УДК 512.544

V. A. Chupordia*

* *Днепропетровский национальный университет им. Олеся Гончара,
Днепропетровск 49010. E-mail: vchupordya@mail.ru*

On the structure of artinian-by-(finite rank) modules over generalized soluble groups

Нехай R – кільце, G – група. R-модуль A називається модулем скінченного рангу над артіновим, якщо $\mathrm{Tor}_R(A)$ є артіновим і $A/\mathrm{Tor}_R(A)$ має скінченний R-ранг. Досліджуються модулі A над груповими кільцями $\mathbb{Z}_{p^\infty}G$ такі, що $A/C_A(H)$ є модулем скінченного рангу над артіновим (як $\mathbb{Z}_{p^\infty}$-модуль) для кожної власної підгрупи H.

Ключові слова: модуль, групове кільце, модуль над груповим кільцем, узагальнено розв'язна група, радикал групи, артіновий модуль, узагальнено радикальна група, модуль скінченного рангу.

Пусть R – кольцо, G – группа. R-модуль A называется модулем конечного ранга над артиновым, если $\mathrm{Tor}_R(A)$ является артиновым и $A/\mathrm{Tor}_R(A)$ имеет конечный R-ранг. Исследуются модули A над групповыми кольцами $\mathbb{Z}_{p^\infty}G$ такие, что $A/C_A(H)$ является модулем конечного ранга над артиновым (как $\mathbb{Z}_{p^\infty}$-модуль) для кажной собственной подгруппы H.

Ключевые слова: модуль, групповое кольцо, модуль над групповым кольцом, обобщенно разрешимая группа, радикал группы, артинов модуль, обобщенно радикальная группа, модуль конечного ранга.

Let R be a ring and G a group. An R-module A is said to be *artinian-by-(finite rank)*, if $\mathrm{Tor}_R(A)$ is artinian and $A/\mathrm{Tor}_R(A)$ has finite R-rank. In this paper modules A over a group ring $\mathbb{Z}_{p^\infty}G$ such that $A/C_A(H)$ is artinian-by-(finite rank) (as an $\mathbb{Z}_{p^\infty}$-module) for every proper subgroup H are investigated.

Key words: modules, group rings, modules over group rings, generalized soluble groups, radical groups, artinian modules, generalized radical groups, modules of finite rank.

1. Introduction

Let R be a ring, G a group and A an RG–module. The modules over group rings are classic objects of study with well established links to various areas of algebra. The case when G is a finite group has been studying in sufficient details for a long time. For the case when G is an infinite group, the situation is different. Thus study modules over group rings of infinite groups requires some different approaches and restrictions. For instance, the classical finiteness conditions are largely employed and popular. The very first restrictions here were those who came from ring theory, namely the conditions like "to be noetherian"and "to be artinian". Noetherian and artinian modules over group rings are also very well investigated. Many aspects of the theory of artinian modules

over group rings are treated in the book [5]. Recently the so–called finitary approach begun to be employed intensively in the theory of infinite dimensional linear groups where it brings many interesting promising results.

Let A be a module over group ring RG, if H is a subgroup of G, then consider the centralizer $C_A(H) = \{a \in A \mid ah = a$ for each element $h \in H\}$ of H in A. Clearly $C_A(H)$ is an RH-submodule of A and H really acts on $A/C_A(H)$. The R-factor-module $A/C_A(H)$ is called the *cocentralizer of H in A*. Then $H/C_H(A/C_A(H))$ is isomorphic to a subgroup of automorphism group of an R-module $A/C_A(H)$. It is not hard to see that $C_H(A/C_A(H))$ is abelian, and therefore the structure of the automorphism group of the R–module $A/C_A(H)$ defines the structure of whole group H.

Let $\mathfrak{M}$ be a class of R-modules. We say that A is an $\mathfrak{M}$-*finitary module over RG*, if $A/C_A(x) \in \mathfrak{M}$ for each element $x \in G$. If R is a field, $C_G(A) = \langle 1 \rangle$ and $\mathfrak{M}$ is a class of all finite dimensional vector spaces over R, then we come to the finitary linear groups. The theory of finitary linear groups is quite well developed (see, the survey [9]). B.A.F. Wehrfritz began to consider the cases when $\mathfrak{M}$ is the class of finite R-modules [11, 13, 14, 16], when $\mathfrak{M}$ is the class of noetherian R-modules [12], when $\mathfrak{M}$ is the class of artinian R-modules [14, 15, 16, 17, 18]. The artinian-finitary modules have been considered also in the paper [6]. The notion of an minimax module extends the notions of noetherian and artinian modules. An R-module A is said to be *minimax*, if A has a finite series of submodules, whose factors are either noetherian or artinian. It is not hard to show that if R is an integral domain, then every minimax R-module A includes a noetherian submodule B such that A/B is artinian. The first natural case here is the case when $R = \mathbb{Z}$ is the ring of all integers. This case has very important applications in generalized soluble groups. Every $\mathbb{Z}$-minimax module M has he following important property: $\mathbf{r}_{\mathbb{Z}}(M)$ is finite and $\mathbf{Tor}(M)$ is an artinian $\mathbb{Z}$-module.

Let R be an integral domain and A be an R-module. An analogue of the concept of a dimension for modules over integral domains is the concept of R-rank. One of the essential differences of R-modules and vector spaces is that some elements of A can have a non-zero annihilator in the ring. Put $\mathbf{Tor}_R(A) = \{a \in A \mid \mathbf{Ann}_R(a) \neq \langle 0 \rangle\}$. It is not hard to see that $\mathbf{Tor}_R(A)$ is an R-submodule of A. Actually, the concept of R-rank works only for the factor-module $A/\mathbf{Tor}_R(A)$. In particular, the finiteness of R-rank does not affect the submodule $\mathbf{Tor}_R(A)$. We say that an R-module A is an *artinian-by-(finite rank)*, if $\mathbf{Tor}_R(A)$ is artinian and $A/\mathbf{Tor}_R(A)$ has finite R-rank. In particular, if an artinian-by-(finite rank) module A is R-torsion-free, then it could be embedded into a finite dimensional vector space (over the field of fractions of R). If A is R-periodic, then it is artinian.

Let G be a group, A an RG-module, and $\mathfrak{M}$ a class of R-modules. Put

$$\mathcal{C}_{\mathfrak{M}}(G) = \{H \mid H \text{ is a subgroup of } G \text{ such that } A/C_A(H) \in \mathfrak{M}\}$$

If A is an $\mathfrak{M}$-finitary module, then $\mathcal{C}_{\mathfrak{M}}(G)$ contains every cyclic subgroup (moreover, every finitely generated subgroup whenever $\mathfrak{M}$ satisfies some natural restrictions). It is clear that the structure of G depends significantly on which important subfamilies of the family $\Lambda(G)$ of all proper subgroups of G include $\mathcal{C}_{\mathfrak{M}}(G)$. The first natural

question that arises here is the following: What is the structure of a group G in which $\Lambda(G) = \mathcal{C}_{\mathfrak{M}}(G)$ (in other words, the cocentralizer of every proper subgroup of G belongs to $\mathfrak{M}$)? In [7] it was considered the case when $R = \mathbb{Z}$ and $\mathfrak{M}$ is the class of all artinian-by-(finite rank) modules. The next natural generalization of the case when $R = \mathbb{Z}$ is the case when $R = \mathbb{Z}_{p^\infty}$ – the ring of p-adic integer, where p is prime. The proofs of the results we used the same technique as in [7].

Recall that a group G is called *generalized radical*, if G has an ascending series whose factors are locally nilpotent or locally finite.

The following results were obtained:

Theorem 1. *Let G be a locally generalized radical group and A a $\mathbb{Z}_{p^\infty}G$-module. If the factor-module $A/C_A(H)$ is artinian-by-(finite rank) for every proper subgroup H of G, then either $A/C_A(G)$ is artinian-by-(finite rank) or $G/C_G(A)$ is a cyclic or quasicyclic q-group for some prime q.*

Corollary 1. *Let G be a locally generalized radical group and A a $\mathbb{Z}_{p^\infty}G$-module. If a factor-module $A/C_A(H)$ is minimax for every proper subgroup H of G, then either $A/C_A(G)$ is minimax or $G/C_G(A)$ is a cyclic or quasicyclic q-group for some prime q.*

Corollary 2. *Let G be a locally generalized radical group and A a $\mathbb{Z}_{p^\infty}G$-module. If a factor-module $A/C_A(H)$ is finitely generated for every proper subgroup H of G, then either $A/C_A(G)$ is finitely generated or $G/C_G(A)$ is a cyclic or quasicyclic q-group for some prime q.*

Corollary 3. *Let G be a locally generalized radical group and A a $\mathbb{Z}_{p^\infty}G$-module. If a factor-module $A/C_A(H)$ is artinian for every proper subgroup H of G, then either $A/C_A(G)$ is artinian or $G/C_G(A)$ is a cyclic or quasicyclic q-group for some prime q.*

2. Some preparatory results

Lemma 1 ([7]). *Let R be a ring, G a group and A an RG–module. If L, H are subgroups of G, whose cocentralizers are artinian–by–(finite rank) modules, then $A/C_A(\langle H, L \rangle)$ is also artinian–by–(finite rank).*

A group G is said to be $\mathfrak{F}$-*perfect* if G does not include proper subgroups of finite index.

Let G be a generalized radical group. Then either G has an ascendant locally nilpotent subgroup or it has an ascendant locally finite subgroup. In the first case, the locally nilpotent radical $\mathbf{Lnr}(G)$ of G is non-identity. In the second case, it is not hard to see that G includes a non-trivial normal locally finite subgroup. Clearly in every group G the subgroup $\mathbf{Lfr}(G)$ generated by all normal locally finite subgroups is the largest normal locally finite subgroup (the *locally finite radical*). Thus every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors.

Observe also that a periodic generalized radical group is locally finite, and hence a periodic locally generalized radical group is also locally finite.

Let q be a prime and A is an additive abelian q-group. For each positive integer n we define n^{th} layer $\Omega_n(A)$ by the following rule: $\Omega_n(A) = \{a \in A \mid q^n a = 0\}$. Clearly $\Omega_n(A)$ is a characteristic subgroup of A.

Futher by $\mathbf{Dr}_{\lambda \in \Lambda} G_\lambda$ we denote a direct product of groups G_λ, $\lambda \in \Lambda$.

Lemma 2. *Let G be a locally generalized radical group and A be a $\mathbb{Z}_{p^\infty} G$-module. Suppose that A includes a $\mathbb{Z}_{p^\infty} G$-submodule B which is artinian-by-(finite rank). Then the following assertions hold:*

(i) $G/C_G(B)$ is soluble-by-finite.

(ii) If $G/C_G(B)$ is periodic, then it is nilpotent-by-finite.

(iii) If $G/C_G(B)$ is $\mathfrak{F}$-perfect and periodic, then it is abelian. Moreover $[[B,G],G] = \langle 0 \rangle$.

Proof. Without loss of generality we can suppose that $C_G(B) = \langle 1 \rangle$. We recall that the additive group of artinian $\mathbb{Z}_{p^\infty}$-module is Chernikov, that is $K = \mathbf{Tor}_{\mathbb{Z}_{p^\infty}}(B)$ includes a divisible subgroup D, which is a direct sum of quasicyclic subgroups such that K/D is finite. The additive group of B/K is torsion-free and has finite $\mathbb{Z}_{p^\infty}$-rank. In particular, the $\Pi(D) = \{p\}$. Clearly D is G-invariant. The factor-group $G/C_G(D)$ is isomorphic to a subgroup of $\mathbf{GL}_m(\mathbb{Q}_{p^\infty})$ where $\mathbb{Q}_{p^\infty}$ is the field of fractions of $\mathbb{Z}_{p^\infty}$ and m satisfies $p^m = |\mathbf{\Omega}_1(D)|$. Let $\mathbb{Q}_{p^\infty}$ be a field of fractions of $\mathbb{Z}_{p^\infty}$, then $G/C_G(D)$ is isomorphic to a subgroup of $\mathbf{GL}_m(\mathbb{Q}_{p^\infty})$. Note that $\mathbf{char}(\mathbb{Q}_{p^\infty}) = 0$. Being locally generalized radical, $G/C_G(D)$ does not include the non-cyclic free subgroup; thus an application of Tits Theorem (see, for example, [10, Corollary 10.17]) shows that $G/C_G(D)$ is soluble-by-finite. If G is periodic, then $G/C_G(D)$ is finite (see, for example, [10, Theorem 9.33]). Since K/D is finite, $G/C_G(K/D)$ is finite. Finally, $G/C_G(B/K)$ is isomorphic to a subgroup of $\mathbf{GL}_r(\mathbb{Q}_{p^\infty})$, where $\mathbf{r} = \mathbf{r}_{\mathbb{Z}_{p^\infty}}(B/K)$. Using again the fact that $G/C_G(A/K)$ does not include the non-cyclic free subgroup and Tits Theorem or Theorem 9.33 of the book [10] (for periodic G), we obtain that $G/C_G(B/K)$ is soluble-by-finite (respectively finite whenever G is periodic). Put $Z = C_G(D) \cap C_G(K/D) \cap C_G(B/K)$. Then G/Z is embedded in $G/C_G(D) \cap G/C_G(K/D) \cap G/C_G(B/K)$, in particular, G/Z is soluble-by-finite (respectively finite). If $x \in Z$, then x acts trivially on every factor of the series $\langle 0 \rangle \leq D \leq K \leq A$. Then Z is nilpotent [4]. It follows that G is soluble-by-finite (respectively, for periodic G, it is nilpotent–by–finite). This completes the proof of (i) and (ii).

Now we prove (iii). Suppose now that G is an $\mathfrak{F}$-perfect group. Again consider the series of G-invariant subgroups $\langle 0 \rangle \leq K \leq B$. Being abelian and Chernikov, K is an union of ascending series

$$\langle 0 \rangle = K_0 \leq K_1 \leq \ldots \leq K_n \leq K_{n+1} \leq \ldots$$

of G-invariant finite subgroups K_n, $n \in \mathbb{N}$. Then the factor-group $G/C_G(K_n)$ is finite, $n \in \mathbb{N}$. Since G is $\mathfrak{F}$-perfect, $G = C_G(K_n)$ for each $n \in \mathbb{N}$. The equation $K = \bigcup_{n \in \mathbb{N}} K_n$

implies that $G = C_G(K)$. By the above $G/C_G(B/K)$ is soluble-by-finite, and being $\mathfrak{F}$-perfect, it is soluble. Then $G/C_G(B/K)$ includes normal subgroups U, V such that $C_G(B/K) \leq U \leq V$, $U/C_G(B/K)$ is isomorphic to a subgroup of $\mathbf{UT}_r(\mathbb{Q}_{p^\infty})$, V/U includes a free abelian subgroup of finite index [1, Theorem 2]. Since $G/C_G(B/K)$ is $\mathfrak{F}$-perfect, it follows that $G/C_G(B/K)$ is torsion-free. Being periodic, $G/C_G(B/K)$ must be identity. In other words, $G = C_G(B/K)$. Hence G acts trivially on every factor of the series $\langle 0 \rangle \leq K \leq A$, so that $[[B,G],G] = \langle 0 \rangle$ and we obtain that G is abelian [4].

Corollary 4. *Let G be a group and A a $\mathbb{Z}_{p^\infty}G$-module. If the factor-module $A/C_A(G)$ is artinian-by-(finite rank), then every locally generalized radical subgroup of $G/C_G(A)$ is soluble-by-finite, and every periodic subgroup of $G/C_G(A)$ is nilpotent-by-finite.*

Proof. Indeed, Lemma 2 shows that $G/C_G(A/C_A(G))$ is soluble-by-finite. Every element $x \in C_G(A/C_A(G))$ acts trivially in the factors of the series $\langle 0 \rangle \leq C_A(G) \leq A$. It follows that $C_G(A/C_A(G))$ is abelian. Suppose now that $H/C_G(A)$ is a periodic subgroup. Since $A/C_A(G)$ is artinian–by–(finite rank), A has a series of H-invariant subgroups $\langle 0 \rangle \leq C_A(G) \leq D \leq K \leq A$ where $D/C_A(G)$ is a divisible Chernikov subgroup, K/D is finite and A/K is torsion-free and has finite $\mathbb{Z}_{p^\infty}$-rank. In Lemma 2 we have already proved that $G/C_G(D/C_A(G))$, $G/C_G(K/D)$ and $G/C_G(A/K)$ are finite. Let $Z = C_G(D/C_A(G)) \cap C_G(K/D) \cap C_G(A/K)$. Then G/Z is finite. If $x \in Z$, then x acts trivially on every factor of the series $\langle 0 \rangle \leq C_A(G) \leq D \leq K \leq A$. therefore Z is nilpotent [4].

Next result is well-known, but it is not able to find an appropriate reference.

Lemma 3. *Let G be an abelian group. Suppose that $G \neq KL$ for arbitrary proper subgroups K, L. Then G is a cyclic or quasicyclic q-group for some prime q.*

Proof. If G is finite, then it is not hard to see that G is a cyclic q-group for some prime q. Therefore suppose that G is infinite. If G is periodic, then obviously G is a q-group for some prime q. Let B be a basic subgroup of G, that is B is a pure subgroup of G such that B is a direct product of cyclic q-subgroups and G/B is divisible. The existence of such subgroups follows from [3, Theorem 32.3]. Since G/B is divisible, $G/B = \mathbf{Dr}_{\lambda \in \Lambda} D_\lambda$ where D_λ is a quasicyclic subgroup for every $\lambda \in \Lambda$ (see, for example, [3, Theorem 23.1]). Our condition shows that G/B is a quasicyclic group. In particular, if $B = \langle 1 \rangle$, then G is a quasicyclic group. Assume that $B \neq \langle 1 \rangle$. If B is a bounded subgroup, then $G = B \times C$ for some subgroup C (see, for example, [3, Theorem 27.5]), and we obtain a contradiction. Suppose that B is not bounded. Then B includes a subgroup $C = \mathbf{Dr}_{n \in \mathbb{N}} \langle c_n \rangle$ such that $B = C \times U$ for some subgroup U and $|c_n| = q^n$, $n \in \mathbb{N}$. Let $E = \langle c_n^{-1} \cdot c_{n+1}^q \mid n \in \mathbb{N} \rangle$. Then the factor-group C/E is quasicyclic, so that B/EU is also quasicyclic. It follows that G/EU is a direct product of two quasicyclic subgroups, which yields a contradiction. This shows that $B = \langle 1 \rangle$, which proves our result.

Corollary 5. *Let G be a soluble group. Suppose that G is not finitely generated and $G \neq \langle K, L \rangle$ for arbitrary proper subgroups K, L. Then $G/[G,G]$ is a quasicyclic q-group for some prime q.*

If G is a group, then by $\mathbf{Tor}(G)$ we will denote the maximal normal periodic subgroup of G. We recall that if G is a locally nilpotent group, then $\mathbf{Tor}(G)$ is a (characteristic) subgroup of G and $G/\mathbf{Tor}(G)$ is torsion-free.

3. Proof of main Theorem

Again suppose that $C_G(A) = \langle 1 \rangle$. Suppose that G is a finitely generated group. Then we can choose a finite subset M such that $G = \langle M \rangle$, but $G \neq \langle S \rangle$ for every subset $S \neq M$. If $|M| > 1$, then $M = \{x\} \cup S$ where $x \notin S$ and $S \neq \varnothing$. It follows that $\langle S \rangle = U \neq G$, thus $A/C_A(U)$ is artinian-by-(finite rank). The factor $A/C_A(x)$ is also artinian-by-(finite rank), and Lemma 1 shows that $\langle x, U \rangle = \langle x, S \rangle = G$ has an artinian-by-(finite rank) cocentralizer.

Suppose that $M = \{y\}$, that is $G = \langle y \rangle$ is a cyclic group. If y has infinite order, then $\langle y \rangle = \langle y^{p_1} \rangle \langle y^{p_2} \rangle$ where p_1, p_2 are primes, $p_1 \neq p_2$, and Lemma 1 again implies that $A/C_A(G)$ is artinian-by-(finite rank). Finally, if y has finite order, but this order is not a prime power, then $\langle y \rangle$ is a product of two proper subgroups, and Lemma 1 implies that $A/C_A(G)$ is artinian-by-(finite rank).

Assume now that G is not finitely generated and $A/C_A(G)$ is not artinian-by-(finite rank). Suppose that G includes a proper subgroup of finite index. Then G includes a proper normal subgroup H of finite index. We can choose a finitely generated subgroup F such that $G = HF$. Since G is not finitely generated, $F \neq G$. It follows that cocentralizers of both subgroups H and F are artinian-by-(finite rank). Lemma 1 shows that $FH = G$ has an artinian-by-(finite rank) cocentralizer, and we obtain a contradiction. This contradiction shows that G is an $\mathfrak{F}$-perfect group.

If H is a proper subgroup of G, then Corollary 4 shows that H is soluble-by-finite. In particular, G is locally-(soluble-by-finite). By Theorem A of the paper [2], G includes a normal locally soluble subgroup L such that G/L is finite or locally finite simple group. Since G is an $\mathfrak{F}$-perfect group, then in the first case $G = L$, i.e. G is locally soluble. Consider the second case. Put $C = C_A(L)$. In a natural way, we can consider C as $\mathbb{Z}_{p^\infty}(G/L)$-module. $C_{G/L}(C)$ is a normal subgroup of G/L. Since G/L is a simple group, then either $C_{G/L}(C)$ is the identity subgroup or $C_{G/L}(C) = G/L$. In the second case $C \leq C_A(G)$ and $A/C_A(G)$ is artinian-by-(finite rank). This contradiction shows that $C_{G/L}(C) = < 1 >$. Let H/L be an arbitrary proper subgroup of G/L. Then H is a proper subgroup of G, therefore $A/C_A(H)$ is artinian-by-(finite rank). It follows that $C/(C \cap C_A(H))$ is also artinian-by-(finite rank). Clearly $C_C(H/L) \leq C \cap C_A(H)$, so that $C/C_C(H/L)$ is artinian-by-(finite rank). Since H/L is periodic, it is nilpotent-by-finite by Corollary 4. In other words, every proper subgroup of G/L is nilpotent-by-finite. Using now Theorem A of the paper [8], we obtain that either G/L is soluble-by-finite or a q-group for some prime q. In any case, G/L cannot be an infinite simple group. This contradiction shows that G is locally soluble. Being an infinite locally soluble group, G has a non-identity proper normal subgroup. Corollary 4 shows that this subgroup is soluble. It follows that G includes a non-identity normal abelian subgroup. In turn, it follows that the locally nilpotent radical R_1 of G is non-identity. Suppose that $G \neq R_1$.

Being $\mathfrak{F}$-perfect, G/R_1 is infinite. Using the above arguments, we obtain that the locally nilpotent radical R_2/R_1 of G/R_1 is non-identity. If $G \neq R_2$, then the locally nilpotent radical R_3/R_2 of G/R_2 is non-identity, and so on. Using ordinary induction, we obtain that G is a radical group. Suppose that the upper radical series of G is infinite and consider its term R_ω, where ω is the first infinite ordinal. By its choice, R_ω is not soluble. Then Corollary 4 shows that $R_\omega = G$.

Since R_n is a proper subgroup of G, $A/C_A(R_n)$ is artinian-by-(finite rank), $n \in \mathbb{N}$. R_n is normal in G, therefore $C_A(R_n)$ is a $\mathbb{Z}_{p^\infty}G$-submodule. Lemma 2 (iii) shows that $G/C_G(A/C_A(R_n))$ is abelian. Suppose that there exists a positive integer m such that $G \neq C_G(A/C_A(R_m))$, then $[G,G]$ is a proper subgroup of G. An application of Corollary 4 to Lemma 2 shows that $[G,G]$ is soluble, thus even G is soluble. This contradiction proves the equality $G = C_G(A/C_A(R_n))$. In other words, $[A,G] \leq C_A(R_n)$. Since it is valid for each $n \in \mathbb{N}$, $[A,G] \leq \bigcap_{n \in \mathbb{N}} C_A(R_n)$. The equation $G = \bigcup_{n \in \mathbb{N}} R_n$ implies that $C_A(G) = \bigcap_{n \in \mathbb{N}} C_A(R_n)$. Hence $[A,G] \leq C_A(G)$. Thus G acts trivially on both factors $C_A(G)$ and $A/C_A(G)$, which follows that G is abelian [4]. Contradiction. This contradiction proves that G is soluble.

Let $D = [G,G]$. Then by Corollary 5 G/D is a quasicyclic q-group for some prime q. It follows that G has an ascending series of normal subgroups

$$D = K_0 \leq K_1 \leq \ldots \leq K_n \leq K_{n+1} \leq \ldots$$

such that K_n/D is a cyclic group of order q^n, $n \in \mathbb{N}$, and $G = \bigcup_{n \in \mathbb{N}} K_n$. Every subgroup K_n is proper and normal in G, therefore $C_A(K_n)$ is a $\mathbb{Z}_{p^\infty}G$-submodule and $A/C_A(K_n)$ is artinian-by-(finite rank). Lemma 2 shows that $[[A,G],G] \leq C_A(K_n)$. It is valid for each $n \in \mathbb{N}$, and therefore $[[A,G],G] \leq \bigcap_{n \in \mathbb{N}} C_A(K_n)$. The equation $G = \bigcup_{n \in \mathbb{N}} K_n$ implies that $C_A(G) = \bigcap_{n \in \mathbb{N}} C_A(R_n)$. Hence $[[A,G],G] \leq C_A(G)$. It follows that G acts trivially on factors $C_A(G)$, $[A,G]/C_A(G)$ and $A/[A,G]$. It follows that G is nilpotent of class at most 2 [4].

If G is abelian, then Lemma 3 shows that G is a cyclic or quasicyclic q–group for some prime q. Suppose that G is non-abelian. Let $T = \mathbf{Tor}(G)$. If we suppose that $T \neq G$, then G/T is a non-identity torsion-free nilpotent group. In particular, G/T has a non-identity torsion-free abelian factor-group, which contradicts Corollary 5. This contradiction shows that G is a periodic group. Moreover, G is a q-group. Since G is nilpotent of class 2, then $[G,G] \leq \zeta(G)$. In particular, $G/\zeta(G)$ is a quasicyclic group. In this case, $[G,G]$ is a Chernikov subgroup (see, for example, [5, Theorem 23.1]). It follows that whole group G is Chernikov. Being $\mathfrak{F}$-perfect, G is abelian, which completes the proof. It is not hard to proof that in the Theorem 1 if $G/C_G(A)$ is a quasicyclic q-group for some prime q than $q = p$.

References

1. *Charin V.S.* On the automorphism groups of nilpotent groups // Ukrain Math. J. – 1954, – 6, № 3. – P. 295-304.
2. *Dixon M.R, Evans M.J, Smith H.* Groups with all proper subgroups soluble-by-finite rank // Journal of Algebra 2005, – 289, – P. 135-147.

3. *Fuchs L.* Infinite abelian groups. Vol. 1. New York: Academic Press, 1970.

4. *Kaloujnine L.A.* Uber gewisse Beziehungen zwischen eine Gruppe und ihren Automorphismen // Bericht Math Tagung Berlin 1953. – P. 164-172.

5. *Kurdachenko L.A, Otal J, Subbotin I.Ya.* Artinian modules over group rings. Basel: Birkhauser, 2007.

6. *Kurdachenko L.A, Subbotin I.Ya, Chupordya V.A.* On bounded artinian finitary modules. International Journal of Algebra and Computation. – 2007. – 17, № 4. – P. 881-893.

7. *Kurdachenko L.A, Subbotin I.Ya, Chupordya V.A.* On the structure of some modules over generalized soluble groups // Turkish Journal of Mathematics. – 2014. – 38. – P. 52-59.

8. *Napolitani F, Pegoraro E.* On groups with nilpotent by Chernikov proper subgroup // Arch. Math. – 1997. – 69. – P. 89-94.

9. *Phillips R.* Finitary linear groups: a survey. "Finite and locally finite groups"1995; NATO ASI ser C 471 Dordrecht, Kluwer. – P. 111-146.

10. *Wehrfritz B.A.F.* Infinite linear groups. Berlin: Springer, 1973.

11. *Wehrfritz B.A.F.* Finite-finitary groups of automorphisms // Journal of Algebra and Its Applications. – 2002. – 4. – P. 375-389.

12. *Wehrfritz B.A.F.* On generalized finitary groups // Journal of Algebra. – 2002. – 247. – P. 707-727.

13. *Wehrfritz B.A.F.* Finitary automorphism groups over commutative rings // Journal Pure Appl Algebra. – 2002. – 172. – P. 337-346.

14. *Wehrfritz B.A.F.* Finitary and artinian-finitary groups over the integers Z // Ukrainian Math Journal. – 2002. – 54. – P. 924-936.

15. *Wehrfritz B.A.F.* Artinian-finitary groups over commutative rings // Illinois Journal of Math. – 2003. – 47. – P. 551-565.

16. *Wehrfritz B.A.F.* Finitary and artinian-finitary groups over commutative rings // Journal Group Theory. – 2004. – 7. – P. 243-253.

17. *Wehrfritz B.A.F.* Artinian-finitary groups over commutative rings and non-commutative rings // Journal London Math. Soc. – 2004. – 70.– P. 325-340.

18. *Wehrfritz B.A.F.* Artinian-finitary groups are locally normal-finitary // Journal of Algebra. – 2005. – 287. – P. 417-431.