Researches in Mathematics
Res. Math. **29**(1), 2021, p. 31–42
doi:10.15421/242104

https://vestnmath.dnu.dp.ua
ISSN (Print) 2664-4991
ISSN (Online) 2664-5009

UDK 517.5

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The uniqueness of the best non-symmetric L_1-approximant for continuous functions with values in $\mathbb{R}_p^m$

Abstract. The article considers the questions of the uniqueness of the best non-symmetric L_1-approximations of continuous functions with values in $\mathbb{R}_p^m, p \in (1;+\infty)$ by elements of the two-dimensional subspace $H_2 = \mathrm{span}\{1, g_{a,b}\}$, where

$$g_{a,b}(x) = \begin{cases} -b \cdot (x-1)^2, & x \in [0;1), \\ 0, & x \in [1; a-1), \quad (a \geq 2, b > 0), \\ (x-a+1)^2, & x \in [a-1, a], \end{cases}$$

It is obtained that when $b \in (0;1) \cup (1;+\infty), a \geq 2,$, the subspace H_2 is a unicity space of the best (α, β)-approximations for continuous on the $[0;a]$ functions with values in the space $\mathbb{R}_p^m, p \in (1;+\infty)$. In case $b = 1$, $a \geq 4$ it is proved that the subspace H_2 is not a unicity subspace of the best non-symmetric approximations for these functions. Received the results summarize the previously obtained Strauss results for the real functions in the case $\alpha = \beta = 1$, as well as the results of Babenko and Glushko for the the best (α, β)-approximation for continuous functions on a segment with values in the space $\mathbb{R}_p^m, p \in (1;+\infty)$.

Key words: non-symmetric approximation, unicity space of the best non-symmetric approximations, vector-valued functions, integral metric

Анотація. У статті розглядаються питання єдиності елемента найкращого несиметричного L_1-наближення неперервних функцій зі значеннями у просторі $\mathbb{R}_p^m, p \in (1;+\infty)$ елементами двовимірного підпростору $H_2 = \mathrm{span}\{1, g_{a,b}\}$, де

$$g_{a,b}(x) = \begin{cases} -b \cdot (x-1)^2, & x \in [0;1), \\ 0, & x \in [1; a-1), \quad (a \geq 2, b > 0), \\ (x-a+1)^2, & x \in [a-1, a], \end{cases}$$

Отримано, що, коли $b \in (0;1) \cup (1;+\infty), a \geq 2$, підпростір H_2 є простором єдиності елемента найкращого (α, β)-наближення для неперервних на відрізку $[0;a]$ функцій зі значеннями у просторі $\mathbb{R}_p^m, p \in (1;+\infty)$. У випадку, коли $b = 1$, $a \geq 4$ доведено, що підпростір H_2 не є підпростором єдиності елемента найкращого несиметричного наближення для вказаних функцій. Отримані результати узагальнюють отримані раніше результати Штрауса для дійсних функцій у випадку $\alpha = \beta = 1$, а також результати Бабенка й Глушко на випадок найкращого (α, β)-наближення для неперервних на відрізку функцій зі значеннями у просторі $\mathbb{R}_p^m, p \in (1;+\infty)$.

Ключові слова: несиметричне наближення, простір єдиності елемента найкращого несиметричного наближення, векторнозначні функції, інтегральна метрика

MSC2020: Pri 41A52, Sec 41A65, 46B40

Let X be a partially ordered set and its order is consistent with algebraic operations.

The following definitions are given in [4] .

Let $E \subset X$ be a non-empty set. The element $y \in X$ is called supremum (infimum) of the set E and is denoted by $\sup E$ ($\inf E$) if the following conditions hold:

1) $x \le y$ ($x \ge y$) $\forall x \in E$;

2) for any element $z \in X$ such that $x \le z$ ($x \ge z$), it follows that $y \le z$ ($y \ge z$).

The supremum of the set E is denoted by $x_1 \vee x_2 \vee ... \vee x_n$ and the infimum of the set E is denoted by $x_1 \wedge x_2 \wedge ... \wedge x_n$ if the set E consists of elements $x_1, x_2, ..., x_n$.

Suppose in the space X for any two elements $x, y \in X$ there exists their supremum $x \vee y$; then the element $x_+ = x \vee 0$ is called the positive part of the element $x \in X$, the element $x_- = (-x) \vee 0$ is its negative part, and the element $|x| = x_+ + x_-$ is the module of the element x.

Let a order of a partially ordered vector space X is consistent with algebraic operations and for any two elements $x, y \in X$ there exists their supremum $x \vee y$. Then a space X is called a KN-lineal if in X the monotone norm is defined, i. e., $|x| \le |y| \Rightarrow ||x||_X \le ||y||_X$.

A KN-lineal is called a KN-space (or $K_\sigma N$-space) if for any (or any numbered) non-empty set bounded above or below there exists the its upper or lower bound respectively.

A $K_\sigma N$-space is called a KB-space if its norm satisfies two conditions:

1) $||x_n||_X \to 0$ if $x_n \downarrow 0$;

2) $||x_n||_X \to +\infty$ if $x_n \uparrow +\infty$ ($x_n \ge 0$).

Let Q be a metric compact set with metric ρ, Σ be a σ-field of Borel subsets of Q, μ be a non-atomic non-negative finite measure. Furthermore, assume that μ be positive on each non-empty open subset of Σ.

Let X be a KB-space with the norm $|| \cdot ||_X$.

By $C(Q, X)$ denote the space of continuous functions $f : Q \to X$.

For any $x \in Q$ and positive numbers α, β put

$$|f(x)|_{\alpha,\beta} = \alpha \cdot f_+(x) + \beta \cdot f_-(x),$$

$$||f(x)||_{X;\alpha,\beta} = ||\alpha \cdot f_+(x) + \beta \cdot f_-(x)||_X,$$

where $f_\pm(x) = (\pm f(x)) \vee 0$.

Suppose the space $C(Q, X)$ is supplied with the non-symmetric L_1-norm:

$$||f||_{1;\alpha,\beta} = \int\limits_Q ||f(x)||_{X;\alpha,\beta} d\mu(x).$$

For $f \in C(Q,X)$, $H \subset C(Q,X)$ the quantity

$$E(f,H)_{1;\alpha,\beta} = \inf_{g \in H} \|f-g\|_{1;\alpha,\beta} \tag{1}$$

is called the best (α,β)-approximation of a function f by a set H in the metric L_1. The function $g^* \in H$ is the best (α,β)-approximant of a function f by elements of a set H in the metric L_1 if g^* realizes the greatest lower bound in the equality (1). By Z_f denote the set of zeros for a function f, and $N_f = Q \setminus Z_f$.

For $f, g \in C(Q,X)$, $x \in Q$ put

$$\tau_-^{(\alpha,\beta)}(f(x), g(x))_X = \lim_{t \to -0} \frac{||(f+tg)(x)||_{X;\alpha,\beta} - ||f(x)||_{X;\alpha,\beta}}{t}.$$

For $\alpha = \beta = 1$ such functional was considered in [5] and [1].

The following theorem was proved in [3].

Theorem 1. ([3]) *Let H be a subspace of $C(Q,X)$. An element g^* is the best (α,β)-approximant of a function $f \in C(Q,X)$ by elements from H in the metric L_1 iff $\forall g \in H$*

$$\int_{N_{f-g^*}} \tau_-^{(\alpha,\beta)}(f-g^*, g)_X d\mu(x) \le \int_{Z_{f-g^*}} ||g(x)||_{X;\beta,\alpha} d\mu(x). \tag{2}$$

A normalized space X is called strictly convex if for any $x, y \in X$ such that $||x+y|| = ||x|| + ||y||$, it follows that there is $\lambda \in \mathbb{R}$ such that $y = \lambda \cdot x$.

Let X be a strictly convex KB-space with a strictly monotone norm, i.e., $|x| < |y| \Rightarrow ||x||_X < ||y||_X$.

Let H be a subspace of the space $C(Q,X)$. We set

$$H' = \{h \in C(Q,X) : \quad \exists g_h \in H \quad \forall x \in Q \quad h(x) = \pm g_h(x)\}.$$

Originaly such sets were introduced by Hans Strauss [2] for $X = \mathbb{R}$, $Q = [a,b]$.

Using the methods of Strauss [2] was the following theorem is proved in [3], it generalizes results of [2], [5], [6], [8].

Theorem 2. ([3]) *Let X be a strictly convex KB-space with a strictly monotone norm, H be a subspace of $C(Q,X)$. Each function $f \in C(Q,X)$ has at most one best (α,β)-approximant by elements from H iff each function $h \in H'$ has at most one best (α,β)-approximant by subspace H.*

The Corollary 1 follows from Theorem 1 and Theorem 2.

Corollary 1. *Let X be a strictly convex KB-space with a strictly monotone norm, H be a subspace of space $C(Q,X)$. Each function $f \in C(Q,X)$ has at most one the best (α,β)-approximant by elements from H iff for any function $h \in H' \setminus \{0\}$ there exists a function $g_0 \in H$ such that*

$$\int_{N_h} \tau_-^{(\alpha,\beta)}(h(x), g_0(x))_X d\mu(x) > \int_{Z_h} ||g_0(x)||_{X;\beta,\alpha} d\mu(x).$$

The results stated above extend the known results Strauss (see [2]) for the case of nonsymmetric approximation of functions from the space $C(Q,X)$. In 1994, Babenko and Glushko (see [6]) indicated another a class of functions that has the same characteristic property, but is constructed independently of the form of functions of the approximating subspace.Their results were also generalized to the case of an nonsymmetric approximation of the function from the space $C(Q,X)$ in [3]. Namely, have been proven the following results.

The subspaces of the following type are considered as approximating subspaces. Let $\{u_i(t)\}_{i=1}^n$ be a system of linearly independent functions from $C(Q,\mathbf{R})$. We set

$$H_n = \{p(x) = \sum_{i=1}^{n} a_i u_i(x) \ : \quad a_i \in X, \quad i = 1, ..., n\}.$$

Note that H_n is a subspace of weak dimension n. The weak dimension was introduced in [7].

The following theorem was proved in [3].

Theorem 3. ([3]) *Let X -be a KB-space. Then the subspace H_n of the space $C(Q,X)$ is the set of existence of the best (α,β)-approximant for any function $g \in C(Q,X)$.*

By $\omega(u,x)$ we denote the modulus of continuity of the function $u \in C(Q,X)$.

Let Q be a metrically convex compact set, i.e. , for any $x_0, x_1 \in Q$ and for any $\lambda \in (0;1)$ there exists a point $x_\lambda \in Q$ such that $\rho(x_0,x_\lambda) = \lambda\rho(x_0,x_1)$ and $\rho(x_\lambda,x_1) = (1-\lambda)\cdot\rho(x_0,x_1)$.

For $g \in C(Q,X)$ put $\overline{g}(t) = \frac{g(t)}{\|g(t)\|_X}$, if $t \in Q \backslash Z_g$, $\overline{g}(t) = 0$, if $t \in Z_g$.

Let also $\omega(x) = \max\limits_{i=1,...,n} \omega(u_i,x)$ and for a non-empty set $M \subset Q$

$$E(x,M) = \inf_{y \in M} \rho(x,y).$$

Put

$$H'' = \{h \in C(Q,X) \ : \quad \exists p_h \in H_n \quad \forall x \in Q \quad h(x) = \pm\overline{p}_h(x)\cdot\omega(E(x,Z_{p_h}))\}.$$

The following theorem is a generalization of Theorem 2 from [6] and Theorem 5 from [8] for the case of nonsymmetric approximation of functions from $C(Q,X)$ by elements of the subspace H_n.

Theorem 4. ([3]) *Let X be a strictly convex KB-space with strictly monotone norm, Q is a metrically convex compact. Each function $f \in C(Q,X)$ has a unique best (α,β)-approximant in H_n iff each function $h \in H''$ has a unique best (α,β)-approximant in H_n.*

Corollary 2. ([3]) *Let X be a strictly convex KB-space with a strictly monotone norm, Q is a metrically convex compact. Each function $f \in C(Q, X)$ has a unique the best (α,β)-approximant by elements from H_n iff for any function $h \in H'' \setminus \{0\}$ there exists a function $p \in H_n$ such that*

$$\int\limits_{N_h} \tau_-^{(\alpha,\beta)}(h(x), p(x))_X d\mu(x) > \int\limits_{Z_h} ||p(x)||_{X;\beta,\alpha} d\mu(x).$$

Now let $X = \mathbb{R}_p^m$ be a space of vectors $\mathbf{f} = (f^1, f^2, ..., f^m)$ with the norm

$$\|\mathbf{f}\|_{\mathbf{R}_p^m} = \left(\sum_{j=1}^{m} |f^j|^p \right)^{\frac{1}{p}}, \quad (1 < p < +\infty).$$

Let $\|\mathbf{f}\|_{\mathbf{R}_p^m;\alpha,\beta} = \left(\sum\limits_{j=1}^{m} |f^j|_{\alpha,\beta}^p \right)^{\frac{1}{p}}$ be (α,β)-norm.

The derivative $\tau_-^{(\alpha,\beta)}(f,g)$ in the space $\mathbb{R}_p^m$ will have the form

$$\tau_-^{(\alpha,\beta)}(f,g)(x) = \frac{\sum\limits_{j=1}^{m} g^j(x)|f^j(x)|_{\alpha,\beta}^{p-1} \cdot sgn_{\alpha,\beta} f^j(x)}{\left(\sum\limits_{j=1}^{m} |f^j(x)|_{\alpha,\beta}^p \right)^{1-1/p}}, \quad x \in Q \setminus Z_{\mathbf{f}},$$

where $sgn_{\alpha,\beta} f^j(x) = \alpha \cdot sgn f_+(x) - \beta \cdot sgn f_-(x)$, $\mathbf{f} = (f^1, f^2, ..., f^m)$, $\mathbf{g} = (g^1, g^2, ..., g^m) \in C(Q, \mathbf{R}_p^m)$.

In what follows, we consider the linear span of set of functions

$$g(x) = 1, \forall x \in [0, a],$$

$$g_{a,b}(x) = \begin{cases} -b \cdot (x-1)^2, & x \in [0;1), \\ 0, & x \in [1; a-1), \quad (a \geq 2, b > 0), \\ (x-a+1)^2, & x \in [a-1, a], \end{cases}$$

as an approximating subspace.

Such a subspace was considered by Strauss in [2]. He proved that span $\{g, g_{a,b}\}$ is a weakly Chebyshev subspace.

Then the subspace H_n can be written in the form

$$H_2 = \left\{ \mathbf{p} = (p^1, p^2, ..., p^m) : p^j(x) = c_1^j + c_2^j \cdot g_{a,b}(x) \right\}.$$

Let's consider the cases:

1) $b \neq 1, a \geq 2$. Let us show that in this case, for any function $f \in C([0;a], \mathbb{R}_p^m)$ there is unique the best (α,β)-approximant in H_2 in the metric L_1, i.e. that $\forall \mathbf{h} \in H'' \setminus \{0\} \quad \exists \mathbf{p}_0 \in H_2 :$

$$\int\limits_{N_h} \tau_-^{(\alpha,\beta)}(\mathbf{h}(x), \mathbf{p}_0(x))_X dx > \int\limits_{Z_{\mathbf{h}}} ||\mathbf{p}_0(x)||_{\mathbf{R}_p^m;\beta,\alpha} dx.$$

Let us introduce the notation:

$$I = \left\{ j : \quad h^j(x) \geq 0, \quad \forall x \in [0; a] \right\},$$

$$J = \left\{ j : \quad h^j(x) \leq 0, \quad \forall x \in [0; a] \right\},$$

$$M = \left\{ j : \quad h^j(x) \quad \text{has a change of sign on} \quad [0; a] \right\}.$$

Let $M \neq \varnothing$. Then for $j \in M \ \forall h^j \quad \exists p_h^j : \quad \operatorname{sgn} h^j(x) = \operatorname{sgn} p_h^j(x), \forall x \in [0; a]$. Take $\mathbf{p}_0 = (p_0^1, ..., p_0^m)$, where

$$p_0^j = \begin{cases} p_h^j, & j \in M, \\ 0, & j \notin M. \end{cases}$$

Then, since $M \neq \varnothing$, then $\mathbf{p}_0 \neq \mathbf{0}$ and

$$\int\limits_{N_h} \tau_-^{(\alpha,\beta)}(\mathbf{h}(x), \mathbf{p}_0(x))_X dx = \int\limits_{[0;a] \setminus Z_{\mathbf{h}}} \frac{\sum\limits_{j=1}^{m} p_0^j(x) |h^j(x)|_{\alpha,\beta}^{p-1} \cdot sgn_{\alpha,\beta} h^j(x)}{\left(\sum\limits_{j=1}^{m} |h^j(x)|_{\alpha,\beta}^p \right)^{1-1/p}} dx =$$

$$= \int\limits_{[0;a] \setminus Z_{\mathbf{h}}} \frac{\sum\limits_{j \in M} p_h^j(x) |h^j(x)|_{\alpha,\beta}^{p-1} \cdot sgn_{\alpha,\beta} h^j(x)}{\left(\sum\limits_{j=1}^{m} |h^j(x)|_{\alpha,\beta}^p \right)^{1-1/p}} dx = \int\limits_{[0;a] \setminus Z_{\mathbf{h}}} \frac{\sum\limits_{j \in M} |p_h^j(x)|_{\alpha,\beta} \cdot |h^j(x)|_{\alpha,\beta}^{p-1}}{\left(\sum\limits_{j=1}^{m} |h^j(x)|_{\alpha,\beta}^p \right)^{1-1/p}} dx > 0.$$

On the other hand, $Z_{\mathbf{h}} = Z_{p_h^1} \cap Z_{p_h^2} \cap ... \cap Z_{p_h^m}$. If $\exists j_0 : \quad p_h^{j_0} = c_1^{j_0} + c_2^{j_0} \cdot g_{a,b}(x)$ and $c_1^{j_0} \neq 0$, then $\operatorname{card} Z_{\mathbf{h}} \leq 1$. If for all $j = 1, 2, ..., m$ $p_h^j(x) = c_2^j \cdot g_{a,b}(x)$, then $Z_{p_h^j} = [1; a-1], j = 1, 2, ..., m$ and $Z_{\mathbf{h}} = [1; a-1]$. Therefore

$$\int\limits_{Z_{\mathbf{h}}} \left(\sum\limits_{j=1}^{m} |p_0^j(x)|_{\beta,\alpha}^p \right)^{1/p} dx = \int\limits_{Z_{\mathbf{h}}} \left(\sum\limits_{j \in M} |p_h^j(x)|_{\beta,\alpha}^p \right)^{1/p} dx = 0.$$

Now let $M = \varnothing$. Consider an arbitrary function $h \in H'' \setminus \{0\}$. By the definition of the set H'' exist $\mathbf{p}_h = (p_h^1, p_h^2, ..., p_h^m)$ such that

$$h^j(x) = \begin{cases} \pm \frac{p_h^j(x)}{||\mathbf{p_h}(x)||_{\mathbf{R}_p^m}} \cdot \omega(E(x, Z_{\mathbf{p_h}})), & x \notin Z_{\mathbf{p_h}}, \\ 0, & x \in Z_{\mathbf{p_h}}. \end{cases}$$

Let's consider two cases:

(a) If among the indices $j = 1, 2, ..., m \quad \exists j_0$ such that $p_h^{j_0} = c_1^{j_0} + c_2^{j_0} \cdot g_{a,b}(x)$ and $c_1^{j_0} \neq 0$, then $card Z_{\mathbf{h}} \leq 1$. Then, as $\mathbf{p}_0 = (p_0^1, p_0^2, ..., p_0^m)$ we take: for $j \in I$ $p_0^j(x) = p_1(x)$, where $p_1(x)$ is any positive function on $[0; a]$ from H_2; for $j \in J$ $p_0^j(x) = p_2(x)$, where $p_2(x)$ is any negative function on $[0; a]$ from H_2.

Then $\forall j \in \{1, 2, ..., m\} \quad p_0^j(x) \cdot sgn_{\alpha,\beta} h^j(x) = |p_0^j(x)|_{\alpha,\beta}$ a.e. by $[0; a] \setminus Z_{\mathbf{h}}$.

Then for $h \in H'' \setminus \{0\}$ we have

$$\int\limits_{N_h} \tau_-^{(\alpha,\beta)}(\mathbf{h}(x), \mathbf{p}_0(x))_X dx = \int\limits_{[0;a]\setminus Z_{\mathbf{h}}} \frac{\sum\limits_{j=1}^{m} p_0^j(x)|h^j(x)|_{\alpha,\beta}^{p-1} \cdot sgn_{\alpha,\beta} h^j(x)}{\left(\sum\limits_{j=1}^{m} |h^j(x)|_{\alpha,\beta}^p\right)^{1-1/p}} dx =$$

$$= \int\limits_{[0;a]\setminus Z_{\mathbf{h}}} \frac{\sum\limits_{j=1}^{m} |p_0^j(x)|_{\alpha,\beta} \cdot |h^j(x)|_{\alpha,\beta}^{p-1}}{\left(\sum\limits_{j=1}^{m} |h^j(x)|_{\alpha,\beta}^p\right)^{1-1/p}} dx > 0.$$

On the other hand, since $card Z_{\mathbf{h}} \leq 1$, then

$$\int\limits_{Z_{\mathbf{h}}} \left(\sum_{j=1}^{m} |p_0^j(x)|_{\beta,\alpha}^p\right)^{1/p} dx = 0.$$

By Corollary 2 H_2 is the subspace of uniqueness of the best (α,β)-approximant for $C([0;a], \mathbb{R}_p^m)$.

(b) If for all indices $j = 1, 2, ..., m$ $\quad p_h^j(x) = c_2^j \cdot g_{a,b}(x)$, then

$$h^j(x) = \begin{cases} \pm \frac{c_2^j}{||\mathbf{c}_2||_{\mathbb{R}_p^m}} \cdot \omega(E(x, Z_{g_{a,b}})), & x \notin Z_{g_{a,b}}, \\ 0, & x \in Z_{g_{a,b}}, \end{cases}$$

where $\mathbf{c}_2 = (c_2^1, ..., c_2^m)$. In this case, for $b \in (0;1)$ we choose the function $\mathbf{p}_0 = (p_0^1, ..., p_0^m)$ such that

$$p_0^j(x) = \begin{cases} g_{a,b}, & j \in I, \\ -g_{a,b}, & j \in J, \end{cases}$$

and for $b \in (1;+\infty)$ such that

$$p_0^j(x) = \begin{cases} -g_{a,b}, & j \in I, \\ g_{a,b}, & j \in J, \end{cases}$$

Now for $b \in (0;1)$

$$\int\limits_{[0;a]\setminus Z_{\mathbf{h}}} \frac{\sum\limits_{j=1}^{m} p_0^j(x)|h^j(x)|_{\alpha,\beta}^{p-1} \cdot sgn_{\alpha,\beta} h^j(x)}{\left(\sum\limits_{j=1}^{m} |h^j(x)|_{\alpha,\beta}^p\right)^{1-1/p}} dx =$$

$$= \int\limits_{[0;a]\setminus Z_{\mathbf{h}}} \frac{\alpha g_{a,b}(x) \sum\limits_{j\in I} |h^j(x)|^{p-1}_{\alpha,\beta} + \beta g_{a,b}(x) \sum\limits_{j\in J} |h^j(x)|^{p-1}_{\alpha,\beta}}{\left(\sum\limits_{j=1}^{m} |h^j(x)|^{p}_{\alpha,\beta}\right)^{1-1/p}} dx =$$

$$= \int\limits_0^1 \frac{-\alpha \cdot b \cdot (x-1)^2 \sum\limits_{j\in I} \left(\alpha \cdot |c_2^j|\right)^{p-1} + \beta(-b\cdot(x-1)^2)\cdot \sum\limits_{j\in J} \left(\beta \cdot |c_2^j|\right)^{p-1}}{\left(\sum\limits_{j\in I} \left(\alpha \cdot |c_2^j|\right)^{p} + \sum\limits_{j\in J} \left(\beta \cdot |c_2^j|\right)^{p}\right)^{1-1/p}} dx +$$

$$+ \int\limits_{a-1}^{a} \frac{-\alpha \cdot (x-a+1)^2 \sum\limits_{j\in I} \left(\alpha \cdot |c_2^j|\right)^{p-1} + \beta \cdot (x-a+1)^2 \cdot \sum\limits_{j\in J} \left(\beta \cdot |c_2^j|\right)^{p-1}}{\left(\sum\limits_{j\in I} \left(\alpha \cdot |c_2^j|\right)^{p} + \sum\limits_{j\in J} \left(\beta \cdot |c_2^j|\right)^{p}\right)^{1-1/p}} dx =$$

$$= \frac{-b\cdot\left(\alpha^p \cdot \sum\limits_{j\in I} |c_2^j|^{p-1} + \beta^p \cdot \sum\limits_{j\in J} |c_2^j|^{p-1}\right)}{\left(\sum\limits_{j\in I} \left(\alpha \cdot |c_2^j|\right)^{p} + \sum\limits_{j\in J} \left(\beta \cdot |c_2^j|\right)^{p}\right)^{1-1/p}} \cdot \int\limits_0^1 (x-1)^2 dx +$$

$$+ \frac{\alpha^p \cdot \sum\limits_{j\in I} |c_2^j|^{p-1} + \beta^p \cdot \sum\limits_{j\in J} |c_2^j|^{p-1}}{\left(\sum\limits_{j\in I} \left(\alpha \cdot |c_2^j|\right)^{p} + \sum\limits_{j\in J} \left(\beta \cdot |c_2^j|\right)^{p}\right)^{1-1/p}} \cdot \int\limits_{a-1}^{a} (x-a+1)^2 dx =$$

$$= \frac{1}{3} \cdot \frac{\alpha^p \cdot \sum\limits_{j\in I} |c_2^j|^{p-1} + \beta^p \cdot \sum\limits_{j\in J} |c_2^j|^{p-1}}{\left(\sum\limits_{j\in I} \left(\alpha \cdot |c_2^j|\right)^{p} + \sum\limits_{j\in J} \left(\beta \cdot |c_2^j|\right)^{p}\right)^{1-1/p}} \cdot (1-b) > 0.$$

For $b > 1$

$$\int\limits_{[0;a]\setminus Z_{\mathbf{h}}} \frac{\sum\limits_{j=1}^{m} p_0^j(x) |h^j(x)|^{p-1}_{\alpha,\beta} \cdot sgn_{\alpha,\beta} h^j(x)}{\left(\sum\limits_{j=1}^{m} |h^j(x)|^{p}_{\alpha,\beta}\right)^{1-1/p}} dx =$$

$$= -\int\limits_{[0;a]\setminus Z_{\mathbf{h}}} \frac{\alpha g_{a,b}(x) \sum\limits_{j\in I} |h^j(x)|^{p-1}_{\alpha,\beta} + \beta g_{a,b}(x) \sum\limits_{j\in J} |h^j(x)|^{p-1}_{\alpha,\beta}}{\left(\sum\limits_{j=1}^{m} |h^j(x)|^{p}_{\alpha,\beta}\right)^{1-1/p}} dx =$$

$$= \frac{1}{3} \cdot \frac{\alpha^p \cdot \sum\limits_{j\in I} |c_2^j|^{p-1} + \beta^p \cdot \sum\limits_{j\in J} |c_2^j|^{p-1}}{\left(\sum\limits_{j\in I} \left(\alpha \cdot |c_2^j|\right)^p + \sum\limits_{j\in J} \left(\beta \cdot |c_2^j|\right)^p \right)^{1-1/p}} \cdot (b-1) > 0.$$

On the other hand, for $b \in (0;1)\cup(1;+\infty)$ $\quad p_0^j(x) = 0, \forall x \in [1; a-1], j = 1, 2, ..., m,$ $Z_{\mathbf{h}} = [1; a-1]$ and $\int\limits_{Z_{\mathbf{h}}} \left(\sum\limits_{j=1}^m |p_0^j(x)|^p_{\beta,\alpha} \right)^{1/p} dx = 0.$

Therefore, the subspace H_2 for all $b \in (0;1) \cup (1 : +\infty)$ is the uniqueness space of the best (α, β)-approximations for the functions from the space $C([0;a], \mathbb{R}^m_p)$.

2) Now let $b = 1, a \geq 4$.

In this case, the subspace H_2 is not a uniqueness set of the best non-symmetric approximations for functions from space $C([0;a], \mathbb{R}^m_p)$ in the metric L_1. Let us show this using Corollary 2, that is, i.e. we show that there exists a function $\mathbf{h} = (h^1, h^2, ..., h^m) \in H''$ such that for any $\mathbf{p} = (p^1, p^2, ..., p^m) \in H_2$ the condition

$$\int\limits_{N_h} \tau_-^{(\alpha,\beta)}(h(x), p(x))_X dx \leq \int\limits_{Z_h} ||p(x)||_{X;\beta,\alpha} dx$$

is true.

Take $\mathbf{h} = (h^1, h^2, ..., h^m)$, in which

$$h^j(x) = \frac{1}{m^{1/p}} \cdot \omega(E(x, Z_{g_{a,1}})), \quad j = 1, 2, ..., m.$$

It is clear that $\mathbf{h} \in H''$, since as $\mathbf{p_h}$ we can take a vector function of the form $\mathbf{p_h} = (g_{a,1}, g_{a,1}, ..., g_{a,1}) \in H_2$. Then $\|\mathbf{p_h}\|_{\mathbb{R}^m_p} = \left(\sum\limits_{j=1}^m |g_{a,1}(x)|^p \right)^{\frac{1}{p}} = m^{1/p} \cdot |g_{a,1}(x)|$ and $\frac{p_h^j(x)}{\|\mathbf{p_h}\|_{\mathbb{R}^m_p}} = \frac{g_{a,1}(x)}{m^{1/p}|g_{a,1}(x)|} = m^{-1/p} \cdot sgn g_{a,1}(x).$

Note that $Z_{\mathbf{h}} = Z_{\omega(E(x, Z_{g_{a,1}}))} = Z_{g_{a,1}} = [1; a-1]$, and also that $h^j(x) \geq 0, \forall x \in [0; a], j = 1, 2, ..., m.$

Now, for the indicated function $\mathbf{h}$ and an arbitrary function $\mathbf{p} = (p^1, p^2, ..., p^m) \in H_2$, where $p^j(x) = c_1^j + c_2^j \cdot g_{a,1}(x)$, we have

$$\int\limits_{[0;a]\setminus Z_{\mathbf{h}}} \tau_-^{(\alpha,\beta)}(h(x), p(x))_X dx = \int\limits_{[0;a]\setminus Z_{\mathbf{h}}} \frac{\sum\limits_{j=1}^m p^j(x) |h^j(x)|^{p-1}_{\alpha,\beta} \cdot sgn_{\alpha,\beta} h^j(x)}{\left(\sum\limits_{j=1}^m |h^j(x)|^p_{\alpha,\beta} \right)^{1-1/p}} dx =$$

$$= \int\limits_0^1 \frac{\sum\limits_{j=1}^m (c_1^j - c_2^j (x-1)^2) \cdot (\alpha \cdot m^{-1/p} \omega(E(x, Z_{g_{a,1}})))^{p-1} \cdot \alpha}{\left(\sum\limits_{j=1}^m (\alpha \cdot m^{-1/p} \omega(E(x, Z_{g_{a,1}})))^p \right)^{1-1/p}} dx +$$

$$+\int\limits_{a-1}^{a}\frac{\sum\limits_{j=1}^{m}(c_1^j+c_2^j(x-a+1)^2)\cdot(\alpha\cdot m^{-1/p}\omega(E(x,Z_{g_{a,1}})))^{p-1}\cdot\alpha}{\left(\sum\limits_{j=1}^{m}(\alpha\cdot m^{-1/p}\omega(E(x,Z_{g_{a,1}})))^p\right)^{1-1/p}}dx=$$

$$=\frac{\alpha}{m^{1-1/p}}\cdot\int\limits_{0}^{1}\sum_{j=1}^{m}(c_1^j-c_2^j(x-1)^2)dx+\frac{\alpha}{m^{1-1/p}}\cdot\int\limits_{a-1}^{a}\sum_{j=1}^{m}(c_1^j+c_2^j(x-a+1)^2)dx=$$

$$=\frac{\alpha}{m^{1-1/p}}\cdot\sum_{j=1}^{m}2c_1^j\leq\frac{2\alpha}{m^{1-1/p}}\cdot\sum_{j=1}^{m}|c_1^j|$$

Let $M_+=\{j:\quad c_1^j\geq 0\}$, $M_-=\{j:\quad c_1^j<0\}$, $b^j=1$, for $j\in M_+$, $b^j=-1$, for $j\in M_-$.

Then the equality

$$\sum_{j=1}^{m}|c_1^j|=\sum_{j=1}^{m}|c_1^j|_{\beta,\alpha}\cdot|b^j|_{\beta^{-1},\alpha^{-1}}$$

is true.

Next, we apply Hölder's inequality, which for non-symmetric norms has view:

$$\sum_{j=1}^{m}|x_i|_{\alpha,\beta}\cdot|y_i|_{\alpha^{-1},\beta^{-1}}\leq\left(\sum_{j=1}^{m}|x_i|^p_{\alpha,\beta}\right)^{\frac{1}{p}}\cdot\left(\sum_{j=1}^{m}|y_i|^q_{\alpha^{-1},\beta^{-1}}\right)^{1/q},$$

where $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\sum_{j=1}^{m}|c_1^j|_{\beta,\alpha}\cdot|b^j|_{\beta^{-1},\alpha^{-1}}\leq\left(\sum_{j=1}^{m}|c_1^j|^p_{\beta,\alpha}\right)^{1/p}\cdot\left(\sum_{j=1}^{m}|b^j|^q_{\beta^{-1},\alpha^{-1}}\right)^{1/q}=$$

$$=\left(\sum_{j=1}^{m}|c_1^j|^p_{\beta,\alpha}\right)^{1/p}\cdot\left(\sum_{j\in M_+}(\beta^{-1})^q+\sum_{j\in M_-}(\alpha^{-1})^q\right)^{1/q}=$$

$$=\left(\sum_{j=1}^{m}|c_1^j|^p_{\beta,\alpha}\right)^{1/p}\cdot\left(\frac{1}{\beta^q}cardM_++\frac{1}{\alpha^q}cardM_-\right)^{1/q}\leq$$

$$\leq\left(\sum_{j=1}^{m}|c_1^j|^p_{\beta,\alpha}\right)^{1/p}\cdot\left(\max\left(\frac{1}{\beta^q};\frac{1}{\alpha^q}\right)\right)^{1/q}\cdot m^{1/q}=$$

$$=\left(\sum_{j=1}^{m}|c_1^j|^p_{\beta,\alpha}\right)^{1/p}\cdot\max\left(\frac{1}{\beta};\frac{1}{\alpha}\right)\cdot m^{1/q}.$$

Then we get

$$\int\limits_{[0;a]\setminus Z_{\mathbf{h}}} \tau_-^{(\alpha,\beta)}(h(x),p(x))_X dx \leq \frac{2\alpha}{m^{1-1/p}} \cdot \sum_{j=1}^{m} |c_1^j| \leq$$

$$\leq 2\alpha \cdot \left(\sum_{j=1}^{m} |c_1^j|^p_{\beta,\alpha} \right)^{1/p} \cdot \max\left(\frac{1}{\beta}; \frac{1}{\alpha} \right).$$

On the other hand,

$$\int\limits_{Z_h} ||p(x)||_{\mathbb{R}_p^m;\beta,\alpha} dx = \int\limits_{1}^{a-1} \left(\sum_{j=1}^{m} |c_1^j|^p_{\beta,\alpha} \right)^{1/p} dx = (a-2) \left(\sum_{j=1}^{m} |c_1^j|^p_{\beta,\alpha} \right)^{1/p}.$$

Comparing the values of the last two integrals, we find that for $a \geq 2 + 2\alpha \cdot \max\left(\frac{1}{\beta}; \frac{1}{\alpha}\right)$ there are functions from the space $C([0;a], \mathbb{R}_p^m)$ that have at least two the best (α,β)-approximants in the subspace H_2 in metric L_1.

Further, let us take as $\mathbf{h} = (h^1, h^2, ..., h^m)$, where

$$h^j(x) = -\frac{1}{m^{1/p}} \cdot \omega(E(x, Z_{g_{a,1}})), \quad j = 1, 2, ..., m,$$

and carry out similar reasoning. We get

$$\int\limits_{[0;a]\setminus Z_{\mathbf{h}}} \tau_-^{(\alpha,\beta)}(h(x),p(x))_X dx =$$

$$= \int\limits_{0}^{1} \frac{\sum\limits_{j=1}^{m} (c_1^j - c_2^j (x-1)^2) \cdot (\beta \cdot m^{-1/p} \omega(E(x, Z_{g_{a,1}})))^{p-1} \cdot (-\beta)}{\left(\sum\limits_{j=1}^{m} (\beta \cdot m^{-1/p} \omega(E(x, Z_{g_{a,1}})))^p \right)^{1-1/p}} dx +$$

$$+ \int\limits_{a-1}^{a} \frac{\sum\limits_{j=1}^{m} (c_1^j + c_2^j (x-a+1)^2) \cdot (\beta \cdot m^{-1/p} \omega(E(x, Z_{g_{a,1}})))^{p-1} \cdot (-\beta)}{\left(\sum\limits_{j=1}^{m} (\beta \cdot m^{-1/p} \omega(E(x, Z_{g_{a,1}})))^p \right)^{1-1/p}} dx =$$

$$= -\frac{\beta}{m^{1-1/p}} \cdot \sum_{j=1}^{m} 2c_1^j \leq \frac{2\beta}{m^{1-1/p}} \cdot \sum_{j=1}^{m} |c_1^j| \leq 2\beta \cdot \left(\sum_{j=1}^{m} |c_1^j|^p_{\beta,\alpha} \right)^{1/p} \cdot \max\left(\frac{1}{\beta}; \frac{1}{\alpha} \right).$$

On the other hand, also

$$\int\limits_{Z_h} ||p(x)||_{\mathbb{R}_p^m;\beta,\alpha} dx = (a-2) \left(\sum_{j=1}^{m} |c_1^j|^p_{\beta,\alpha} \right)^{1/p}.$$

Now, after comparing the values of the last two integrals, we get that for $a \geq 2 + 2\beta \cdot \max\left(\frac{1}{\beta}; \frac{1}{\alpha}\right)$ there also exist functions from the space two $C([0;a], \mathbb{R}_p^m)$ that have at least two the best (α, β)-approximants in the subspace H_2 in the metric L_1.

Combining the obtained intervals, we get that the subspace H_2 is not the unicity set of the best (α, β)-approximations for the space $C([0;a], (R)_p^m)$, in the case when $a \geq 2 + 2 \cdot \min\{\alpha; \beta\} \cdot \max\left(\frac{1}{\beta}; \frac{1}{\alpha}\right)$ or, which is the same, $a \geq 4$.

Thus, we got a statement.

Theorem 5. *Subspace H_2*

1) is a subspace of uniqueness of the best (α, β)-approximants for functions from the space $C([0;a], \mathbb{R}_p^m)$ in the metric L_1 for all $b \in (0;1) \cup (1 : +\infty), a \in [2 : +\infty)$;

2) is not a subspace of uniqueness of the best (α, β)-approximants for functions from the space $C([0;a], \mathbb{R}_p^m)$ in the metric L_1 for $b = 1$, $a \geq 4$.

A similar result was obtained by H. Strauss in [2] for real functions and $\alpha = \beta = 1$, V.F. Babenko, V.N. Glushko [6] for (α, β)-approximation of real functions. Our given results were extended to the case of non-symmetric approximation of vector-functions with values in the space $\mathbb{R}_p^m, (p \in (1; +\infty))$.

Our result was not obtained for all values $a \geq 2$ for $b = 1$. For the case $b = 1, 2 \leq a < 4$, the question of the uniqueness of the best non-symmetric L_1-approximant for continuous functions by H_2 has not yet been studied.

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Received: 20.05.2021. *Accepted:* 27.06.2021