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Criterion of the best non-symmetric approximant for multivariable functions in space $L_{1,p_2,\ldots,p_n}$

Abstract. The criterion of the best non-symmetric approximant for n-variable functions in the space $L_{1,p_2,\ldots,p_n}$ $(1 < p_i < +\infty, i = 2,3,...,n)$ with (α,β)-norm

$$\|f\|_{1,p_2,\ldots,p_n;\alpha,\beta} = \left[\int\limits_{a_n}^{b_n} \cdots \left[\int\limits_{a_2}^{b_2}\left[\int\limits_{a_1}^{b_1} |f(x)|_{\alpha,\beta}dx_1\right]^{p_2} dx_2\right]^{\frac{p_3}{p_2}} \cdots dx_n\right]^{\frac{1}{p_n}},$$

where $0 < \alpha,\beta < \infty$, $f_+(x) = \max\{f(x),0\}$, $f_-(x) = \max\{-f(x),0\}$, $\mathrm{sgn}_{\alpha,\beta} f(x) = \alpha\cdot \mathrm{sgn} f_+(x) - \beta \cdot \mathrm{sgn} f_-(x)$, $|f|_{\alpha,\beta} = \alpha\cdot f_+ + \beta\cdot f_- = f(x)\cdot \mathrm{sgn}_{\alpha,\beta} f(x)$, is obtained in the article.
It is proved that if $P_m = \sum\limits_{k=1}^{m} c_k\varphi_k$, where $\{\varphi_k\}_{k=1}^m$ is a linearly independent system functions of $L_{1,p_2,\ldots,p_n}$, c_k are real numbers, then the polynomial P_m^* is the best (α,β)-approximant for f in the space $L_{1,p_2,\ldots,p_n}$ $(1 < p_i < \infty$, $i = 2,3,...,n)$, if and only if, for any polynomial P_m

$$\int\limits_K P_m\cdot F_0^* dx \le \int\limits_{a_n}^{b_n} ... \int\limits_{a_2}^{b_2} \int\limits_{e_{x_2,\ldots,x_n}} |P_m|_{\beta,\alpha} dx_1 \cdot \operatorname*{ess\,sup}_{x_1\in[a_1,b_1]} |F_0^*|_{\frac{1}{\alpha},\frac{1}{\beta}} dx_2...dx_n,$$

where $K = [a_1,b_1]\times\ldots\times[a_n,b_n]$, $e_{x_2,\ldots,x_n} = \{x_1\in[a_1,b_1] : f - P_m^* = 0\}$,

$$F_0^* = \frac{|R_m^*|_{1;\alpha,\beta}^{p_2-1}|R_m^*|_{1,p_2;\alpha,\beta}^{p_3-p_2}\cdot\ldots\cdot|R_m^*|_{1,p_2,\ldots,p_{n-1};\alpha,\beta}^{p_n-p_{n-1}}\mathrm{sgn}_{\alpha,\beta}R_m^*}{||R_m^*||_{1,p_2,\ldots,p_n;\alpha,\beta}^{p_n-1}},$$

$$|f|_{p_k,\ldots,p_i;\alpha,\beta} = \left[\int\limits_{a_i}^{b_i} \cdots \left[\int\limits_{a_{k+1}}^{b_{k+1}}\left[\int\limits_{a_k}^{b_k} |f|_{\alpha,\beta}^{p_k} dx_k\right]^{\frac{p_{k+1}}{p_k}} dx_{k+1}\right]^{\frac{p_{k+2}}{p_{k+1}}} \ldots dx_i\right]^{\frac{1}{p_i}},$$

$(1\le k < i \le n)$, $R_m^* = f - P_m^*$.
This criterion is a generalization of the known Smirnov's criterion for functions of two variables, when $\alpha=\beta=1$.

Key words: mixed integral metric, polynomial, the best non-symmetric approximant

Анотація. У статті отримано критерій елемента найкращого несиметричного наближення для функцій n-змінних у метриці простору $L_{1,p_2,...,p_n}$, де $1 < p_i < +\infty, i = 2,3,...,n$, з (α,β)-нормою

$$\|f\|_{1,p_2,...,p_n;\alpha,\beta} = \left[\int\limits_{a_n}^{b_n} \cdots \left[\int\limits_{a_2}^{b_2} \left[\int\limits_{a_1}^{b_1} |f(x)|_{\alpha,\beta} dx_1\right]^{p_2} dx_2\right]^{\frac{p_3}{p_2}} \cdots dx_n\right]^{\frac{1}{p_n}},$$

де $0 < \alpha,\beta < \infty$, $f_+(x) = \max\{f(x),0\}$, $f_-(x) = \max\{-f(x),0\}$, $\mathrm{sgn}_{\alpha,\beta} f(x) = \alpha \cdot \mathrm{sgn} f_+(x) - \beta \cdot \mathrm{sgn} f_-(x)$, $|f|_{\alpha,\beta} = \alpha \cdot f_+ + \beta \cdot f_- = f(x) \cdot \mathrm{sgn}_{\alpha,\beta} f(x)$.

Доведено, що, якщо $P_m = \sum\limits_{k=1}^{m} c_k\varphi_k$, де $\{\varphi_k\}_{k=1}^m$ – лінійно незалежна система функцій із $L_{1,p_2,...,p_n}$, c_k – дійсні числа, то полiном P_m^* є поліномом найкращого (α,β)-наближення для функції f у метриці простору $L_{1,p_2,...,p_n}$ $(1 < p_i < \infty,\ i = 2,3,...,n)$, тоді й тільки тоді, коли для будь-якого полінома P_m

$$\int\limits_K P_m \cdot F_0^* dx \le \int\limits_{a_n}^{b_n} ... \int\limits_{a_2}^{b_2} \int\limits_{e_{x_2,...,x_n}} |P_m|_{\beta,\alpha} dx_1 \cdot \operatorname*{ess\,sup}_{x_1\in[a_1,b_1]} |F_0^*|_{\frac{1}{\alpha},\frac{1}{\beta}} dx_2...dx_n,$$

де $K = [a_1,b_1] \times \ldots \times [a_n,b_n]$, $e(x_2,...,x_n) = \{x_1 \in [a_1,b_1] : f - P_m^* = 0\}$,

$$F_0^* = \frac{|R_m^*|_{1;\alpha,\beta}^{p_2-1} |R_m^*|_{1,p_2;\alpha,\beta}^{p_3-p_2} \cdot \ldots \cdot |R_m^*|_{1,p_2,...,p_{n-1};\alpha,\beta}^{p_n-p_{n-1}} \mathrm{sgn}_{\alpha,\beta} R_m^*}{||R_m^*||_{1,p_2,...,p_n;\alpha,\beta}^{p_n-1}},$$

$$|f|_{p_k,...,p_i;\alpha,\beta} = \left[\int\limits_{a_i}^{b_i} \cdots \left[\int\limits_{a_{k+1}}^{b_{k+1}} \left[\int\limits_{a_k}^{b_k} |f|_{\alpha,\beta}^{p_k} dx_k\right]^{\frac{p_{k+1}}{p_k}} dx_{k+1}\right]^{\frac{p_{k+2}}{p_{k+1}}} \ldots dx_i\right]^{\frac{1}{p_i}},$$

$(1 \le k < i \le n)$, $R_m^* = f - P_m^*$.

Даний критерій є узагальненням відомого критерію, доведеного Г.С. Смірновим для функцій двох змінних у випадку, коли $\alpha = \beta = 1$.

Ключові слова: змішана інтегральна метрика, поліном, елемент найкращого несиметричного наближення

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Let $L_{p_1,...,p_n}(1 \le p_i < \infty, 1 \le i \le n)$ be a space of real-valued summable functions of n-variables $f(x) = f(x_1,\ldots,x_n)$, defined on $K = I_1 \times I_2 \times \ldots \times I_n$, where $I_i = [a_i,b_i], 1 \le i \le n$, with the norm

$$\|f\|_{p_1,...,p_n} = \left[\int\limits_{I_n} \cdots \left[\int\limits_{I_2} \left[\int\limits_{I_1} |f(x)|^{p_1} dx_1\right]^{\frac{p_2}{p_1}} dx_2\right]^{\frac{p_3}{p_2}} \ldots dx_n\right]^{\frac{1}{p_n}}.$$

Let's put,

$$|f|_{p_k,\dots,p_i} = \left[\int\limits_{I_i} \dots \left[\int\limits_{I_{k+1}} \left[\int\limits_{I_k} |f(x)|^{p_k} dx_k \right]^{\frac{p_{k+1}}{p_k}} dx_{k+1} \right]^{\frac{p_{k+2}}{p_{k+1}}} \dots dx_i \right]^{\frac{1}{p_i}},$$

where $1 \le k < i \le n$.

We also consider the classes $L_{p_1,\dots,p_n}$ (where at least one $p_i = \infty$) of the functions f, the finite norms of which are determined by formulas

$$\|f\|_{p_1,\dots,p_{n-1},\infty} = \operatorname*{ess\,sup}_{x_n \in I_n} |f|_{p_1,\dots,p_{n-1}},$$

$$\|f\|_{p_1,\dots,p_{i-1},\infty,p_{i+1},\dots,p_n} =$$

$$= \left[\int\limits_{I_n} \dots \left[\int\limits_{I_{i+1}} \left(\operatorname*{ess\,sup}_{x_i \in I_i} |f|_{p_1,\dots,p_{i-1}} \right)^{p_{i+1}} dx_{i+1} \right]^{\frac{p_{i+2}}{p_{i+1}}} \dots dx_n \right]^{\frac{1}{p_n}},$$

where $1 \le i < n$.

If $0 < \alpha, \beta < \infty$, for $f(x)$ we set

$$f_+(x) = \max\{f(x), 0\}, \ \ f_-(x) = \max\{-f(x), 0\},$$

$$\mathrm{sgn}_{\alpha,\beta} f(x) = \alpha \cdot \mathrm{sgn} f_+(x) - \beta \cdot \mathrm{sgn} f_-(x),$$

$$|f|_{\alpha,\beta} = \alpha \cdot f_+ + \beta \cdot f_-.$$

Let us define a non-symmetric norm in $L_{p_1,\dots,p_n}$:

$$\|f\|_{p_1,\dots,p_n;\alpha,\beta} = \left[\int\limits_{I_n} \cdots \left[\int\limits_{I_2} \left[\int\limits_{I_1} |f(x)|^{p_1}_{\alpha,\beta} dx_1 \right]^{\frac{p_2}{p_1}} dx_2 \right]^{\frac{p_3}{p_2}} \cdots dx_n \right]^{\frac{1}{p_n}},$$

also put

$$|f|_{p_k,\dots,p_i;\alpha,\beta} = \left[\int\limits_{I_i} \dots \left[\int\limits_{I_{k+1}} \left[\int\limits_{I_k} |f(x)|^{p_k}_{\alpha,\beta} dx_k \right]^{\frac{p_{k+1}}{p_k}} dx_{k+1} \right]^{\frac{p_{k+2}}{p_{k+1}}} \dots dx_i \right]^{\frac{1}{p_i}},$$

where $1 \le k < i \le n$.

Let $H_m = \text{span}\{\varphi_1, \ldots, \varphi_m\}$ for some system of linearly independent functions $\{\varphi_1, \ldots, \varphi_m\} \subset L_{p_1,\ldots,p_n}$. Then the elements of the set H_m (polynomials P_m) are given in view

$$P_m = \sum_{k=1}^{m} \lambda_k \varphi_k. \tag{1}$$

The quantity

$$E_m(f)_{p_1,\ldots,p_n;\alpha,\beta} = \inf_{P_m \in H_m} \|f - P_m\|_{p_1,\ldots,p_n;\alpha,\beta} \tag{2}$$

we will call the best (α, β)-approximation of the function $f \in L_{p_1,\ldots,p_n}$ by set H_m in $L_{p_1,\ldots,p_n}$. The polynomial P_m^*, which realizes inf in the right-hand side of (2), is called a best (α, β)-approximant of the function f by the set H_m.

In 1973, G.S. Smirnov in [1], formulated and proved the criterion of a best approximant in spaces with mixed integral metrics for functions of two variables. V.M. Traktynska in [3] expended the result by Smirnov in the case of functions of many variables. She obtained the criterion of a best approximant for the function $f \in L_{p_1,\ldots,p_n}$ with the condition $1 < p_i < \infty, 1 \le i \le n$. In the case when at least one of $p_i = 1$, this criterion proved to be true, when the $f - P_m^* \ne 0$ almost everywhere on $I_1 \times I_2 \times \ldots \times I_n$. In [5] these restrictions have been removed by disseminating the results of G.S. Smirnov [2] in the case of approximation of functions of many variables. In this paper, the results of [5] are generalized to the case of an non-symmetric approximation. Note that for $1 < p_i < \infty, 1 \le i \le n$, the criterion of the best non-symmetric approximant for the functions $f \in L_{p_1,\ldots,p_n}$ were obtained in [4].

Let $f \in L_{1,p_2,\ldots,p_n}$ with $||f||_{1,p_2,\ldots,p_n;\alpha,\beta} = |\int\limits_{I_1} |f|_{\alpha,\beta} dx_1|_{p_2,\ldots,p_n} > 0$, and $g \in L_{\infty,q_2,\ldots,q_n}$ with $||g||_{\infty,q_2,\ldots,q_n;\alpha^{-1},\beta^{-1}} = |\operatorname*{ess\,sup}\limits_{x_1 \in I_1} |g|_{\alpha^{-1},\beta^{-1}}|_{q_2,\ldots,q_n} \le 1$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $1 < p_i < \infty$, $i = 2, 3, \ldots, n$.

Using Hölder's inequality and Fubini's theorem, we obtain

$$\int\limits_K f(x) g(x) dx_1 \ldots dx_n \le \int\limits_{I_n} \ldots \int\limits_{I_1} |f|_{\alpha,\beta} |g|_{\alpha^{-1},\beta^{-1}} dx_1 \ldots dx_n \le$$

$$\le \int\limits_{I_n} \ldots \int\limits_{I_2} \left[\int\limits_{I_1} |f|_{\alpha,\beta} dx_1 \right] \left[\operatorname*{ess\,sup}_{x_1 \in I_1} |g|_{\alpha^{-1},\beta^{-1}} \right] dx_2 \ldots dx_n \le$$

$$\int\limits_{I_n} \ldots \int\limits_{I_3} \left[\int\limits_{I_2} \left[\int\limits_{I_1} |f|_{\alpha,\beta} dx_1 \right]^{p_2} dx_2 \right]^{\frac{1}{p_2}} \left[\int\limits_{I_2} \left[\operatorname*{ess\,sup}_{x_1 \in I_1} |g|_{\frac{1}{\alpha},\frac{1}{\beta}} \right]^{q_2} dx_2 \right]^{\frac{1}{q_2}} dx_3 \ldots dx_n \le$$

$$\le ||f||_{1,p_2,\ldots,p_n;\alpha,\beta} \cdot ||g||_{\infty,q_2,\ldots,q_n;\alpha^{-1},\beta^{-1}} \le ||f||_{1,p_2,\ldots,p_n;\alpha,\beta},$$

i.e.

$$\int\limits_K f(x)g(x)dx_1\dots dx_n \le ||f||_{1,p_2,\dots,p_n;\alpha,\beta}. \quad (3)$$

It is easy to verify that for the function $g_0 \in L_{\infty,q_2,\dots,q_n}$ of the form

$$g_0 = ||f||^{1-p_n}_{1,p_2,\dots,p_n;\alpha,\beta}|f|^{p_2-1}_{1;\alpha,\beta}|f|^{p_3-p_2}_{1,p_2;\alpha,\beta}|f|^{p_4-p_3}_{1,p_2,p_3;\alpha,\beta}\cdot\ldots\cdot|f|^{p_n-p_{n-1}}_{1,p_2,\dots,p_{n-1};\alpha,\beta}\cdot\mathrm{sgn}_{\alpha,\beta}f$$

in inequality (3) the sign of equality is achieved, and that all other functions $g \in L_{\infty,q_2,\dots,q_n}$ with $||g||_{\infty,q_2,\dots,q_n;\alpha^{-1},\beta^{-1}} = 1$, which give the sign of equality in (3), coincide with g_0 almost everywhere where $f(x) \neq 0$, and $\operatorname*{ess\,sup}_{x_1\in I_1}|g(x)|_{\alpha^{-1},\beta^{-1}} = \operatorname*{ess\,sup}_{x_1\in I_1}|g_0(x)|_{\alpha^{-1},\beta^{-1}}$ for almost every $(x_2,\dots,x_n) \in I_2\times\ldots\times I_n$.

Next we need the statement obtained in the article of V.M. Traktynska and M.Ye. Tkachenko [4].

Theorem 1. *The best (α,β)-approximation of the function $f \in L_{p_1,\dots,p_n}$ $(1 \le p_i < \infty)$ by set H_m in $L_{p_1,\dots,p_n}$ is*

$$E_m(f)_{p_1,\dots,p_n;\alpha,\beta} = \sup\int\limits_K f(x)g(x)dx_1...dx_n,$$

where sup *extends to the function $g \in L_{q_1,\dots,q_n}$ such that $||g||_{q_1,\dots,q_n;\alpha^{-1},\beta^{-1}} \le 1$ and $\int\limits_K P_m(x)\cdot g(x)dx_1...dx_n = 0$ for any polynomials P_m of the form* (1), *and* sup *is achieved on some functions $\varphi \in L_{q_1,\dots,q_n}$ with norm $||\varphi||_{q_1,\dots,q_n;\alpha^{-1},\beta^{-1}} = 1$.*

Let also

$$P_m = \sum_{k=1}^m c_k\varphi_k, \quad (4)$$

where $\{\varphi_k\}_{k=1}^m$ is a linearly independent system of functions from $L_{1,p_2,\dots,p_n}$ $(1 < p_i < \infty,\ i = 2,3,\dots,n)$, and c_k are real numbers.

Let $f \in L_{1,p_2,\dots,p_n}$ be such that for each P_m the norm $||f - P_m|| > 0$.

We introduce the following functions:

$$F_0 = |R^*_m|^{p_2-1}_{1;\alpha,\beta}|R^*_m|^{p_3-p_2}_{1,p_2;\alpha,\beta}|R^*_m|^{p_4-p_3}_{1,p_2,p_3;\alpha,\beta}\cdot\ldots\cdot|R^*_m|^{p_n-p_{n-1}}_{1,p_2,\dots,p_{n-1};\alpha,\beta}\mathrm{sgn}_{\alpha,\beta}R^*_m,$$

$$F^*_0 = \frac{F_0}{||R^*_m||^{p_n-1}_{1,p_2,\dots,p_n;\alpha,\beta}},$$

where $R^*_m = f - P^*_m$, and P^*_m is some polynomial of the form (4).

It is easy to notice that F^*_0 has the form of a function for which in the inequality of the form (3) the sign of equality is achieved: $F^*_0 \in L_{\infty,q_2,\dots,q_n}$, $||F^*_0||_{\infty,q_2,\dots,q_n;\alpha^{-1},\beta^{-1}} = 1$ and

$$\int\limits_K (f - P^*_m)F^*_0dx_1...dx_n = ||f - P^*_m||_{1,p_2,\dots,p_n;\alpha,\beta}.$$

Denote for each $(x_2,...,x_n) \in I_2 \times \ldots \times I_n$

$$e(x_2,...,x_n) = \{x_1 \in I_1 : f - P_m^* = 0\}.$$

Such a theorem holds.

Theorem 2. *The polynomial P_m^* is a best (α,β)-approximant for the function f in the metric $L_{1,p_2,...,p_n}$ $(1 < p_i < \infty,\ i = 2,3,...,n)$ if and only if the condition*

$$\int\limits_K P_m F_0^* dx_1...dx_n \le \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{e(x_2,...,x_n)} |P_m|_{\beta,\alpha} dx_1 \operatorname*{ess\,sup}_{x_1\in I_1} |F_0^*|_{\frac{1}{\alpha},\frac{1}{\beta}} dx_2...dx_n \tag{5}$$

holds for any polynomial P_m of the form (4).

Proof. We will prove the necessity. Let P_m^* be the best (α,β)-approximant for the function f in the metric $L_{1,p_2,...,p_n}$. Then by Theorem 1 we find such a function $F_1 \in L_{\infty,q_2,...,q_n}$, that:

1) $||F_1||_{\infty,q_2,...,q_n;\alpha^{-1},\beta^{-1}} = 1$;

2) for each P_m of the form (4) $\int\limits_K P_m \cdot F_1 dx_1...dx_n = 0$;

3) $\int\limits_K f \cdot F_1 dx_1...dx_n = \int\limits_K (f - P_m^*) F_1 dx_1...dx_n = ||f - P_m^*||_{1,p_2,...,p_n;\alpha,\beta}$.

The last equation holds only when F_1 satisfies the condition of equality in the inequality (3), so $F_0^*(x) = F_1(x)$ for all points x for which $f - P_m^* \ne 0$, and $\operatorname*{ess\,sup}_{x_1\in I_1} |F_0^*| = \operatorname*{ess\,sup}_{x_1\in I_1} |F_1|$ almost everywhere on $I_2 \times \ldots \times I_n$. Then for each polynomial P_m we have

$$\int\limits_K P_m \cdot F_0^* dx_1...dx_n = \int\limits_K P_m \cdot (F_0^* - F_1) dx_1...dx_n =$$

$$= \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{e(x_2,...,x_n)} P_m \cdot (F_0^* - F_1) dx_1 dx_2...dx_n =$$

$$= \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{e(x_2,...,x_n)} P_m \cdot (-F_1) dx_1 dx_2...dx_n \le$$

$$\le \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{e(x_2,...,x_n)} |P_m|_{\beta,\alpha} \cdot |-F_1|_{\beta^{-1},\alpha^{-1}} dx_1 dx_2...dx_n =$$

$$= \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{e(x_2,...,x_n)} |P_m|_{\beta,\alpha} \cdot |F_1|_{\alpha^{-1},\beta^{-1}} dx_1 dx_2...dx_n \le$$

$$\le \int\limits_{I_n} ... \int\limits_{I_2} \left[\left(\int\limits_{e(x_2,...,x_n)} |P_m|_{\beta,\alpha} dx_1 \right) \left(\operatorname*{ess\,sup}_{x_1\in I_1} |F_0^*|_{\alpha^{-1},\beta^{-1}} \right) \right] dx_2...dx_n.$$

Necessity proved.

We will prove sufficiency.

Suppose that for every polynomial P_m of the form (4) holds condition (5). Then, using inequality (5) at the end, for each polynomial $Q_m = P_m^* + P_m$ we get

$$||f - Q_m||_{1,p_2,...,p_n;\alpha,\beta} = ||f - P_m^* - P_m||_{1,p_2,...,p_n;\alpha,\beta} \cdot ||F_0^*||_{\infty,q_2,...,q_n;\alpha^{-1},\beta^{-1}} \geq$$

$$\geq \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{I_1} |f - P_m^* - P_m|_{\alpha,\beta} dx_1 \cdot \operatorname*{ess\,sup}_{x_1 \in I_1} |F_0^*|_{\alpha^{-1},\beta^{-1}} dx_2...dx_n =$$

$$= \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{I_1 \setminus e(x_2,...,x_n)} |f - P_m^* - P_m|_{\alpha,\beta} dx_1 \cdot \operatorname*{ess\,sup}_{x_1 \in I_1} |F_0^*|_{\alpha^{-1},\beta^{-1}} dx_2...dx_n +$$

$$+ \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{e(x_2,...,x_n)} |-P_m|_{\alpha,\beta} dx_1 \cdot \operatorname*{ess\,sup}_{x_1 \in I_1} |F_0^*|_{\alpha^{-1},\beta^{-1}} dx_2...dx_n \geq$$

$$\geq \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{I_1 \setminus e(x_2,...,x_n)} |f - P_m^* - P_m|_{\alpha,\beta} \cdot |F_0^*|_{\alpha^{-1},\beta^{-1}} dx_1...dx_n +$$

$$+ \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{e(x_2,...,x_n)} |P_m|_{\beta,\alpha} dx_1 \cdot \operatorname*{ess\,sup}_{x_1 \in I_1} |F_0^*|_{\alpha^{-1},\beta^{-1}} dx_2...dx_n \geq$$

$$\geq \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{I_1 \setminus e(x_2,...,x_n)} (f - P_m^* - P_m) \cdot F_0^* dx_1...dx_n +$$

$$+ \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{e(x_2,...,x_n)} |P_m|_{\beta,\alpha} dx_1 \cdot \operatorname*{ess\,sup}_{x_1 \in I_1} |F_0^*|_{\alpha^{-1},\beta^{-1}} dx_2...dx_n =$$

$$= \int\limits_K (f - P_m^*) F_0^* dx_1...dx_n - \int\limits_K P_m F_0^* dx_1...dx_n +$$

$$+ \int\limits_{I_n} ... \int\limits_{I_2} \int\limits_{e(x_2,...,x_n)} |P_m|_{\beta,\alpha} dx_1 \cdot \operatorname*{ess\,sup}_{x_1 \in I_1} |F_0^*|_{\alpha^{-1},\beta^{-1}} dx_2...dx_n \geq$$

$$\geq \int\limits_K (f - P_m^*) F_0^* dx_1...dx_n = ||f - P_m^*||_{1,p_2,...,p_n;\alpha,\beta}.$$

This means that P_m^* is the best (α,β)-approximant for f in the metric $L_{1,p_2,...,p_n}$. Theorem proved.

The condition of Theorem 2 can be strengthened.

Theorem 3. *The polynomial* P_m^* *is a best* (α,β)*-approximant for the function* f *in the metric* $L_{1,p_2,...,p_n}$ $(1 < p_i < \infty,\ i = 2,3,...,n)$ *if and only if the condition*

$$-\int\limits_{I_n}\dots\int\limits_{I_2}\ \int\limits_{e(x_2,...,x_n)} |P_m|_{\alpha,\beta}dx_1\cdot\operatorname*{ess\,sup}_{x_1\in I_1}|F_0^*|_{\frac{1}{\alpha},\frac{1}{\beta}}dx_2...dx_n \le \int\limits_K P_mF_0^*dx_1...dx_n \le$$

$$\le \int\limits_{I_n}\dots\int\limits_{I_2}\ \int\limits_{e(x_2,...,x_n)} |P_m|_{\beta,\alpha}dx_1\cdot\operatorname*{ess\,sup}_{x_1\in I_1}|F_0^*|_{\frac{1}{\alpha},\frac{1}{\beta}}dx_2...dx_n.$$

holds for any polynomial P_m *of the form* (4).

To prove this statement, it suffices to note that for the function F_1 from the proof of the necessary condition of Theorem 2 we have:

$$\int\limits_K P_m\cdot F_0^*dx_1...dx_n = \int\limits_{I_n}\dots\int\limits_{I_2}\ \int\limits_{e(x_2,...,x_n)} P_m\cdot(-F_1)dx_1dx_2...dx_n =$$

$$= -\int\limits_{I_n}\dots\int\limits_{I_2}\ \int\limits_{e(x_2,...,x_n)} P_m\cdot F_1dx_1dx_2...dx_n \ge$$

$$\ge -\int\limits_{I_n}\dots\int\limits_{I_2}\ \int\limits_{e(x_2,...,x_n)} |P_m|_{\alpha,\beta}\cdot|F_1|_{\alpha^{-1},\beta^{-1}}dx_1dx_2...dx_n \ge$$

$$\ge -\int\limits_{I_n}\dots\int\limits_{I_2}\ \int\limits_{e(x_2,...,x_n)} |P_m|_{\alpha,\beta}dx_1\cdot\operatorname*{ess\,sup}_{x_1\in I_1}|F_0^*|_{\alpha^{-1},\beta^{-1}}dx_2...dx_n.$$

In this form, the criterion is a generalization of the criterion of the best approximant in the case $\alpha = \beta = 1$ from [5].

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