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On asymptotically optimal cubatures for multidimensional Sobolev spaces

Abstract. We find an asymptotically optimal method of recovery of the weighted integral for the classes of multivariate functions that are defined via restrictions on their (distributional) gradient.
Key words: Asymptotically optimal recovery methods, weighted cubature formulae, multidimensional Sobolev space.

Анотація. Знайдено асимптотично оптимальний метод відновлення інтеграла з вагою для класів функцій багатьох змінних, що задаються обмеженнями на (узагальнений) градієнт.
Ключові слова: Асимптотично оптимальні методи відновлення, вагові кубатурні формули, Соболевські класи функцій багатьох змінних.

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1. Introduction

Let $Q \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a nonempty, bounded, open set. By $W^{1,p}(Q)$, $p \in [1,\infty]$, we denote the Sobolev space of functions $f\colon Q \to \mathbb{R}$, such that f and all their (distributional) first order partial derivatives belong to $L_p(Q)$. For $x = (x^1, \dots, x^d) \in \mathbb{R}^d$ and $q \in [1,\infty)$, we set $|x|_q := \left(\sum\limits_{k=1}^{d} |x^k|^q\right)^{\frac{1}{q}}$, $|x|_\infty := \max\limits_{k=1,\dots,d} |x^k|$. It is clear that for all $f \in W^{1,p}(Q)$ we have $\| \, |\nabla f|_1 \, \|_{L_p(Q)} < \infty$. For $p \in [1,\infty]$ set $W^{\nabla}_p(Q) := \{f \in W^{1,p}(Q)\colon \| \, |\nabla f|_1 \, \|_p \le 1\}$.

We say, that a set $Q \subset \mathbb{R}^d$ is *composed of* $m \in \mathbb{N}$ *convex domains*, if there exist disjoint convex domains $Q_1, \dots, Q_m$ such that $Q = \bigcup\limits_{k=1}^{m} Q_k$.

It is known (see e.g. §§1–3 of Chapter 1 in [7]) that every bounded convex set is Jordan measurable, and hence a bounded set composed of a finite number of convex domains is also Jordan measurable.

Everywhere in the article we consider only domains Q composed of a finite number of convex domains. Such domains satisfy the so-called cone condition and hence for all $p > d$ each function from $W^{1,p}(Q)$ has a continuous representation, see e.g. Chapter 4 and Theorem 4.12 in [1].

Everywhere below we assume that $p > d \geq 2$. For $h > 0$ we set

$$\square_h^d := \{x \in \mathbb{R}^d \colon |x|_\infty \leq h\}.$$

For a finite set A, by $|A|$ we denote the number of its elements.

Let a measurable bounded set $Q \subset \mathbb{R}^d$, a class X of continuous on Q functions and $n \in \mathbb{N}$ be given. Let also w be a non-negative integrable on Q weight function. We consider the problem of optimal recovery of the integral $\int_Q w(x)f(x)dx$ of a function $f \in X$ based on n function's values at points from Q. Arbitrary function $\Phi \colon \mathbb{R}^n \to \mathbb{R}$ is called a method of recovery. For given points $x_1, \dots, x_n \in Q$ the error of recovery of the integral by the method Φ is defined by the equality

$$e(X, \Phi, w, x_1, \dots, x_n) := \sup_{f \in X} \left| \int\limits_Q w(x)f(x)dx - \Phi(f(x_1), \dots, f(x_n)) \right|.$$

The problem of the optimal recovery of the integral is to find the best error of recovery

$$E_n(X, w) := \inf_{x_1, \dots, x_n \in Q} \inf_{\Phi \colon \mathbb{R}^n \to \mathbb{R}} e(X, \Phi, w, x_1, \dots, x_n), \qquad (1)$$

the best method of recovery and the best position of the informational set $x_1, \dots, x_n$ (i.e. such method $\tilde{\Phi} \colon \mathbb{R}^n \to \mathbb{R}$ and points $\tilde{x_1}, \dots, \tilde{x_n} \in Q$, for which the infima in (1) are attained), if such points and such a method exist. In the case when $w(x) \equiv 1$ on Q we write $E_n(X)$ instead of $E_n(X, w)$.

In many cases it is hard to find the best error of recovery and optimal recovery method; in such situations it is interesting to find asymptotically optimal method of recovery, i.e. such sequence of methods $\Phi_n \colon \mathbb{R}^n \to \mathbb{R}$ and information sets $\{x_1^n, \dots, x_n^n\}$, $n \in \mathbb{N}$, that

$$\lim_{n \to \infty} \frac{E_n(X, w)}{e(X, \Phi_n, w, x_1^n, \dots, x_n^n)} = 1.$$

In the case $X = W_p^\nabla(Q)$ it is sufficient to consider only linear methods of recovery in (1). More precisely, the following lemma holds.

Lemma 1. *For given points $x_1, \dots, x_n \in Q$*

$$\inf_{\Phi \colon \mathbb{R}^n \to \mathbb{R}} \sup_{f \in W_p^\nabla(Q)} \left| \int\limits_Q w(x)f(x)dx - \Phi(f(x_1), \dots, f(x_n)) \right|$$

$$= \inf_{\substack{c_k\in\mathbb{R},\\ k=1,\dots,n}} \sup_{f\in W_p^{\nabla}(Q)} \left| \int\limits_Q w(x)f(x)dx - \sum_{k=1}^n c_k f(x_k) \right|.$$

The existence of an optimal linear method of recovery is well known in many situations, in particular for convex centrally symmetric classes X. See for example Chapter 1 §3 in [14].

Problems of optimal recovery are heavily studied and have a broad literature. We refer to the monographs [10, 14, 13, 12] for many results and further references.

This article is intimately connected with [4], where, in particular, the quantity $E_n(W_p^{\nabla}(Q))$ for the case of convex bounded domain Q was studied. The goal of this article is to find asymptotically optimal methods of the integral recovery with non-constant weight function.

Some results regarding optimal recovery of integrals with weights can be found in [11, 2, 3, 5, 6]

The article is organized as follows. In Section 2 we adduce necessary results from [4]. In Sections 3 and 4 we find the asymptotically optimal methods of the integral recovery for some classes of weight functions.

2. Necessary known results

2.1. Extremal functions

Let $p > d$, p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $h > 0$. Consider the function $f_e\colon \square_h^d \to \mathbb{R}$

$$f_e(y) = f_{e,h}(y) = \int\limits_0^{|y|_\infty} \left| \frac{h^{d-1}}{u^{d-1}} - \frac{u}{h} \right|^{p'-1} du. \tag{2}$$

In the proof of Theorem 3 in [4] it was shown, that $f_{e,h} \in W^{1,p}(\square_h^d)$ and

$$|\nabla f_{e,h}(y)|_1 = \left(\frac{h^{d-1}}{|y|_\infty^{d-1}} - \frac{|y|_\infty}{h} \right)^{p'-1} \quad \text{a.e. on } \square_h^d. \tag{3}$$

Since $p(p'-1) = p'$,

$$\| |\nabla f_{e,1}|_1 \|_{L_p(\square_1^d)}^p = \left\| \frac{1}{|\cdot|_\infty^{d-1}} - |\cdot|_\infty \right\|_{L_{p'}(\square_1^d)}^{p'}.$$

Set

$$c(d,p) := \frac{1}{d} \left\| \frac{1}{|\cdot|_\infty^{d-1}} - |\cdot|_\infty \right\|_{L_{p'}(\square_1^d)}. \tag{4}$$

The restriction of the function $f_{e,h}$, to the boundary $\partial\Box_h^d$ is constant; hence we can continuously continue the function $f_{e,h}$ to whole $\mathbb{R}^d$ by setting $f_{e,h}(y)$ equal to the value of $f_{e,h}$ on the boundary of $\Box_h^d$ for all $y\notin\Box_h^d$.

Let $Q\subset\mathbb{R}^d$ be a bounded region, $x_1,\dots,x_n\in Q$ and $h>0$. For all $y\in\mathbb{R}^d$ set

$$f_h(x_1,\dots,x_n;y):=\min_{k=1,\dots,n} f_{e,h}(y-x_k). \tag{5}$$

It is easy to see that $f_h(x_1,\dots,x_n)\in W^{1,p}(Q)$ for all $p\in(d,\infty]$.

For $x\in\mathbb{R}^d$ and $h>0$ set

$$\Box_h^d(x):=\{y\in\mathbb{R}^d\colon |x-y|\le h\}.$$

The following lemma is contained in [4], see Lemma 2.

Lemma 2. *Let a bounded measurable domain $Q\subset\mathbb{R}^d$, $n\in\mathbb{N}$ and $h>0$ be given. If the points $\bar{x}_1,\dots\bar{x}_n\in Q$ are such that the sets $\Box_h^d(\bar{x}_1),\dots,\Box_h^d(\bar{x}_n)$ are pairwise disjoint subsets of Q, then for all $x_1,x_2,\dots,x_n\in Q$*

$$\int\limits_Q f_h(x_1,\dots,x_n;y)dy\ge\int\limits_Q f_h(\bar{x}_1,\dots,\bar{x}_n;y)dy\ge 0 \tag{6}$$

and for all $p\in(d,\infty]$

$$\|\,|\nabla f_h(x_1,\dots,x_n)|_1\,\|_{L_p(Q)}\le\|\,|\nabla f_h(\bar{x}_1,\dots,\bar{x}_n)|_1\,\|_{L_p(Q)}. \tag{7}$$

2.2. Asymptotically optimal information sets and weights

In this section we recall the construction of the optimal information sets and weight for the problem of the integral optimal recovery in the case when Q is Jordan measurable, see [4].

For $h>0$, we consider the lattice Λ in $\mathbb{R}^d$ generated by the vectors $(2h,0,0,\dots,0)$, $(0,2h,0,0,\dots,0),\dots,$ $(0,\dots,0,2h)\in\mathbb{R}^d$. By $P_k(h)$ we denote the cubes

$$2hk+[0,2h]^d,\ k\in\mathbb{Z}^d;$$

their volumes are equal to $(2h)^d$. By $A(h)$, we denote the set of all cubes $P_k(h)$ that are contained in Q. Let $a(h)$ be the set of the centers of the cubes from $A(h)$. By $B(h)$ we denote the set of all cubes $P_k(h)$ that have non-empty set of common with Q interior points. Let $b(h)$ be the set of the centers of the cubes from $B(h)$. Since Q is Jordan measurable, we have $\lim\limits_{h\to 0}|A(h)|\cdot(2h)^d=\lim\limits_{h\to 0}|B(h)|\cdot(2h)^d=\operatorname{mes}Q$ or, equivalently,

$$|A(h)|=\frac{\text{mes Q}}{(2h)^d}+o\left(\frac{1}{h^d}\right)\ \text{and}$$

$$|B(h)| = \frac{\operatorname{mes} \mathrm{Q}}{(2h)^d} + o\left(\frac{1}{h^d}\right) \text{ as } h \to 0. \quad (8)$$

Let $n \in \mathbb{N}$ be fixed. We set

$$h_n := \frac{1}{2}\left(\frac{\operatorname{mes} \mathrm{Q}}{n}\right)^{\frac{1}{d}}. \quad (9)$$

Then, due to (8),

$$|A(h_n)| = n + o(n) \text{ and } |B(h_n)| = n + o(n) \text{ as } n \to \infty. \quad (10)$$

For each cube P from the set $B(3h_n)$, we choose a point z according to the following rule: z is the center of the cube P if it belongs to $a(h_n)$; otherwise, z is an arbitrary point from $P \cap a(h_n)$ if the intersection is not empty, and z is an arbitrary internal point of $Q \cap P$ if the intersection is empty.

By $S_1(n)$ we denote the set of such points z. From (8) it follows that the number $|S_1(n)|$ of points in $S_1(n)$ satisfies

$$|S_1(n)| = \frac{n}{3^d} + o(n), \text{ as } n \to \infty. \quad (11)$$

Since for all $n \in \mathbb{N}$ $a(3h_n) \subset a(h_n)$, we obtain that

$$|S_1(n) \setminus a(h_n)| \leq |B(3h_n)| - |A(3h_n)| = o(n), \text{ as } n \to \infty. \quad (12)$$

Denote by $S_2(n)$ arbitrary subset of the set $a(h_n) \setminus S_1(n)$, that contains $n - |S_1(n)|$ points (for large enough n this number is positive; if $|a(h_n) \setminus S_1(n)| \leq n - |S_1(n)|$, then we set $S_2(n) := a(h_n) \setminus S_1(n)$). Set $S(n) := S_1(n) \cup S_2(n)$.

Let $S(n) = \{x_1^*, \ldots, x_{|S(n)|}^*\}$. For each $k = 1, \ldots, |S(n)|$, we define the set

$$V_k = V_k(n) := \{x \in Q \cap P(3h_n; x_k^*) \colon |x - x_k^*|_\infty < |x - x_s^*|_\infty, \quad s \neq k\}, \quad (13)$$

where $P(3h_n; x_k^*)$ is the cube from $B(3h_n)$ that contains x_k^*. The sets V_k are pairwise disjoint, $\bigcup_{k=1}^{|S(n)|} V_k \subset Q$, and $\operatorname{mes}\left(Q \setminus \bigcup_{k=1}^{|S(n)|} V_k\right) = 0$.

We set

$$c_k^* := \operatorname{mes} V_k, k = 1, \ldots, |S(n)|. \quad (14)$$

2.3. Optimal recovery formulae

An asymptotically optimal solution of the integral optimal recovery problem in the case of a convex domain and unit weight function is given by the following theorem, see [4].

Theorem 1. *Let $d, n \in \mathbb{N}$, $p \in (d, \infty]$ and a bounded convex set Q be given. Then*

$$E_n\left(W_p^\nabla(Q)\right) = c(d,p)\left(\frac{\operatorname{mes} Q}{2^d}\right)^{\frac{1}{d}+\frac{1}{p'}} \cdot \frac{1+o(1)}{n^{\frac{1}{d}}},\ n \to \infty,$$

where the constant $c(d,p)$ is defined by (4). The asymptotically optimal information set is $S(n)$, defined in Section 2.2, and the optimal recovery method is

$$\tilde{\Phi}_n(f(x_1), \dots, f(x_{|S(n)|})) = \sum_{k=1}^{|S(n)|} c_k^* f(x_k),$$

where the weights c_k^* are defined by (14).

3. Step weight functions

Let $Q \subset \mathbb{R}^d$. We say that the weight function $w(x)$ is a *step function*, if there exists a number $m \in \mathbb{N}$ and disjoint sets $Q_1, \dots, Q_m$ such that $Q = \bigcup\limits_{k=1}^{m} Q_k$ and $w(x) = w_k > 0$, for $x \in Q_k$, $k = 1, \dots, m$.

The goal of this section is to solve the problem of optimal integral recovery in the case, when the weight function is a step function and the sets Q_k are convex.

3.1. Optimal recovery with step weight function

For each $j = 1, \dots, m$ and positive integer n_j define the information set $S_j(n_j) = \{x_k^j\}_{k=1}^{|S_j(n_j)|}$ and weights c_k^j, $k = 1, \dots, |S_j(n_j)|$, the same way as it was done in Section 2.2, with the domain Q substituted by Q_j and the number n by n_j.

The main result of this section is the following theorem.

Theorem 2. *Let $p > d$, w be a positive step function on the set $Q \subset \mathbb{R}^d$, i.e. $w(x) = w_k > 0$, $x \in Q_k$, $k = 1, \dots, m$, where $Q = \bigcup\limits_{k=1}^{m} Q_k$. Suppose that each of the sets Q_k is convex. Then*

$$E_n\left(W_p^\nabla(Q), w\right) = \frac{c(d,p)(1+o(1))}{2^{1+\frac{d}{p'}} n^{\frac{1}{d}}} \|w\|_{L_q(Q)},\ n \to \infty,$$

where $q = \frac{p'd}{p'+d}$ and $c(d,p)$ is defined by (4).

The asymptotically optimal information set is the set $S(n) = \bigcup\limits_{j=1}^{m} S_j(n_j)$ and the asymptotically optimal recovery method is

$$\tilde{\Phi}_n(f(x_1), \dots, f(x_{|S(n)|})) = \sum_{j=1}^{m} w_j \sum_{k=1}^{|S_j(n_j)|} c_k^j f(x_k^j),$$

where the sets $S_j(n_j)$ and numbers c_k^j are defined above,

$$n_j = \left[|S(n)| \frac{w_j^q \operatorname{mes} Q_j}{\sum\limits_{k=1}^{m} w_k^q \operatorname{mes} Q_k} \right], \; j = 1, \ldots, m,$$

and $[x]$ denotes the biggest integer not exceeding the real number x.

It is easy to see, that for an asymptotically optimal information set we have $n_k \to \infty$ as $n \to \infty$.

The proof of the theorem will be given in several subsections below.

3.2. Auxiliary results

We need the following lemma, which was used in several works, see e.g. [2]. We omit its technical proof.

Lemma 3. *Let $m \in \mathbb{N}$, positive numbers $a_1, \ldots, a_m$, $n_1, \ldots, n_m$, α and n such that $\sum\limits_{k=1}^{m} n_k = n$ be given. Then*

$$\sum_{k=1}^{m} \frac{a_k}{n_k^\alpha} \geq \frac{1}{n^\alpha} \left(\sum_{k=1}^{m} a_k^{\frac{1}{\alpha+1}} \right)^{\alpha+1}.$$

The inequality turns into equality if for $k = 1, \ldots, m$

$$n_k = n \frac{a_k^{\frac{1}{\alpha+1}}}{\sum\limits_{k=1}^{m} a_k^{\frac{1}{\alpha+1}}}. \tag{15}$$

Lemma 4. *Let a Jordan measurable set $Q \subset \mathbb{R}^d$ be given. For a given $h > 0$ consider the set*

$$Q^h := \{x \in Q : \inf_{y \in \partial Q} |x - y|_\infty \geq h\}.$$

Then $\operatorname{mes} Q^h = \operatorname{mes} Q + o(1)$*, $h \to 0$.*

Proof. Using notations from Section 2.2 and equalities (8) we have $|B(h) \setminus A(h)| = o(h^{-d})$, $h \to 0$. For each cube $P \in B(h) \setminus A(h)$ (with length of the edge equal to $2h$) consider the cube $\tilde{P}$ with the same center and the length of the edge equal to $4h$. Set $\tilde{Q} := Q \setminus \bigcup\limits_{P \in B(h) \setminus A(h)} \tilde{P}$. Then $Q^h \supset \tilde{Q}$ and $\operatorname{mes} \tilde{Q} \geq \operatorname{mes} Q - o(h^{-d}) \cdot (4h)^d = \operatorname{mes} Q + o(1)$, $h \to 0$. The lemma is proved.

3.3. Estimate from above

For each set of quantities n_k of information nodes in the sets Q_k, $k = 1,\dots,m$, choose the information nodes x_k^i and coefficients c_k^i, $i = 1,\dots,n_k$, to be optimal for each of the set Q_k. Using Lemma 1, Theorem 1, Holder's inequality and Lemma 3 we obtain (as $n\to\infty$)

$$E_n\left(W_p^\nabla(Q), w\right) \le \inf_{n_i} \sup_{f\in W_p^\nabla(Q)} \sum_{i=1}^m w_i \left| \int\limits_{Q_i} f(x)dx - \sum_{k=1}^{n_i} c_k^i f(x_k^i)\right|$$

$$\le \inf_{n_i} \sup_{f\in W_p^\nabla(Q)} \sum_{i=1}^m w_i \| \, |\nabla f|_1 \, \|_{L_p(Q_i)} \sup_{g\in W_p^\nabla(Q_i)} \left| \int\limits_{Q_i} g(x)dx - \sum_{k=1}^{n_i} c_k^i g(x_k^i)\right|$$

$$= \inf_{n_i} \sup_{f\in W_p^\nabla(Q)} \sum_{i=1}^m w_i \| \, |\nabla f|_1 \, \|_{L_p(Q_i)} E_{n_i}(W_p^\nabla(Q_i))$$

(by Theorem 1)

$$= \inf_{n_i} \sup_{f\in W_p^\nabla(Q)} \sum_{i=1}^m w_i c(d,p) \left(\frac{\operatorname{mes} Q_i}{2^d}\right)^{\frac{1}{d}+\frac{1}{p'}} \frac{1+o(1)}{n_i^{\frac{1}{d}}} \| \, |\nabla f|_1 \, \|_{L_p(Q_i)}$$

(applying Holder's inequality)

$$\le c(d,p)(1+o(1)) \inf_{n_i} \sup_{f\in W_p^\nabla(Q)} \left(\sum_{i=1}^m \frac{w_i^{p'}}{n_i^{\frac{p'}{d}}} \left(\frac{\operatorname{mes} Q_i}{2^d}\right)^{\frac{p'}{d}+1}\right)^{\frac{1}{p'}}$$

$$\cdot \left(\sum_{i=1}^m \| \, |\nabla f|_1 \, \|_{L_p(Q_i)}^p \right)^{\frac{1}{p}}$$

(using Lemma 3)

$$= \frac{c(d,p)(1+o(1))}{n^{\frac{1}{d}}} \left(\sum_{i=1}^m w_i^{\frac{p'}{1+\frac{p'}{d}}} \frac{\operatorname{mes} Q_i}{2^d}\right)^{\frac{1+\frac{p'}{d}}{p'}} = \frac{c(d,p)(1+o(1))}{2^{1+\frac{d}{p'}} n^{\frac{1}{d}}} \|w\|_{L_q(Q)}.$$

3.4. Glued extremal functions

For each configuration of the information set $x_1,\dots,x_n$, and weights $w_1,\dots,w_n$, we define a function $f\colon Q\to\mathbb{R}$ by the following construction. For each $i = 1,\dots,m$, define a function $f_i(x)\colon Q_i\to\mathbb{R}$ by the formula

$$f_i(x) := f_{h_i}(x_1^i,\dots,x_{n_i}^i; x),$$

where, the points $x_1^i,\dots,x_{n_i}^i$ are the points from information set that belong to Q_i, $h_i := \frac{1}{2}\left(\frac{\operatorname{mes} Q_i}{n_i}\right)^{\frac{1}{d}}$ and the functions f_h are defined by (5).

Set

$$g(x) := \min_{y \in \bigcup_{k=1}^{m} \partial Q_k} |x - y|$$

and

$$f(x) = w_i^{\frac{d}{pd+p-d}} \min\{g(x), f_i(x)\},\ x \in Q_i,\ i = 1, \dots, m.$$

Due to the triangle inequality,

$$|g(x) - g(y)| \le |x - y| \tag{16}$$

for all $x, y \in Q$, hence the function g is Lipschitz. Thus, due to the Rademacher theorem, there exists (classical) gradient of g almost everywhere. Moreover, it agrees with the distributional gradient almost everywhere. Hence, due to (16), $g \in W^{1,\infty}(Q)$, see for example [8], Sections 3 and 4. Therefore $f \in W^{1,p}(Q)$ and $\frac{f}{\| |\nabla f|_1 \|_{L_p(Q)}} \in W_p^{\nabla}(Q)$.

Note, that $f \equiv w_i^{\frac{d}{pd+p-d}} f_{h_i}(x_1^i, \dots, x_{n_i}^i)$ on $Q_i^{H_i}$, where $H_i \to 0$ as $h_i \to 0$ and the notation Q^h is defined in Lemma 4.

Denote by $\tilde{f}$ the function f with $x_1, \dots, x_n$ chosen in such a way, that for all $i = 1, \dots, m$, the subsets $x_1^i, \dots, x_{n_i}^i$ are optimal information sets in Theorem 1 (with Q substituted by Q_i and n by n_i). Set $\tilde{Q} = \tilde{Q}(h_1, \dots, h_m) := \bigcup_{i=1}^{m} Q_i^{H_i}$. By Lemma 4, $\operatorname{mes} \tilde{Q} = \operatorname{mes} Q + o(1)$, $n \to \infty$.

We need the following lemma.

Lemma 5. *For each $i = 1, \dots, m$ as $n \to \infty$*

$$\| |\nabla \tilde{f}|_1 \|_{L_p(Q_i^{H_i})}^p = (1 + o(1)) w_i^{\frac{dp}{dp+p-d}} \frac{\operatorname{mes} Q_i^{H_i}}{2^d} \int_{\square_1^d} \left(\frac{1}{|x|_\infty^{d-1}} - |x|_\infty \right)^{p'} dx.$$

Proof. By the definition of the function $\tilde{f}$ and Lemma 4 we have

$$\| |\nabla \tilde{f}|_1 \|_{L_p(Q_i^{H_i})}^p = (1 + o(1)) n_i w_i^{\frac{dp}{dp+p-d}} \| |\nabla f_{e,h_i}|_1 \|_{L_p(\square_{h_i}^d)}^p.$$

Since $(p' - 1)p = p'$, from (3) we obtain

$$\| |\nabla f_{e,h_i}|_1 \|_{L_p(\square_{h_i}^d)}^p = \int_{\square_{h_i}^d} \left(\frac{h_i^{d-1}}{|x|_\infty^{d-1}} - \frac{|x|_\infty}{h_i} \right)^{p'} dx$$

$$= h_i^d \int_{\square_1^d} \left(\frac{1}{|y|_\infty^{d-1}} - |y|_\infty \right)^{p'} dy.$$

To finish the proof of the lemma it is sufficient to notice that $n_i h_i^d = \frac{\operatorname{mes} Q_i}{2^d}$. The lemma is proved.

3.5. Estimate from below

For each information set $x_1, \dots, x_n$ and weights $c_1, \dots, c_n$,

$$\sup_{g\in W_p^{\nabla}(Q)}\left|\int\limits_Q w(x)g(x)dx-\sum_{k=1}^{n}c_kg(x_k)\right|\geq\frac{1}{\|\,|\nabla f|_1\,\|_{L_p(Q)}}\int\limits_Q w(x)f(x)dx$$

$$\text{(by Lemma 2)}\geq\frac{1}{\|\,|\nabla\tilde{f}|_1\,\|_{L_p(Q)}}\sum_{i=1}^{m}w_i\int\limits_{Q_i}\tilde{f}_i(x)dx$$

$$\text{(by Lemma 4)}=\frac{1+o(1)}{\|\,|\nabla\tilde{f}|_1\,\|_{L_p(\tilde{Q})}}\sum_{i=1}^{m}w_i\int\limits_{Q_i^{H_i}}\tilde{f}_i(x)dx$$

(by extremality of the function $\tilde{f}$ in Theorem 1)

$$=\frac{1+o(1)}{\|\,|\nabla\tilde{f}|_1\,\|_{L_p(\tilde{Q})}}\sum_{i=1}^{m}w_ic(d,p)\left(\frac{\operatorname{mes}Q_i^{H_i}}{2^d}\right)^{\frac{1}{d}+\frac{1}{p'}}\|\,|\nabla\tilde{f}|_1\,\|_{L_p(Q_i^{H_i})}n_i^{-\frac{1}{d}}$$

(by Lemma 5)

$$=\frac{c(d,p)(1+o(1))}{\left(\sum\limits_{i=1}^{m}w_i^{\frac{pd}{pd+p-d}}\frac{\operatorname{mes}Q_i^{H_i}}{2^d}\right)^{\frac{1}{p}}}\sum_{i=1}^{m}w_i^{\frac{pd+p}{pd+p-d}}\left(\frac{\operatorname{mes}Q_i^{H_i}}{2^d}\right)^{\frac{1}{d}+\frac{1}{p'}}\left(\frac{\operatorname{mes}Q_i^{H_i}}{2^d}\right)^{\frac{1}{p}}n_i^{-\frac{1}{d}}$$

$$=\frac{c(d,p)(1+o(1))}{\left(\sum\limits_{i=1}^{m}w_i^{\frac{pd}{pd+p-d}}\frac{\operatorname{mes}Q_i^{H_i}}{2^d}\right)^{\frac{1}{p}}}\sum_{i=1}^{m}w_i^{\frac{pd+p}{pd+p-d}}\left(\frac{\operatorname{mes}Q_i^{H_i}}{2^d}\right)^{\frac{1}{d}+1}n_i^{-\frac{1}{d}}$$

$$\text{(by Lemma 3)}\geq\frac{c(d,p)(1+o(1))}{n^{\frac{1}{d}}\left(\sum\limits_{i=1}^{m}w_i^{\frac{pd}{pd+p-d}}\frac{\operatorname{mes}Q_i^{H_i}}{2^d}\right)^{\frac{1}{p}}}\left(\sum_{i=1}^{m}w_i^{\frac{pd}{pd+p-d}}\frac{\operatorname{mes}Q_i^{H_i}}{2^d}\right)^{\frac{d+1}{d}}$$

$$=\frac{c(d,p)(1+o(1))}{n^{\frac{1}{d}}}\left(\sum_{i=1}^{m}w_i^{\frac{pd}{pd+p-d}}\frac{\operatorname{mes}Q_i^{H_i}}{2^d}\right)^{\frac{pd+p-d}{pd}}$$

$$=\frac{c(d,p)(1+o(1))}{2^{1+\frac{d}{p'}}n^{\frac{1}{d}}}\|w\|_{L_q(Q)}.$$

4. Main results

The goal of this section is to solve the problem of asymptotically optimal integral recovery in the case when the weight function is admissible.

4.1. Definition and some examples

Let a bounded set $Q \subset \mathbb{R}^d$ be given. We call a function $w\colon Q \to \mathbb{R}$ *admissible* if it is positive almost everywhere, bounded, and there exists $M \in \mathbb{N}$, such that for all $C_1 < C_2$ the set $\{x \in Q\colon w(x) \in [C_1, C_2]\}$ is composed of $m \le M$ convex domains.

First of all note, that the step functions are admissible, if the sets Q_k are composed of convex domains. Another set of admissible functions can be constructed as follows.

Let $\varphi\colon [0,1] \to \mathbb{R}$ be a monotone function. Consider the function $w = w(\varphi)\colon \square_1^d \to \mathbb{R}$, defined by the formula $w(x) = \varphi(|x|_\infty)$. For arbitrary $C_1 < C_2$, the set $\varphi^{-1}[C_1, C_2]$ is equal to $[\alpha, \beta] \setminus X$, where $0 \le \alpha \le \beta \le 1$ and $X \subset \{\alpha, \beta\}$. Hence the set $w^{-1}[C_1, C_2]$ is equal to $\square_\beta^d \setminus \square_\alpha^d$ (with appropriately included boundaries). The latter set is a union of $2d$ convex sets. Indeed, let for definiteness $X = \emptyset$. Then the convex sets can be chosen as

$$T_1^{\pm} = \square_\beta^d \cap \{\pm x_1 \in [\alpha, \beta]\},$$

and for $k = 2, \ldots, d$

$$T_k^{\pm} = \square_\beta^d \cap \{\pm x_k \in [\alpha, \beta], |x_1|, \ldots, |x_{k-1}| < \alpha\}.$$

hence the weight w is admissible and M from the definition can be set equal to $2d$.

Using the arguments above, one can prove that the function $w(\varphi)$ remains admissible, if one or several assumptions are modified:

1. Request φ to be only a piecewise-monotone function; i.e. such that there exists a partition $0 = t_0 < t_1 < \ldots < t_n = 1$, so that φ is monotone on each of the intervals (t_{i-1}, t_i), $i = 1, \ldots, n$.

2. Substitute the set $\square_1^d$ by an arbitrary convex domain Q.

3. Substitute the norm $|\cdot|_\infty$ by the norm $|\cdot|_1$.

4.2. Solution of the integral optimal recovery problem

Theorem 3. *Let $p > d$, w be admissible function on $Q \subset \mathbb{R}^d$. Then*

$$E_n\left(W_p^\nabla(Q), w\right) = \frac{c(d,p)(1+o(1))}{2^{1+\frac{d}{p'}} n^{\frac{1}{d}}} \|w\|_{L_q(Q)},\ n \to \infty,$$

where $q = \frac{p'd}{p'+d}$ and $c(d,p)$ is defined by (4).

Proof. Note, that if for all $x \in Q$ we have $w(x) \le v(x)$, then

$$E_n\left(W_p^\nabla(Q), w\right) \le E_n\left(W_p^\nabla(Q), v\right).$$

This inequality follows from the existence of an optimal linear method of recovery (see Lemma 1) and the equality

$$\inf_{\substack{c_k\in\mathbb{R},\\ k=1,\dots,n}} \sup_{f\in W_p^{\nabla}(Q)} \left| \int_Q w(x)f(x)dx - \sum_{k=1}^n c_k f(x_k) \right| = \sup_{\substack{f\in W_p^{\nabla}(Q),\\ f(x_k)=0,\, k=1,\dots,n}} \int_Q w(x)f(x)dx,$$

which can be proved using the standard arguments involving the Hahn-Banach theorem, see e.g. [9, Theorem 1.3.4].

Let $c \le w(x) \le C$ for all $x \in Q$. Fix $m \in \mathbb{N}$, and for $k = 1, \dots, m$ set

$$Q_{k,m} := \left\{ x \in Q \colon w(x) \in \left[c + (k-1)\frac{C-c}{m}, c + k\frac{C-c}{m} \right] \right\}.$$

Consider the function $\overline{w}_m(x) = c + k\frac{C-c}{m}$ for all $x \in Q_{k,m}$, $k = 1, \dots, m$. Then $w(x) \le \overline{w}_m(x)$ for all $x \in Q$ and using Theorem 2 we obtain

$$\lim_{n\to\infty} E_n\left(W_p^{\nabla}(Q), w\right) \cdot \left(\frac{c(d,p)}{2^{1+\frac{d}{p'}} n^{\frac{1}{d}}} \|w\|_{L_q(Q)} \right)^{-1} \le \lim_{n\to\infty} E_n\left(W_p^{\nabla}(Q), \overline{w}_m\right) \cdot \left(\frac{c(d,p)}{2^{1+\frac{d}{p'}} n^{\frac{1}{d}}} \|\overline{w}_m\|_{L_q(Q)} \right)^{-1} \frac{\|\overline{w}_m\|_{L_q(Q)}}{\|w\|_{L_q(Q)}} = \frac{\|\overline{w}_m\|_{L_q(Q)}}{\|w\|_{L_q(Q)}}.$$

Since $\|\overline{w}_m\|_{L_q(Q)} \to \|w\|_{L_q(Q)}$ as $m \to \infty$ we obtain

$$\lim_{n\to\infty} E_n\left(W_p^{\nabla}(Q), w\right) \cdot \left(\frac{c(d,p)}{2^{1+\frac{d}{p'}} n^{\frac{1}{d}}} \|w\|_{L_q(Q)} \right)^{-1} \le 1.$$

Applying the same arguments to the function $\underline{w}_m(x) = c + (k-1)\frac{C-c}{m}$ for all $x \in Q_{k,m}$, $k = 1, \dots, m$, we can get the inequality

$$\lim_{n\to\infty} E_n\left(W_p^{\nabla}(Q), w\right) \cdot \left(\frac{c(d,p)}{2^{1+\frac{d}{p'}} n^{\frac{1}{d}}} \|w\|_{L_q(Q)} \right)^{-1} \ge 1.$$

The theorem is proved.

References

1. *Adams R.A., Fournier J.J.F.*: Sobolev Spaces, ISSN. Elsevier Science, 2003.

2. *Babenko V.F.*: Faithful asymptotics of remainders optimal for some classes of functions with cubic weight formulas. Math. Notes **20(4)** (1976), 887–890.
3. *Babenko V.F.*: On optimization of weight quadrature formulas. Ukrainian Math. J. **47(8)** (1995), 1157–1168.
4. *Babenko V.F., Babenko Yu.V., Kovalenko O.V.*: On multivariate Ostrowski type inequalities and their applications. Math. Ineq. Appl. **23(2)** (2020), 569–583.
5. *Chernaya E.V.*: Asymptotically exact estimation of the error of weighted cubature formulas optimal in some classes of continuous functions. Ukrainian Math. J. **47(10)** (1995), 1606–1618.
6. *Chernaya E.V.*: On the optimization of weighted cubature formulae on certain classes of continuous functions. East J. Approx. **1** (1995), 47–60.
7. *Gruber P.M., Lekkerkerker C.G.*: Geometry of Numbers. Bibliotheca mathematica. North-Holland, 1987.
8. *Heinonen J.*: Lectures on Lipschitz Analysis. Bericht (Jyväskylän yliopisto. Matematiikan ja tilastotieteen laitos). University of Jyväskylä, 2005.
9. *Korneichuk N.*: Exact Constants in Approximation Theory. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1991.
10. *Nikolsky S.M.*: Quadrature Formulae. Nauka, Moscow, 1979.
11. *Polovinkin V.I.*: Some estimates of norms of functionals of cubature formula errors. Math. Notes **5(3)** (1969), 192–195.
12. *Sobolev S.L., Vaskevich V.L.*: The Theory of Cubature Formulas. Mathematics and Its Applications. Springer Netherlands, 2013.
13. *Traub J.F., Wozhniakovski H.*: A general theory of optimal algorithms. Acad. Press, 1980.
14. *Zhensykbaev A.A.*: Problems of recovery of operators. Moscow-Izhevsk, 2003.

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