Omega_n$ було введено А.М. Паськом у роботі 2015 р., у якій було побудовано клітинну структуру цих просторів, аналогічну побудованій В.І. Рубаном клітинній структурі просторів Ω_n. У цій статті, задля дослідження гомологічних груп топологічних просторів $\mathbb{C}\Omega_n$, обчислено групи відносних гомологій $H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1})$, виявилося, що у вимірностях $1 \leq k < n$ ці групи відносних гомологій – тривіальні, а при $n \leq k \leq 2n+1$ ці групи дорівнюють $H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1}) = \mathbb{Z}^{C_{n+1}^{k-n}}$, зокрема $H_n(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1}) = \mathbb{Z}$. Відтак, розглянуто точну гомологічну послідовність пари $(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n)$ і доведено рівність нулю її оператора вкладення $i_* : H_k(\mathbb{C}\Omega_n) \to H_k(\mathbb{C}\Omega_{n+1})$. Зважаючи, що групи відносних гомологій $H_k(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n)$ дорівнюють нулю у вимірностях $1 \leq k \leq n$, та оператор включення $i_* = 0$, і використовуючи точну гомологічну послідовність пари топологічних просторів $(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n)$, доведено тривіальність груп гомологій $H_k(\mathbb{C}\Omega_n)$ у вимірностях $1 \leq k < n$. Подібні до описаних міркування дозволяють знайти $H_n(\mathbb{C}\Omega_n)$, що повністю розв'язує задачу обчислення гомологічних груп $H_k(\mathbb{C}\Omega_n), n \geq 2$, для $k = 1, ..., n$.
Ключові слова: групи гомологій, сплайни, клітинний простір

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Let $\omega(t), t \geq 0$, be the non-negative, continuous increasing function, $\omega(0) = 0$, and $n \in \mathbb{N}$. Consider integer $q \geq 0$ and the system of the knots

$$0 = \eta_0 < \eta_1 < \ldots < \eta_q < \eta_{q+1} = 1.$$

For each $q \leq n$ and $0 \leq k \leq q$ we consider the function (ω-spline)

$$F(\eta_k, \eta_{k+1}, s_k, t) = s_k \cdot \min\{\omega(t - \eta_k), \omega(\eta_{k+1} - t)\}, \qquad for \quad t \in [\eta_k, \eta_{k+1}], \tag{1}$$

with $s_k \in \mathbb{C}, |s_k| = 1$, and the subspace $\mathbb{C}\Omega_n$ of the space $L_1[0,1]$ that consists of the splines of the form (1) for $q \leq n$.

Also there are the subspaces $\Omega_n(m) \subset \mathbb{C}\Omega_n$, $n, m \in \mathbb{N}, n \geq 2, m \geq 2$, which consist of the splines (1) with $s_k \in \left\{e^{i\frac{2\pi l}{m}} : l = 0, 1, \ldots, m-1\right\}$. The space $\Omega_n(2)$ coincides with the space Ω_n. The spaces Ω_n have been investigated in [1], [3], [7], [8]. V.I. Ruban [7], [8] has built CW-structure on Ω_n and calculated the cohomologies of the space Ω_n

$$H^k(\Omega_n) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}^{\frac{2^{n+2}+(-1)^{n+1}}{3}}, & k = n, \\ 0, & k \neq 0, n. \end{cases}$$

V.A.Koshcheev [1] has proved that the spaces Ω_n are simply connected. In [3] the author found that the homotopy groups

$$\pi_k(\Omega_n) = \begin{cases} 0, & 2 \leq k \leq n-1, \\ \mathbb{Z}^{\frac{2^{n+2}+(-1)^{n+1}}{3}}, & k = n. \end{cases}$$

The spaces $\Omega_n(m)$ were introduced in [6] as a generalization of the spaces Ω_n. The problem of calculating the homology groups of the spaces $\Omega_n(m)$ was solved in [2] where it was found that the homology groups of the space $\Omega_n(m)$, $m \geq 2$, $n \geq 2$, are equal to

$$H_k\left(\Omega_n(m)\right) = \begin{cases} \mathbb{Z}, & k = 0, \\ 0, & k \neq 0, n, \\ \mathbb{Z}^{\frac{m^{n+2}+(-1)^{n+1}}{m+1}}, & k = n. \end{cases}$$

The $\mathbb{C}\Omega_n$ spaces were introduced in [4], where the structure of the $(2n+1)$-dimensional CW-complex on these spaces is constructed and it is proved that they are simply connected (for $n \geq 2$). In [5] the author found some of the homology groups of the spaces $\mathbb{C}\Omega_n, n \geq 2$,

$$H_k\left(\mathbb{C}\Omega_n\right) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, k = 2n+1; \\ 0, & \text{if } k = 1, k = 2; \\ \mathbb{Z}^{n+\frac{(n-1)(n-2)}{2}}, & \text{if } k = 2n-1; \\ \mathbb{Z}^n, & \text{if } k = 2n. \end{cases}$$

Also in [5] A.M. Pasko found the Euler characteristic $\chi\left(\mathbb{C}\Omega_n\right) = 0$.

Let us describe the CW-structure of the space $\mathbb{C}\Omega_n$ built in [4]. CW-complex is a Hausdorff space E written as a union

$$E = \bigcup_{q=0}^{\infty} \bigcup_{k} e_k^q$$

of the non-overlapping sets e_k^q (q-cells) in such a way that any q-cell e_k^q has a continuous characteristic map $f_k^q : D^q \to E$ of closed q-dimensional ball D^q to E such that the restriction of f_k^q to $\mathrm{Int} D^q$ is a homeomorphism between $\mathrm{Int} D^q$ and e_k^q. Herewith E satisfies the conditions:

(C) the boundary $\dot{e}_k^q = \bar{e}_k^q \setminus e_k^q$ of any q-cell lies in the union of a finite number of j-cells for $j < q$;

(W) subset $F \subset E$ is closed if and only if all the intersections $F \bigcap \bar{e}_k^q$ are closed.

Consider the set of two symbols $A = \{1, e\}$. For any integer $q \geq 0$ and sequence (word) $u = u_0 u_1 ... u_q$ of the symbols $u_k \in A$ let $c^q(u)$ be the set of the splines (1) with exactly q knots $\eta_1, \eta_2, \ldots, \eta_q$,

$$0 = \eta_0 < \eta_1 < \ldots < \eta_q < \eta_{q+1} = 1,$$

and the coefficients

$$s_k = \begin{cases} 1, & \text{if } u_k = 1, \\ e^{i\phi_k}, 0 < \phi_k < 2\pi, & \text{if } u_k = e. \end{cases}$$

Let $h = h(u)$, $0 \le h(u) \le q+1$, be the amount of the symbols u_k of the word $u = u_0 u_1 ... u_q$ such that $u_k = e$. Consider the $(q+h)$-dimensional ball D^{q+h} written as Cartesian product of the standard q-simplex B^q and the h-cube:

$$B^q \times [0, 2\pi]^h = \{(x_0, x_1, \dots, x_q; \theta_1, \theta_2, \dots, \theta_h) : x_k \ge 0, \sum_{k=0}^{q} x_k = 1, 0 \le \theta_j \le 2\pi\}.$$

Let $u_{k_1}, u_{k_2}, \dots, u_{k_h}$, $k_1 < k_2 < \dots < k_h$, be all of the symbols u_k of the word $u = u_0 u_1 ... u_q$ such that $u_k = e$. Consider

$$(x, \theta) = (x_0, x_1, \dots, x_q; \theta_1, \theta_2, \dots, \theta_h) \in B^q \times [0, 2\pi]^h$$

and define the vector of knots $\eta(x) = (\eta_0(x), \eta_1(x), \dots, \eta_{q+1}(x))$,

$$\eta_k(x) = \sum_{j=0}^{k-1} x_j, \ \ k = 1, \dots, q+1, \ \ \eta_0(x) = 0,$$

and the vector of coefficients $s(u, \theta) = (s_0(u, \theta), s_1(u, \theta), \dots, s_q(u, \theta))$,

$$s_k(u, \theta) = \begin{cases} 1, & \text{if } k \ne k_1, \dots, k_h, \\ e^{i\theta_j}, & \text{if } k = k_j, j = 1, \dots, h. \end{cases}$$

So we can define the characteristic map $\psi_u^q : B^q \times [0, 2\pi]^h \longrightarrow \mathbb{C}\Omega_n$ of the $(q+h)$-cell $c^q(u)$ as

$$\psi_u^q(x, \theta) = F(\eta(x), s(u, \theta), t), \quad t \in [0, 1],$$

where F is defined by (1) spline.

So we have the CW-structure of the space $\mathbb{C}\Omega_n$:

$$\mathbb{C}\Omega_n = \bigcup_{k=0}^{2n+1} \bigcup_{q+h(u)=k} c^q(u),$$

the only 0-cell is $c^0(1)$, the 1-cells are $c^0(e), c^1(11)$, the 2-cells are $c^1(e1), c^1(1e), c^2(111)$ and so on, the only $(2n+1)$-cell is $c^n(ee...e)$.

Consider CW-complex E. The set of the q-cells of E may be used as the basis of a free abelian group $C_q(E)$. The elements of $C_q(E)$ are called q-chains. There are the homomorphisms of the groups $\partial = \partial_q : C_q(E) \to C_{q-1}(E)$. This homomorphisms are called boundary operators. Consider the groups $Z_q(E) = \mathrm{Ker}\partial_q$ (the groups of q-cycles) and $B_q(E) = \mathrm{Im}\partial_{q+1}$ (the groups of q-boundaries). The identity $\partial_q \partial_{q+1} = 0$ implies $B_q(E) \subset Z_q(E)$ that allows to define the homology groups $H_q(E) = Z_q(E)/B_q(E)$.

It is known from [4] that the boundary of the $(q+h(u))$-cell $c^q(u)$ of the space $\mathbb{C}\Omega_n$ equals

$$\partial c^q(u_0 \dots u_q) = \sum_{k:\ u_k = 1} (-1)^k c^{q-1}(u_0 \dots \hat{u}_k \dots u_q), \tag{2}$$

where the word $u_0 \dots \hat{u}_k \dots u_q$ is $u_0 \dots u_{k-1} u_{k+1} \dots u_q$.

The goal of this paper is to find the homology groups $H_k(\mathbb{C}\Omega_n), 1 \le k \le n$. The main result of the paper is the following theorem.

Theorem 1. *For each $n \geq 2$ and $1 \leq k \leq n$ the homology group*

$$H_k(\mathbb{C}\Omega_n) = \begin{cases} 0, & 1 \leq k < n; \\ \mathbb{Z}, & k = n, n \text{ is odd}; \\ 0, & k = n, n \text{ is even}. \end{cases} \tag{3}$$

In order to prove the theorem we need the relative homology groups $H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1})$. The following lemma holds.

Lemma 1. *For each $n \geq 2$ and $1 \leq k \leq 2n+1$ the relative homology group $H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1}), n \geq 2$, equal*

$$H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1}) = \begin{cases} 0, & 1 \leq k < n; \\ \mathbb{Z}^{C_{n+1}^{k-n}}, & n \leq k \leq 2n+1. \end{cases}$$

Proof. Let integers $n \geq 2$ and $1 \leq k < n$. It follows from (2) that for every k-chain $c \in C_k(\mathbb{C}\Omega_n)$ its boundary $\partial c \in C_{k-1}(\mathbb{C}\Omega_{n-1})$. Then the boundary operator of the relative chain complex $C_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1})$ is zero. Therefore $H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1})$ are free abelian groups and the basis of the free abelian group $H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1})$ consists of the k-cells of the space $\mathbb{C}\Omega_n$ that don't belong to $\mathbb{C}\Omega_{n-1}$. All such cells are $c^n(u_0 u_1 \ldots u_n), u_l \in A, l = 0, \ldots, n$, the cell of the smallest dimension among them is $c^n(11\ldots1)$, $\dim(c^n(11...1)) = n$. Therefore $H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1}) = 0$, where $0 \leq k < n$.

If $n \leq k \leq 2n+1$ the basis of $H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1})$ consists of the cells $c^n(u_0 u_1 \ldots u_n)$, where $h(u_0 u_1 \ldots u_n) = k - n$ (the number of symbols u_l such that $u_l = e$ equals $k-n$). The number of such words is C_{n+1}^{k-n}, so

$$H_k(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1}) = \mathbb{Z}^{C_{n+1}^{k-n}}.$$

The lemma is proved.

Proof of the theorem. Let integers $n \geq 2$ and $1 \leq k \leq n$. Consider the inclusion $i : \mathbb{C}\Omega_n \longrightarrow \mathbb{C}\Omega_{n+1}$. It induces the homomorphism $\bar{i} : C_k(\mathbb{C}\Omega_n) \longrightarrow C_k(\mathbb{C}\Omega_{n+1})$ of the groups of the chains and the homomorphism $i_* : H_k(\mathbb{C}\Omega_n) \longrightarrow H_k(\mathbb{C}\Omega_{n+1})$ of the homology groups.

In order to research the operator i_* consider the homomorphism

$$D : C_k(\mathbb{C}\Omega_n) \longrightarrow C_{k+1}(\mathbb{C}\Omega_{n+1})$$

defined on the generators by the equality

$$Dc^q(u_0 u_1 ... u_q) = c^{q+1}(1 u_0 u_1 ... u_q),$$

where $q = k - h(u_0 u_1 ... u_q)$. By the virtue of (2)

$$\partial D c^q(u_0 u_1 ... u_q) = c^q(u_0 u_1 ... u_q) - D\partial c^q(u_0 u_1 ... u_q)$$

for every generator $c^q(u_0u_1...u_q)$. Then for every k-chain $\alpha \in C_k(\mathbb{C}\Omega_n)$

$$\partial D\alpha = \bar{i}\alpha - D\partial\alpha. \tag{4}$$

It follows from (4) that for every k-cycle α the chain $\bar{i}\alpha = \partial D\alpha$ is k-boundary, then the homomorphism i_* is trivial: $i_* = 0$.

Consider the exact homology sequence of the pair $(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n)$:

$$... \to H_{k+1}(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n) \xrightarrow{\partial} H_k(\mathbb{C}\Omega_n) \xrightarrow{i_*=0} H_k(\mathbb{C}\Omega_{n+1}) \xrightarrow{j_*}$$

$$\xrightarrow{j_*} H_k(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n) \to ... \tag{5}$$

If $k < n$ then $H_{k+1}(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n) = H_k(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n) = 0$ accordingly to lemma 1. So the sequence (5) turns to the short exact sequence

$$0 \to H_k(\mathbb{C}\Omega_n) \xrightarrow{i_*=0} H_k(\mathbb{C}\Omega_{n+1}) \to 0.$$

It is possible only if $H_k(\mathbb{C}\Omega_n) = H_k(\mathbb{C}\Omega_{n+1}) = 0$. Therefore

$$H_k(\mathbb{C}\Omega_n) = 0,\ 1 \le k < n. \tag{6}$$

In order to complete the proof of the theorem we need to prove (3) in the case $k = n$. Consider $k = n + 1$ in the exact sequence (5):

$$... \to H_{n+1}(\mathbb{C}\Omega_n) \xrightarrow{i_*=0} H_{n+1}(\mathbb{C}\Omega_{n+1}) \xrightarrow{j_*} H_{n+1}(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n) \xrightarrow{\partial}$$

$$\to H_n(\mathbb{C}\Omega_n) \xrightarrow{i_*=0} H_n(\mathbb{C}\Omega_{n+1}) \to \dots.$$

Because $i_* = 0$ we can replace here the groups $H_{n+1}(\mathbb{C}\Omega_n), H_n(\mathbb{C}\Omega_{n+1})$ by the trivial group. So we get the short exact sequence

$$0 \to H_{n+1}(\mathbb{C}\Omega_{n+1}) \xrightarrow{j_*} H_{n+1}(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n) \xrightarrow{\partial} H_n(\mathbb{C}\Omega_n) \to 0. \tag{7}$$

It follows from the proof of the lemma 1 that the group $H_{n+1}(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n)$ has the only generator $c^{n+1}(11...1)$. Let n be even. Accordingly to (2) the boundary of the generator $c^{n+1}(11...1)$ is

$$\partial c^{n+1}(11...1) = 0.$$

The operator ∂ in (7) is epimorphism, so $H_n(\mathbb{C}\Omega_n) = 0$ if n is even.

Let n be odd. Then $n + 1$ is even and $H_{n+1}(\mathbb{C}\Omega_{n+1}) = 0$. Therefore the short exact sequence (7) turns to

$$0 \to H_{n+1}(\mathbb{C}\Omega_{n+1}, \mathbb{C}\Omega_n) \xrightarrow{\partial} H_n(\mathbb{C}\Omega_n) \to 0.$$

It implies that ∂ is isomorphism. By the virtue of lemma 1 $H_n(\mathbb{C}\Omega_n) = \mathbb{Z}$.

Therefore

$$H_n\left(\mathbb{C}\Omega_n\right) = \begin{cases} 0, & n \text{ is even;} \\ \mathbb{Z}, & n \text{ is odd.} \end{cases} \tag{8}$$

By the virtue of (6), (8) the equality (3) holds. The proof of the theorem is completed.

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