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On the Construction of Chaotic Dynamical Systems on the Box Fractal [1]

Abstract. In this paper, our main aim is to obtain two different discrete chaotic dynamical systems on the Box fractal (B). For this goal, we first give two composition functions (which generate Box fractal and filled-square respectively via escape time algorithm) of expanding, folding and translation mappings. In order to examine the properties of these dynamical systems more easily, we use the intrinsic metric which is defined by the code representation of the points on B and express these dynamical systems on the code sets of this fractal. We then obtain that they are chaotic in the sense of Devaney and give an algorithm to compute periodic points.
Key words: Box fractal, code representation, intrinsic metric, chaotic dynamical systems, periodic points, escape time algorithm

Анотація. У цій роботі нашою головною ціллю є отримання двох різних дискретних хаотичних динамічних систем на ящичному фракталі (B). З цією метою ми спочатку надаємо дві функції композиції (які породжують ящичний фрактал та заповнений квадрат відповідно за допомогою алгоритму часу втечі) розширюючих, складаючих та транслюючих відображень. Аби досліджувати властивості цих динамічних систем було простіше, ми використовуємо внутрішню метрику, котра визначається кодовим представленням точок B і виражаємо ці динамічні системи на кодових множинах даного фракталу. Потім ми визначаємо, що вони хаотичні в сенсі Дівейні і надаємо алгоритм для обчислення періодичних точок.
Ключові слова: ящичний фрактал, кодове представлення, внутрішня метрика, хаотичні динамічні системи, періодичні точки, алгоритм часу втечі

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1. Introduction

One of the most popular methods of obtaining fractals is the iterated function systems (IFS). This theory was introduced by Hutchinson in [7]. Using this method, many fractals can be defined as the attractor of an IFS [3, 6, 8, 10].

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Another different method of constructing the fractals is the escape time algorithm. The escape time algorithm is a technique to display of the system behavior under iteration and it determines whether the orbit sequence tends to infinity or not. The filled-in Julia sets and Mandelbrot set are obtained by this method. It is well-known that f on Julia set is a chaotic dynamical system in the sense of Devaney (for details see [4]), where $f : \mathbb{C} \to \mathbb{C}$ is a polynomial function. Moreover, by using the folding, expanding, translation and rotation mappings, many classical fractals are obtained via escape time algorithm (see [1]). It is seen that all functions defined in [1] may indicate the dynamical systems on the related fractals. There are also some studies about defining a dynamical system on the different fractal sets, i.e., Sierpinski gasket, Sierpinski carpet and the Sierpinski tetrahedron (for details, see [2, 3, 5, 15]). In general, expressing these dynamical systems by using the code representations of the points allows us to investigate the properties of the chaotic dynamical systems more easily. This leads to the need to define an intrinsic metric by using the code representations of the points on the associated fractals [2, 9, 11, 12, 13, 14].

Throughout this study, we consider the Box fractal (also known as Vicsek fractal or anticross-stitch curve). Özdemir, Saltan and Demir get an explicit formula for the intrinsic metric on the Box fractal by using the code representations of the points in [9]. Hence, this metric can be useful for analyzing the geometrical and topological properties of any structures defined on the Box fractal, for instance, the authors investigate the geodesics between any two points in the same study.

In this paper, we first give some basic definitions and define both the code representation of a point on B and the code sets of B. Then, we express the function F, which generates Box fractal via escape time algorithm (for details see [1]), by using the code representations of the points on B. Moreover, we illustrate the dynamical system $\{B; F\}$ in the Figure 2 for a better understanding what each mapping exactly does on the code sets of B. We then compute the fixed points and $2-$ periodic points of F. On the other hand, we define a new dynamical system $\{B; G\}$, which generates filled-square via escape time algorithm given in [1], using expanding and folding mappings and state it on the code sets of the Box fractal. We also examine the action of G on the some code sets of B in the Figure 3. Furthermore, we give a general formula for computing the periodic points of G. We also show that the dynamical system $\{B; G\}$ is chaotic in the sense of Devaney.

2. The intrinsic metric and the code representations of points on the Box fractal

We now express the code representations of points on the Box fractal and then, we give the intrinsic metric which is defined on the code set of B. For this aim, we begin with the definition of B, which is the attractor of the IFS

$\{\mathbb{R}^2; w_0, w_1, w_2, w_3, w_4\}$, where $w_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 0, 1, 2, 3, 4)$

$$\begin{aligned} w_0(x,y) &= \left(\frac{x}{3}, \frac{y}{3}\right), \\ w_1(x,y) &= \left(\frac{x}{3} + \frac{1}{3}, \frac{y}{3} + \frac{1}{3}\right), \\ w_2(x,y) &= \left(\frac{x}{3} - \frac{1}{3}, \frac{y}{3} + \frac{1}{3}\right), \\ w_3(x,y) &= \left(\frac{x}{3} - \frac{1}{3}, \frac{y}{3} - \frac{1}{3}\right), \\ w_4(x,y) &= \left(\frac{x}{3} + \frac{1}{3}, \frac{y}{3} - \frac{1}{3}\right). \end{aligned}$$

The IFS of the Box fractal consists of five contraction mappings. We get the central, the upper-right, the upper-left, the lower-left and the lower-right part of this fractal by w_0, w_1, w_2, w_3 and w_4 respectively (see Figure 1).

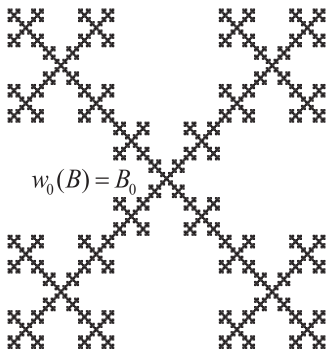

Fig. 1. The Box fractal with contraction mappings

In other words, the subsets $B_0,\ B_1,\ B_2,\ B_3,\ B_4$ of B represents the center, the upper-right part, the upper-left part, the lower-left part and the lower-right part respectively and it is clear that B is the union of these code sets:

$$B = B_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4$$

(see Figure 1).

Note that, for $\sigma = a_1 a_2 \ldots a_n$ and $a_i \in \{0, 1, 2, 3, 4\}$, $(i = 1, 2, \ldots, n)$, $B_\sigma = w_\sigma(B)$ where $w_\sigma = w_{a_n} \circ \ldots \circ w_{a_2} \circ w_{a_1}$. A point, on the attractor of an IFS, can be defined by using the contraction mappings of this IFS. Since $B_{a_1},\ B_{a_1 a_2},\ B_{a_1 a_2 a_3} \ldots$ is a sequence of nested sets that means;

$$B_{a_1} \supset B_{a_1 a_2} \supset B_{a_1 a_2 a_3} \supset \ldots \supset B_{a_1 a_2 \ldots a_n} \supset \ldots$$

by the Cantor intersection theorem, the intersection

$$\bigcap_{n=1}^{\infty} B_\sigma = \{a\}$$

is a point on B, say a. The sequence $a_1a_2\dots a_n\dots$ is called a code representation of the point a (see [9]).

Let us fix $\sigma = a_1a_2\dots a_{k-1}$, ($a_i \in \{0,1,2,3,4\}$ for $i = 1,2,\dots,k-1$). In the general case, $B_{\sigma 0}$ is the center, $B_{\sigma 1}$ is the upper-right part, $B_{\sigma 2}$ is the upper-left part, $B_{\sigma 3}$ is the lower-left part and $B_{\sigma 4}$ is the lower-right part of B_σ.

Thus, B_σ is the smaller piece of B and it is the union of the sets

$$B_{\sigma\omega} = \{\sigma\omega a_{k+1}a_{k+2}a_{k+3}\dots | \omega, a_i \in \{0,1,2,3,4\} \text{ and } i = k+1, k+2, \dots\}.$$

For $i = 0$ and $j \neq 0$, $B_{\sigma i} \cap B_{\sigma j}$ gives only one point. For the general case, if the point a is the intersection of the sub-parts of B_σ such that

$$\{a\} = B_{\sigma\omega} \cap B_{\sigma 0}$$

then this point is included by the following nested sets

$$B_\sigma \supset B_{\sigma\omega} \supset B_{\sigma\omega\omega'} \supset B_{\sigma\omega\omega'\omega'} \supset B_{\sigma\omega\omega'\omega'\omega'} \supset \dots$$

and

$$B_\sigma \supset B_{\sigma 0} \supset B_{\sigma 0\omega} \supset B_{\sigma 0\omega\omega} \supset B_{\sigma 0\omega\omega\omega} \supset \dots.$$

So, a has two code representations such that

$$\sigma\omega\omega'\omega'\omega'\dots \text{ and } \sigma 0\omega\omega\omega\dots$$

where

$$\omega' = \begin{cases} 3, & \omega = 1 \\ 4, & \omega = 2 \\ 1, & \omega = 3 \\ 2, & \omega = 4 \end{cases}$$

ω' is called as the conjugate of the number ω.

3. Expressing the Dynamical System $\{B;F\}$ on the Code Sets of the Box Fractal

In this part, we express the composition function, defined in Example 5 in [1], as a dynamical system on the Box fractal given on the square $[-1/2,1/2] \times [-1/2,1/2]$.

Let $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1,2,3,4,5)$ such that

$$\begin{aligned} f_1(x,y) &= (3x,3y), \\ f_2(x,y) &= (x,-|y|), \\ f_3(x,y) &= (-|x|,y), \\ f_4(x,y) &= \left(-\frac{1}{2}|x+y+1| - \frac{1}{2}(y-x+1), -\frac{1}{2}|x+y+1| + \frac{1}{2}(y-x-1)\right), \\ f_5(x,y) &= (x+1,y+1). \end{aligned}$$

The composition function F is defined by

$$F = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1 \tag{3.1}$$

f_1 is an expanding transformation that triples one point. f_2, f_3 and f_4 are folding mappings. f_2 moves the points from the upper hand sides of the line $y = 0$ to the lower hand side, f_3 takes the points from the right hand sides of the line $x = 0$ to the left hand side and f_4 takes the points from the upper hand side of the line $y = -x - 1$ to the lower hand side. f_5 is a translation mapping that shifts the points one unit up and one unit right.

F states a dynamical system on B and we denoted it by $\{B; F\}$. This dynamical system is expressed by using the code representations of the points on the Box fractal in the following proposition:

Proposition 1. Let the code representations of the points X and Y on B be $x_1x_2x_3\ldots$ and $y_1y_2y_3\ldots$ respectively where $x_i, y_i \in \{0, 1, 2, 3, 4\}$ for $i = 1, 2, 3, \ldots$. The function $F : B \to B$ defined in (3.1) is expressed as follows:

$$F(1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 3 \\ 2, & x_{i+1} = 4 \\ 3, & x_{i+1} = 1 \\ 4, & x_{i+1} = 2 \end{cases} \ (i \geq 1).$$

$$F(2x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 4 \\ 2, & x_{i+1} = 3 \\ 3, & x_{i+1} = 2 \\ 4, & x_{i+1} = 1 \end{cases} \ (i \geq 1).$$

$$F(3x_2x_3\ldots) = x_2x_3x_4\ldots.$$

$$F(4x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 2 \\ 2, & x_{i+1} = 1 \\ 3, & x_{i+1} = 4 \\ 4, & x_{i+1} = 3 \end{cases} \ (i \geq 1).$$

If $x_1 = 0$, there are four situations:

Case 1:

$$F(000\dots01x_{k+1}x_{k+2}x_{k+3}\dots)=y_1y_2y_3\dots,\quad y_i=\begin{cases}0, & x_{i+1}=0\\ 1, & x_{i+1}=1\\ 2, & x_{i+1}=4\\ 3, & x_{i+1}=3\\ 4, & x_{i+1}=2\end{cases}\ (i\geq 1)$$

Case 2:

$$F(000\dots02x_{k+1}x_{k+2}x_{k+3}\dots)=y_1y_2y_3\dots,\quad y_i=\begin{cases}0, & x_{i+1}=0\\ 1, & x_{i+1}=2\\ 2, & x_{i+1}=3\\ 3, & x_{i+1}=4\\ 4, & x_{i+1}=1\end{cases}\ (i\geq 1)$$

Case 3:

$$F(000\dots03x_{k+1}x_{k+2}x_{k+3}\dots)=y_1y_2y_3\dots,\quad y_i=\begin{cases}0, & x_{i+1}=0\\ 1, & x_{i+1}=3\\ 2, & x_{i+1}=2\\ 3, & x_{i+1}=1\\ 4, & x_{i+1}=4\end{cases}\ (i\geq 1)$$

Case 4:

$$F(000\dots04x_{k+1}x_{k+2}x_{k+3}\dots)=y_1y_2y_3\dots,\quad y_i=\begin{cases}0, & x_{i+1}=0\\ 1, & x_{i+1}=4\\ 2, & x_{i+1}=1\\ 3, & x_{i+1}=2\\ 4, & x_{i+1}=3\end{cases}\ (i\geq 1).$$

Then, $\{B;F\}$ is a dynamical system on the code set of the Box fractal.

Proof. The function F must be well-defined on the code set of B. If $X\in B$ has a unique code representation then $F(X)$ has a unique code representation. If X is expressed by two different code representations then $F(X)$ must have two different code representations which state the same point. Therefore, we must check over the images of the following points: $0\overline{1}, 0\overline{2}, 0\overline{3}, 0\overline{4}, 1\overline{3}, 2\overline{4}, 3\overline{1}, 4\overline{2}$, $00\overline{1}, 00\overline{2}, 00\overline{3}, 00\overline{4}, 01\overline{3}, 02\overline{4}, 03\overline{1}, 04\overline{2}, 10\overline{1}, 10\overline{2}, 10\overline{3}, 10\overline{4}, 11\overline{3}, 12\overline{4}, 13\overline{1}, 14\overline{2}, 20\overline{1}$, $20\overline{2}, 20\overline{3}, 20\overline{4}, 21\overline{3}, 22\overline{4}, 23\overline{1}, 24\overline{2}, 30\overline{1}, 30\overline{2}, 30\overline{3}, 30\overline{4}, 31\overline{3}, 32\overline{4}, 33\overline{1}, 34\overline{2}, 40\overline{1}, 40\overline{2}$, $40\overline{3}, 40\overline{4}, 41\overline{3}, 42\overline{4}, 43\overline{1}, 44\overline{2}$.

$$\begin{array}{llll} F(0\overline{1})=\ \overline{1}, & F(0\overline{2})=\ \overline{1}, & F(0\overline{3})=\ \overline{1}, & F(0\overline{4})=\ \overline{1},\\ F(1\overline{3})=\ \overline{1}, & F(2\overline{4})=\ \overline{1}, & F(3\overline{1})=\ \overline{1}, & F(4\overline{2})=\ \overline{1},\end{array}$$

$$\begin{array}{llll} F(00\overline{1})=\ 0\overline{1}, & F(00\overline{2})=\ 0\overline{1}, & F(00\overline{3})=\ 0\overline{1}, & F(00\overline{4})=\ 0\overline{1},\\ F(01\overline{3})=\ 1\overline{3}, & F(02\overline{4})=\ 1\overline{3}, & F(03\overline{1})=\ 1\overline{3}, & F(04\overline{2})=\ 1\overline{3},\end{array}$$

$$\begin{array}{llll}
F(10\overline{1}) = 0\overline{3}, & F(10\overline{2}) = 0\overline{4}, & F(10\overline{3}) = 0\overline{1}, & F(10\overline{4}) = 0\overline{2}, \\
F(11\overline{3}) = 3\overline{1}, & F(12\overline{4}) = 4\overline{2}, & F(13\overline{1}) = 1\overline{3}, & F(14\overline{2}) = 2\overline{4},
\end{array}$$

$$\begin{array}{llll}
F(20\overline{1}) = 0\overline{4}, & F(20\overline{2}) = 0\overline{3}, & F(20\overline{3}) = 0\overline{2}, & F(20\overline{4}) = 0\overline{1}, \\
F(21\overline{3}) = 4\overline{2}, & F(22\overline{4}) = 3\overline{1}, & F(23\overline{1}) = 2\overline{4}, & F(24\overline{2}) = 1\overline{3},
\end{array}$$

$$\begin{array}{llll}
F(30\overline{1}) = 0\overline{1}, & F(30\overline{2}) = 0\overline{2}, & F(30\overline{3}) = 0\overline{3}, & F(30\overline{4}) = 0\overline{4}, \\
F(31\overline{3}) = 1\overline{3}, & F(32\overline{4}) = 2\overline{4}, & F(33\overline{1}) = 3\overline{1}, & F(34\overline{2}) = 4\overline{2},
\end{array}$$

$$\begin{array}{llll}
F(40\overline{1}) = 0\overline{2}, & F(40\overline{2}) = 0\overline{1}, & F(40\overline{3}) = 0\overline{4}, & F(40\overline{4}) = 0\overline{3}, \\
F(41\overline{3}) = 2\overline{4}, & F(42\overline{4}) = 1\overline{3}, & F(43\overline{1}) = 4\overline{2}, & F(44\overline{2}) = 3\overline{1}.
\end{array}$$

In the general case, we can easily conclude that for $\sigma = x_1x_2\ldots x_n$, the images of $\sigma0\overline{1}$ and $\sigma1\overline{3}$, $\sigma0\overline{2}$ and $\sigma2\overline{4}$, $\sigma0\overline{3}$ and $\sigma3\overline{1}$, $\sigma0\overline{4}$ and $\sigma4\overline{2}$ represent the same points.

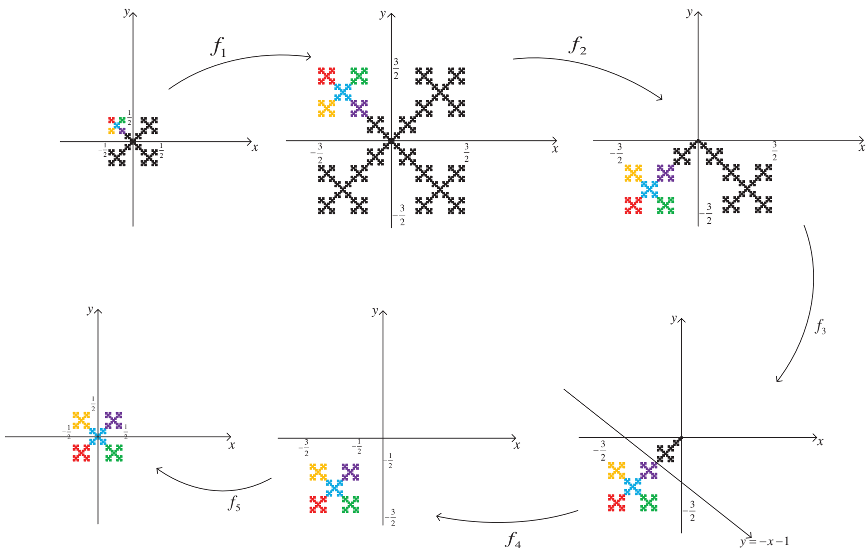

Fig. 2. The action of the function F on the code sets $B_2, B_{2x_2}, B_{2x_2x_3}$

3.1. Periodic points of F

If the equation $F^n(x_1x_2x_3\ldots) = x_1x_2x_3\ldots$ is solved for $n = 1, 2, 3, \ldots$ then we find any $n-$periodic points.
For $n = 1$, the fixed (one-periodic) points are

$\bullet\overline{0} = 000...$, $\bullet\overline{13} = 131313...$, $\bullet\overline{23} = 232323...$, $\bullet\overline{3} = 333...$, $\bullet\overline{43} = 434343...$

The 2-periodic points are

$\bullet\overline{10403020} =$	1040302010403020...,	$\bullet\overline{03} =$	030303...,
$\bullet\overline{10203040} =$	1020304010203040...,	$\bullet\overline{0103} =$	01030103...,
$\bullet\overline{04010203} =$	0401020304010203...,	$\bullet\overline{1030} =$	10301030...,
$\bullet\overline{02010403} =$	0201040302010403...,	$\bullet\overline{10} =$	101010...,

$\bullet\overline{1423} =$	14231423...,	$\bullet\overline{2233} =$	22332233...,	$\bullet\overline{3443} =$	34433443...,
$\bullet\overline{1133} =$	11331133...,	$\bullet\overline{2143} =$	21432143...,	$\bullet\overline{4213} =$	42134213...,
$\bullet\overline{1243} =$	12431243...,	$\bullet\overline{3113} =$	31133113...,	$\bullet\overline{4123} =$	41234123...,
$\bullet\overline{2431} =$	24312431...,	$\bullet\overline{3223} =$	32233223...,	$\bullet\overline{4433} =$	44334433...

4. A Different Dynamical System $\{B;G\}$ on the Box Fractal

By using expanding and folding mappings, we define another dynamical system $\{B;G\}$ on the Box fractal (B):

Let $g_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1,2,3,4,5)$

$$\begin{aligned}
g_1(x,y) &= (3x,3y)\,,\\
g_2(x,y) &= \left(x, \frac{1}{2} - \left|y - \frac{1}{2}\right|\right),\\
g_3(x,y) &= \left(x, -\frac{1}{2} + \left|y + \frac{1}{2}\right|\right),\\
g_4(x,y) &= \left(\frac{1}{2} - \left|x - \frac{1}{2}\right|, y\right),\\
g_5(x,y) &= \left(-\frac{1}{2} + \left|x + \frac{1}{2}\right|, y\right)
\end{aligned}$$

We define the composition function G such as

$$G = g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1 \tag{4.1}$$

The composition function G is a dynamical system on the Box fractal and we now express it on the code sets of B.

Proposition 2. Let the code representations of the points X and Y be $x_1x_2x_3\ldots$, $y_1y_2y_3\ldots$ respectively. The function $G : B \to B$ defined in (4.1) is expressed as follows:

If $x_1 = 0$ then

$$G(x_1x_2x_3\ldots) = x_2x_3x_4\ldots$$

If $x_1 \neq 0$ then

$$G(x_1x_2x_3\ldots) = y_1y_2y_3\ldots, \quad y_i = \begin{cases} 0, & x_{i+1} = 0 \\ 1, & x_{i+1} = 3 \\ 2, & x_{i+1} = 4 \\ 3, & x_{i+1} = 1 \\ 4, & x_{i+1} = 2 \end{cases} \quad (i \geq 1)$$

for $x_i, y_i \in \{0,1,2,3,4\}$ and $i = 1,2,3,\ldots$.

Then, $\{B; G\}$ is a dynamical system on the code set of the Box fractal.

Proof. We must show that the function G is well-defined on the code set of B. If $X \in B$ has a unique code representation then $G(X)$ has a unique code representation. If X has two different code representations then it is enough to check the image of the following points $0\overline{1}, 0\overline{2}, 0\overline{3}, 0\overline{4}, 1\overline{3}, 2\overline{4}, 3\overline{1}, 4\overline{2}, 00\overline{1}, 00\overline{2}, 00\overline{3}, 00\overline{4}, 01\overline{3}, 02\overline{4}, 03\overline{1}, 04\overline{2}, 10\overline{1}, 10\overline{2}, 10\overline{3}, 10\overline{4}, 11\overline{3}, 12\overline{4}, 13\overline{1}, 14\overline{2}, 20\overline{1}, 20\overline{2}, 20\overline{3}, 20\overline{4}, 21\overline{3}, 22\overline{4}, 23\overline{1}, 24\overline{2}, 30\overline{1}, 30\overline{2}, 30\overline{3}, 30\overline{4}, 31\overline{3}, 32\overline{4}, 33\overline{1}, 34\overline{2}, 40\overline{1}, 40\overline{2}, 40\overline{3}, 40\overline{4}, 41\overline{3}, 42\overline{4}, 43\overline{1}, 44\overline{2}$.

$$\begin{array}{llll} G(0\overline{1}) = \overline{1}, & G(0\overline{2}) = \overline{2}, & G(0\overline{3}) = \overline{3}, & G(0\overline{4}) = \overline{4}, \\ G(1\overline{3}) = \overline{1}, & G(2\overline{4}) = \overline{2}, & G(3\overline{1}) = \overline{3}, & G(4\overline{2}) = \overline{4}, \end{array}$$

$$\begin{array}{llll} G(00\overline{1}) = 0\overline{1}, & G(00\overline{2}) = 0\overline{2}, & G(00\overline{3}) = 0\overline{3}, & G(00\overline{4}) = 0\overline{4}, \\ G(01\overline{3}) = 1\overline{3}, & G(02\overline{4}) = 2\overline{4}, & G(03\overline{1}) = 3\overline{1}, & G(04\overline{2}) = 4\overline{2}, \end{array}$$

For $a \neq 0$ $(a \in \{1,2,3,4\})$

$$\begin{array}{llll} G(a0\overline{1}) = 0\overline{3}, & G(a0\overline{2}) = 0\overline{4}, & G(a0\overline{3}) = 0\overline{1}, & G(a0\overline{4}) = 0\overline{2}, \\ G(a1\overline{3}) = 3\overline{1}, & G(a2\overline{4}) = 4\overline{2}, & G(a3\overline{1}) = 1\overline{3}, & G(a4\overline{2}) = 2\overline{4}, \end{array}$$

Let $\sigma = x_1x_2\ldots x_n$, then we can easily conclude that images of $\sigma 0\overline{1}$ and $\sigma 1\overline{3}$, $\sigma 0\overline{2}$ and $\sigma 2\overline{4}$, $\sigma 0\overline{3}$ and $\sigma 3\overline{1}$, $\sigma 0\overline{4}$ and $\sigma 4\overline{2}$ are the different code representations of the same points.

4.1. Periodic points of G

Let the code representation of a point X on the Box fractal be $x_1x_2x_3x_4\ldots$. In order to compute the fixed points of G, we must solve the equation

$$G(x_1x_2x_3x_4\ldots) = x_1x_2x_3x_4\ldots.$$

Easily computations show that there are five fixed points of G such that

$$\begin{array}{lll} \bullet\overline{0} = 000\ldots, & \bullet\overline{24} = 242424\ldots, & \bullet\overline{42} = 424242\ldots. \\ \bullet\overline{13} = 131313\ldots, & \bullet\overline{31} = 313131\ldots, & \end{array}$$

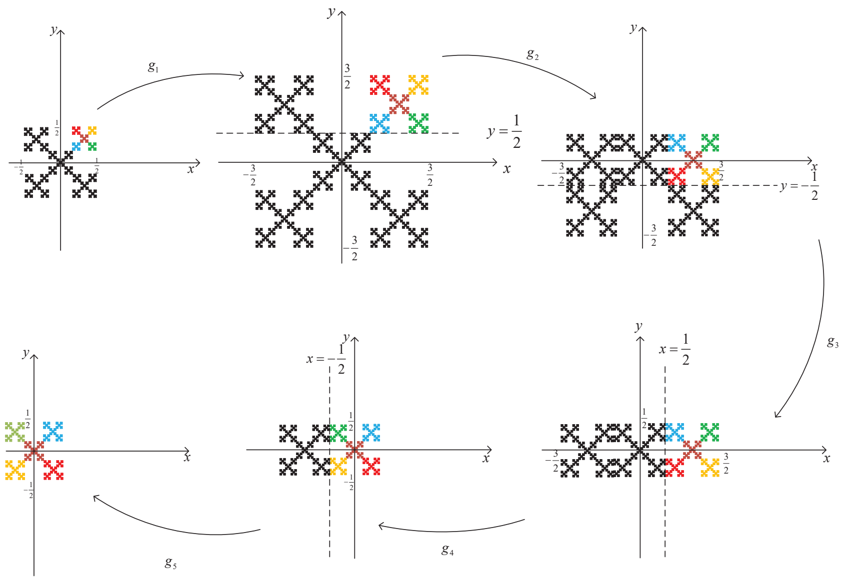

Fig. 3. The action of the function G on the code sets $B_1, B_{1x_2}, B_{1x_2x_3}$

2-periodic points of G are

$\bullet\overline{0103} =$	$01030103\ldots,$	$\bullet\overline{1} =$	$111\ldots,$
$\bullet\overline{0204} =$	$02040204\ldots,$	$\bullet\overline{12} =$	$121212\ldots,$
$\bullet\overline{0301} =$	$03010301\ldots,$	$\bullet\overline{1030} =$	$10301030\ldots,$
$\bullet\overline{0402} =$	$04020402\ldots,$	$\bullet\overline{14} =$	$141414\ldots,$

$\bullet\overline{21} =$	$212121\ldots,$	$\bullet\overline{3} =$	$333\ldots,$	$\bullet\overline{41} =$	$414141\ldots,$
$\bullet\overline{2} =$	$222\ldots,$	$\bullet\overline{32} =$	$323232\ldots,$	$\bullet\overline{43} =$	$434343\ldots,$
$\bullet\overline{23} =$	$232323\ldots,$	$\bullet\overline{34} =$	$343434\ldots,$	$\bullet\overline{4} =$	$4444\ldots,$
$\bullet\overline{2040} =$	$20402040\ldots,$	$\bullet\overline{3010} =$	$30103010\ldots,$	$\bullet\overline{4020} =$	$40204020\ldots$

some of 3-periodic points of G,

$$\bullet\overline{121343} = 121343121343\ldots, \quad \bullet\overline{123341} = 123341123341\ldots$$

$$\bullet\overline{411233} = 411233411233\ldots, \quad \bullet\overline{334112} = 334112334112\ldots$$

some of 4-periodic points of G,

$$\bullet\overline{1243} = 12431243\ldots, \quad \bullet\overline{4213} = 42134213\ldots$$

$$\bullet\overline{1342} = 13421342\ldots, \quad \bullet\overline{4312} = 43124312\ldots$$

some of 5-periodic points of G,

$$\bullet\overline{1243134213}, \quad \bullet\overline{4213224314}, \quad \bullet\overline{1342431242}.$$

A formula for the computation of periodic points of G

For the dynamical system $\{B; G\}$ we obtain a general formula, which computes the periodic points of G.

- If n is odd then n periodic point is the form of

$$\overline{x_1x_2x_3\dots x_nx_1'x_2'x_3'\dots x_n'}$$

- If n is even then n periodic point is the form of

$$\overline{x_1x_2x_3\dots x_n}$$

where $x_1, x_2, \dots, x_n \neq 0$.

- For $n = 2$, $\overline{12}$ is 2-periodic point that means $G^2(\overline{12}) = \overline{12}$ $(G(\overline{12}) = \overline{43}$ and $G(\overline{43}) = \overline{12})$.
- For $n = 3$, let $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, then $\overline{121343}$ is 3-periodic where $\overline{x_1x_2x_3x_1'x_2'x_3'} = \overline{121343}$.
- For $n = 10$, $\overline{1234123412}$ is 10-periodic point.

Theorem 1. *The dynamical system $\{B; G\}$ is chaotic in the sense of Devaney.*

Proof. To show that G is sensitive dependence on initial conditions on B, we find $\epsilon > 0$ such that for any $X \in B$ and any ball $B(X, \delta)$ with radius $\delta > 0$, there is $Y \in B(X, \delta)$ and an integer $n \geq 0$ satisfying $d(G^n(X), G^n(Y)) > \epsilon$. Firstly, let us take an arbitrary point X of B which has the code representation $x_1x_2\dots x_{k-1}x_kx_{k+1}\dots$. Obviously, for any δ, one can find a natural number k such that $\frac{1}{3^{k-2}} \leq \delta$. Now, let us take the point Y with the code representation $x_1x_2\dots x_{k-1}y_ky_{k+1}y_{k+2}\dots$ where $x_k \neq y_k$ and $y_i = y_k$ for $i = k+1, k+2, k+3, \dots$. It is easily shown that $d_{box}(X, Y) \leq \frac{\sqrt{2}}{3^{k-1}} < \delta$ (for the intrinsic metric d_{box} on the Box fractal, see [9]). Since $x_k \neq y_k$, the first terms of the code representations of the points $G^k(X)$ and $G^k(Y)$ are different. Moreover, we get

$$G^k(Y) = \begin{cases} y_k'y_k'y_k'\dots, & \text{if } y_k \neq 0, \\ 000\dots, & \text{if } y_k = 0. \end{cases}$$

This shows that

$$d_{box}(G^k(X), G^k(Y)) > \begin{cases} \frac{\sqrt{2}}{6}, & \text{if } y_k = 0, \\ \frac{\sqrt{2}}{3}, & \text{if } y_k \neq 0. \end{cases}$$

We now show that G is topologically transitive. Let U and V be open subsets of (B, d_{box}). We must obtain a finite natural number n such that $U \cap G^n(V) \neq \emptyset$. Since U is an open set, there exists a natural number k such that $B(X, \frac{\sqrt{2}}{3^{k-1}}) \subseteq U$ where the code representation of $X \in B$ is $x_1x_2\dots x_{k-1}x_kx_{k+1}\dots$. Moreover, it is easily shown that

$$U' = \{x_1x_2x_3\dots x_kz_{k+1}z_{k+2}z_{k+3}\dots | x_1, \dots, x_k \text{ are the first k-terms of X}\}$$

where $z_i \in \{0,1,2,3,4\}$ for $i = k+1, k+2, k+3, \ldots$, is the subset of $B(X, \frac{1}{3^{k-1}})$. Obviously, $G^k(U') = B$ and thus we get $G^k(U) = B$. Consequently, we have the number k such that $U \cap G^k(V) \neq \emptyset$.

In order to show that the periodic points of G are dense in B, we must get periodic points being sufficiently close to any points of B. We take the point A with the code representation $a_1a_2a_3\ldots$ and the open set $B(A, \frac{\sqrt{2}}{3^{k-1}})$ and the sets U, U' which are defined as above. Since for every $i = k+1, k+2, k+3, \ldots$, z_i's are arbitrary, then we can obtain

$$F^k(a_1a_2...a_kz_{k+1}z_{k+2}z_{k+3}...) = z'_{k+1}z'_{k+2}z'_{k+3}... = a_1a_2...a_kz_{k+1}z_{k+2}z_{k+3}...$$

This completes the proof.

Remark 1. By using a similar method of the proof of Theorem 1, one can show that the dynamical system $\{B; F\}$ is chaotic in the sense of Devaney.

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