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Three- and four-term recurrence relations for Horn's hypergeometric function H_4

Abstract. Three- and four-term recurrence relations for hypergeometric functions of the second order (such as hypergeometric functions of Appell, Horn, etc.) are the starting point for constructing branched continued fraction expansions of the ratios of these functions. These relations are essential for obtaining the simplest structure of branched continued fractions (elements of which are simple polynomials) for approximating the solutions of the systems of partial differential equations, as well as some analytical functions of two variables. In this study, three- and four-term recurrence relations for Horn's hypergeometric function H_4 are derived. These relations can be used to construct branched continued fraction expansions for the ratios of this function and they are a generalization of the classical three-term recurrent relations for Gaussian hypergeometric function underlying Gauss' continued fraction.

Key words: hypergeometric function, recurrent relation, branched continued fraction

Анотація. Три- і чотиричленні рекурентні співвідношення гіпергеометричних функцій другого порядку (наприклад, гіпергеометричних функцій Аппеля, Горна та ін.) є основою для побудови гіллястих ланцюгових дробових розвинень відношень цих функцій. Ці співвідношення є важливими для отримання найпростішої структури гіллястих ланцюгових дробів (елементами яких є прості поліноми) для апроксимації розв'язків систем диференціальних рівнянь з частинними похідними, а також деяких аналітичних функцій двох змінних. У цьому дослідженні доведено три- та чотиричленні рекурентні співвідношення гіпергеометричної функції Горна H_4. Співвідношення можуть бути використані для побудови гіллястих ланцюгових дробових розвинень відношень цієї функції; вони є узагальненням класичних тричленних рекурентних співвідношень для гіпергеометричної функції Гауса, що лежить в основі неперервного дробу Гауса.

Ключові слова: гіпергеометрична функція, рекурентне співвідношення, гіллястий ланцюговий дріб

MSC2020: Pri 32A17, Sec 32A05, 65Q30

1. Introduction

Deriving three-, four-, and higher-term recurrent relations of second-order hypergeometric functions has been and remains interesting for many scientists. After all, this is due to the fact that these functions are of importance in many applications, including mathematical physics. The authors discuss various recurrence relations for these functions from the point of view that they are associated with orthogonality properties, differential equations, quantum calculus, etc. In this study, this matter is taken up in connection with the problem of constructing of branched continued fraction expansions for Horn's hypergeometric function ratios. A discussion of some branched continued fraction expansions of the ratios of different generalizations of Gaussian hypergeometric function and the problems associated with them can found [1, 2, 3, 5, 6, 17, 18]. This all is the more remarkable in the view of fact that branched continued fractions are endowed under certain conditions with wide regions of convergence, good convergence rate, and stability of calculations are an effective tool for approximation of certain analytical functions (see [2, 4, 8, 9, 10, 11, 12, 13, 14]).

Horn's hypergeometric function H_4 is defined by double power series (see [16])

$$H_4(a,b;c,d;z_1,z_2)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n}\frac{z_1^m}{m!}\frac{z_2^n}{n!},$$
$$|z_1|<r,\quad |z_2|<s,\quad 4r=(s-1)^2, \tag{1}$$

where a, b and c are complex constants, c and d is not equal to a non-positive integer, z_1 and z_2 are complex variables, $(\cdot)_k$ is the Pochhammer symbol defined for any complex number α and non-negative integers k by $(\alpha)_0=1$ and $(\alpha)_k=\alpha(\alpha+1)\dots(\alpha+k-1)$.

Note that (1) is two-dimensional generalization of Gaussian hypergeometric function. Indeed,

$$H_4(a,b;c,d;0,z_2)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(d)_n}\frac{z_2^n}{n!}={}_2F_1(a,b;d;z_2),$$

$$H_4(a,b;c,d;z_1,0)=H_4(a,0;c,d;z_1,z_2)=\sum_{m=0}^{\infty}\frac{(a)_{2m}}{(c)_m}\frac{z_1^m}{m!}$$
$$=\sum_{m=0}^{\infty}\frac{(a/2)_m((a+1)/2)_m}{(c)_m}\frac{(4z_1)^m}{m!}={}_2F_1(a/2,(a+1)/2;c;4z_1).$$

In addition, the solutions of partial differential equations

$$z_1(1-4z_1)\frac{\partial^2u}{\partial z_1^2}-4z_1z_2\frac{\partial^2u}{\partial z_1\partial z_2}-z_2^2\frac{\partial^2u}{\partial z_2^2}$$
$$+(c-(4a+4)z_1)\frac{\partial u}{\partial z_1}-(3a+2)z_2\frac{\partial u}{\partial z_2}-a(a+1)u=0,$$

$$-2z_1z_2\frac{\partial^2 u}{\partial z_1\partial z_2}+z_2(1-z_2)\frac{\partial^2 u}{\partial z_2^2}-2bz_1\frac{\partial u}{\partial z_1}+(d-(a+b)z_2)\frac{\partial u}{\partial z_2}-abu=0,$$

where u is the unknown function of z_1 and z_2, are expressed by means of Horn's hypergeometric function H_4 [16] (see also [15, p. 235]). Some recurrent relations for this function can be found in [7, 15, 16, 19].

2. Main results

We prove two three-term recurrent relations for the function (1).

Theorem 1. *The following assertion holds true*

$$\begin{aligned}&H_4(a,b;c,d;z_1,z_2)-H_4(a,b+1;c,d;z_1,z_2)\\&\quad=-\frac{a}{d}z_2H_4(a+1,b+1;c,d+1;z_1,z_2).\end{aligned}\tag{2}$$

Proof. By formula (1), we have

$$\begin{aligned}&H_4(a,b;c,d;z_1,z_2)-H_4(a,b+1;c,d;z_1,z_2)\\&\quad=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n}\frac{z_1^m}{m!}\frac{z_2^n}{n!}-\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b+1)_n}{(c)_m(d)_n}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\\&\quad=\sum_{m\geq0,\ n\geq1}\frac{(a)_{2m+n}(b+1)_{n-1}}{(c)_m(d)_n}(b-b-n)\frac{z_1^m}{m!}\frac{z_2^n}{n!}\\&\quad=-\frac{a}{d}z_2\sum_{m\geq0,\ n\geq1}\frac{(a+1)_{2m+n-1}(b+1)_{n-1}}{(c)_m(d+1)_{n-1}}\frac{z_1^m}{m!}\frac{z_2^{n-1}}{(n-1)!}\\&\quad=-\frac{a}{d}z_2\sum_{m\geq0,\ n\geq0}\frac{(a+1)_{2m+n}(b+1)_n}{(c)_m(d+1)_n}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\\&\quad=-\frac{a}{d}z_2H_4(a+1,b+1;c,d+1;z_1,z_2).\end{aligned}$$

Thus, the three-term recurrent relation (2) is proved.

Theorem 2. *The following assertion holds true*

$$\begin{aligned}&H_4(a,b;c,d;z_1,z_2)-H_4(a,b+1;c,d+1;z_1,z_2)\\&\quad=-\frac{a(d-b)}{d(d+1)}z_2H_4(a+1,b+1;c,d+2;z_1,z_2).\end{aligned}$$

Proof. Using the idea of proving relation (2), we obtain

$$\begin{aligned}&H_4(a,b;c,d;z_1,z_2)-H_4(a,b+1;c,d+1;z_1,z_2)\\&\quad=\sum_{m\geq0,\ n\geq1}\frac{(a)_{2m+n}(b+1)_{n-1}}{(c)_m(d+1)_{n-1}}\left(\frac{b}{d}-\frac{b+n}{d+n}\right)\frac{z_1^m}{m!}\frac{z_2^n}{n!}\end{aligned}$$

$$= -\sum_{m\geq 0,\, n\geq 1} \frac{(a)_{2m+n}(b+1)_{n-1}}{(c)_m(d+1)_{n-1}} \frac{n(d-b)}{d(d+n)} \frac{z_1^m}{m!} \frac{z_2^n}{n!}$$
$$= -\frac{a(d-b)}{d(d+1)} z_2 \sum_{m\geq 0,\, n\geq 1} \frac{(a+1)_{2m+n-1}(b+1)_{n-1}}{(c)_m(d+2)_{n-1}} \frac{z_1^m}{m!} \frac{z_2^{n-1}}{(n-1)!}$$
$$= -\frac{a(d-b)}{d(d+1)} z_2 H_4(a+1, b+1; c, d+2; z_1, z_2).$$

The Theorem 2 is proved.

Next, we also prove three four-term recurrent relations for the function (1).

Theorem 3. *The following assertion holds true*

$$\begin{aligned} &H_4(a,b;c,d;z_1,z_2) - H_4(a+1,b;c+1,d;z_1,z_2) \\ &\quad = -\frac{(2c-a)(a+1)}{c(c+1)} z_1 H_4(a+2,b;c+2,d;z_1,z_2) \\ &\qquad - \frac{b}{d} z_2 H_4(a+1,b+1;c+1,d+1;z_1,z_2). \end{aligned} \tag{3}$$

Proof. In the left part (3) using (1) and separating the terms at $m = 0$, we get

$$\begin{aligned} &H_4(a,b;c,d;z_1,z_2) - H_4(a+1,b;c+1,d;z_1,z_2) \\ &= \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!} - \sum_{m,n=0}^{\infty} \frac{(a+1)_{2m+n}(b)_n}{(c+1)_m(d)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(d)_n} \frac{z_2^n}{n!} - \sum_{n=0}^{\infty} \frac{(a+1)_n(b)_n}{(d)_n} \frac{z_2^n}{n!} \\ &\quad + \sum_{m\geq 1,\, n\geq 0} \frac{(a+1)_{2m+n-1}(b)_n}{(c+1)_{m-1}(d)_n} \left(\frac{a}{c} - \frac{a+2m+n}{c+m} \right) \frac{z_1^m}{m!} \frac{z_2^n}{n!} \\ &= \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b)_n}{(d)_n} (a-a-n) \frac{z_2^n}{n!} \\ &\quad + \sum_{m\geq 1,\, n\geq 0} \frac{(a+1)_{2m+n-1}(b)_n}{(c+1)_{m-1}(d)_n} \frac{ma-2mc-nc}{c(c+m)} \frac{z_1^m}{m!} \frac{z_2^n}{n!} \\ &= -\frac{b}{d} z_2 \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b+1)_{n-1}}{(d+1)_{n-1}} \frac{z_2^{n-1}}{n!} n + \sum_{m=1}^{\infty} \frac{(a+1)_{2m-1}}{(c+1)_{m-1}} \frac{m(a-2c)}{c(c+m)} \frac{z_1^m}{m!} \\ &\quad + \sum_{m\geq 1,\, n\geq 1} \frac{(a+1)_{2m+n-1}(b)_n}{(c+1)_{m-1}(d)_n} \frac{ma-2mc-nc}{c(c+m)} \frac{z_1^m}{m!} \frac{z_2^n}{n!} \end{aligned}$$

$$= -\frac{b}{d} z_2 \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(d+1)_n} \frac{z_2^n}{n!}$$

$$- \frac{(2c-a)(a+1)}{c(c+1)} z_1 \sum_{m=1}^{\infty} \frac{(a+2)_{2m-2}}{(c+2)_{m-1}} \frac{z_1^{m-1}}{(m-1)!}$$

$$+ \sum_{m\geq 1,\, n\geq 1} \frac{(a+1)_{2m+n-1}(b)_n}{(c+1)_{m-1}(d)_n} \frac{ma-2mc-nc}{c(c+m)} \frac{z_1^m}{m!} \frac{z_2^n}{n!}$$

$$= -\frac{b}{d} z_2 \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(d+1)_n} \frac{z_2^n}{n!} - \frac{(2c-a)(a+1)}{c(c+1)} z_1 \sum_{m=0}^{\infty} \frac{(a+2)_{2m}}{(c+2)_m} \frac{z_1^m}{m!}$$

$$+ \sum_{m\geq 1,\, n\geq 1} \frac{(a+1)_{2m+n-1}(b)_n}{(c+1)_{m-1}(d)_n} \frac{ma-2mc-nc}{c(c+m)} \frac{z_1^m}{m!} \frac{z_2^n}{n!}.$$

Since

$$\sum_{m\geq 1,\, n\geq 1} \frac{(a+1)_{2m+n-1}(b)_n}{(c+1)_{m-1}(d)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!} \frac{ma-2mc-nc}{c(c+m)}$$

$$= -\frac{(2c-a)(a+1)}{c(c+1)} z_1 \sum_{m\geq 1,\, n\geq 1} \frac{(a+2)_{2(m-1)+n}(b)_n}{(c+2)_{m-1}(d)_n} \frac{z_1^{m-1}}{(m-1)!} \frac{z_2^n}{n!}$$

$$- \frac{b}{d} z_2 \sum_{m\geq 1,\, n\geq 1} \frac{(a+1)_{2m+n-1}(b+1)_{n-1}}{(c+1)_m (d+1)_{n-1}} \frac{z_1^m}{m!} \frac{z_2^{n-1}}{(n-1)!}$$

$$= -\frac{(2c-a)(a+1)}{c(c+1)} z_1 \sum_{m\geq 0,\, n\geq 1} \frac{(a+2)_{2m+n}(b)_n}{(c+2)_m (d)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!}$$

$$- \frac{b}{d} z_2 \sum_{m\geq 1,\, n\geq 0} \frac{(a+1)_{2m+n}(b+1)_n}{(c+1)_m (d+1)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!}$$

and, consequently,

$$-\frac{(2c-a)(a+1)}{c(c+1)} z_1 \sum_{m=0}^{\infty} \frac{(a+2)_{2m}}{(c+2)_m} \frac{z_1^m}{m!}$$

$$- \frac{(2c-a)(a+1)}{c(c+1)} z_1 \sum_{m\geq 0,\, n\geq 1} \frac{(a+2)_{2m+n}(b)_n}{(c+2)_m (d)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!}$$

$$= -\frac{(2c-a)(a+1)}{c(c+1)} z_1 \sum_{m\geq 0,\, n\geq 0} \frac{(a+2)_{2m+n}(b)_n}{(c+2)_m (d)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!}, \tag{4}$$

$$-\frac{b}{d} z_2 \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(d+1)_n} \frac{z_2^n}{n!} - \frac{b}{d} z_2 \sum_{m\geq 1,\, n\geq 0} \frac{(a+1)_{2m+n}(b+1)_n}{(c+1)_m (d+1)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!}$$

$$= -\frac{b}{d} z_2 \sum_{m\geq 0,\, n\geq 0} \frac{(a+1)_{2m+n}(b+1)_n}{(c+1)_m (d+1)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!}, \tag{5}$$

then, adding (4) to (5), we are convinced of the validity of relation (3).

Theorem 4. *The following assertion holds true*

$$\begin{aligned} &H_4(a,b;c,d;z_1,z_2)-H_4(a+1,b;c,d+1;z_1,z_2)\\ &\quad=-\frac{2(a+1)}{c}z_1H_4(a+2,b;c+1,d+1;z_1,z_2)\\ &\qquad-\frac{b(d-a)}{d(d+1)}z_2H_4(a+1,b+1;c,d+2;z_1,z_2). \end{aligned} \tag{6}$$

Proof. Using the idea of proving the previous theorem in the left part (6) with separation of terms at $n=0$, we have

$$\begin{aligned} &H_4(a,b;c,d;z_1,z_2)-H_4(a+1,b;c,d+1;z_1,z_2)\\ &\quad=\sum_{m=0}^{\infty}\frac{(a)_{2m}}{(c)_m}\frac{z_1^m}{m!}-\sum_{m=0}^{\infty}\frac{(a+1)_{2m}}{(c)_m}\frac{z_1^m}{m!}\\ &\qquad+\sum_{m\ge0,\,n\ge1}\frac{(a+1)_{2m+n-1}(b)_n}{(c)_m(d+1)_{n-1}}\left(\frac{a}{d}-\frac{a+2m+n}{d+n}\right)\frac{z_1^m}{m!}\frac{z_2^n}{n!}\\ &\quad=-\frac{2(a+1)}{c}z_1\sum_{m=1}^{\infty}\frac{(a+2)_{2(m-1)}}{(c+1)_{m-1}}\frac{z_1^{m-1}}{(m-1)!}\\ &\qquad-\sum_{m\ge0,\,n\ge1}\frac{(a+1)_{2m+n-1}(b)_n}{(c)_m(d+1)_{n-1}}\frac{2md+nd-na}{d(d+n)}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\\ &\quad=-\frac{2(a+1)}{c}z_1\sum_{m=0}^{\infty}\frac{(a+2)_{2m}}{(c+1)_m}\frac{z_1^m}{m!}\\ &\qquad-\sum_{m\ge0,\,n\ge1}\frac{(a+1)_{2m+n-1}(b)_n}{(c)_m(d+1)_{n-1}}\frac{2m}{d+n}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\\ &\qquad-\frac{b(d-a)}{d(d+1)}z_2\sum_{m\ge0,\,n\ge1}\frac{(a+1)_{2m+n-1}(b+1)_{n-1}}{(c)_m(d+2)_{n-1}}\frac{z_1^m}{m!}\frac{z_2^{n-1}}{(n-1)!}\\ &\quad=-\frac{2(a+1)}{c}z_1\sum_{m=0}^{\infty}\frac{(a+2)_{2m}}{(c+1)_m}\frac{z_1^m}{m!}\\ &\qquad-\frac{2(a+1)}{c}z_1\sum_{m\ge1,\,n\ge1}\frac{(a+2)_{2m+n-2}(b)_n}{(c)_{m-1}(d+1)_n}\frac{z_1^{m-1}}{(m-1)!}\frac{z_2^n}{n!}\\ &\qquad-\frac{b(d-a)}{d(d+1)}z_2\sum_{m\ge0,\,n\ge0}\frac{(a+1)_{2m+n}(b+1)_n}{(c)_m(d+2)_n}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\\ &\quad=-\frac{2(a+1)}{c}z_1\sum_{m=0}^{\infty}\frac{(a+2)_{2m}}{(c+1)_m}\frac{z_1^m}{m!}\\ &\qquad-\frac{2(a+1)}{c}z_1\sum_{m\ge0,\,n\ge1}\frac{(a+2)_{2m+n}(b)_n}{(c+1)_m(d+1)_n}\frac{z_1^m}{m!}\frac{z_2^n}{n!} \end{aligned}$$

$$- \frac{b(d-a)}{d(d+1)} z_2 \sum_{m\geq 0,\, n\geq 0} \frac{(a+1)_{2m+n}(b+1)_n}{(c)_m(d+2)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!}$$

$$= -\frac{2(a+1)}{c} z_1 \sum_{m\geq 0,\, n\geq 0} \frac{(a+2)_{2m+n}(b)_n}{(c+1)_m(d+1)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!}$$

$$- \frac{b(d-a)}{d(d+1)} z_2 \sum_{m\geq 0,\, n\geq 0} \frac{(a+1)_{2m+n}(b+1)_n}{(c)_m(d+2)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!}.$$

At last, in view of formula (1), we directly obtain the recurrent relation (6).

Theorem 5. *The following assertion holds true*

$$\begin{aligned} H_4(a,b;c,d;z_1,z_2) &- H_4(a+1,b;c,d;z_1,z_2) \\ &= -\frac{2(a+1)}{c} z_1 H_4(a+2,b;c+1,d;z_1,z_2) \\ &\quad - \frac{b}{d} z_2 H_4(a+1,b+1;c,d+1;z_1,z_2). \end{aligned} \tag{7}$$

Proof. We have

$$\begin{aligned} &H_4(a,b;c,d;z_1,z_2) - H_4(a+1,b;c,d;z_1,z_2) \\ &= \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!} - \sum_{m,n=0}^{\infty} \frac{(a+1)_{2m+n}(b)_n}{(c)_m(d)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(d)_n} \frac{z_2^n}{n!} - \sum_{n=0}^{\infty} \frac{(a+1)_n(b)_n}{(d)_n} \frac{z_2^n}{n!} \\ &\quad + \sum_{m\geq 1,\, n\geq 0} \frac{(a+1)_{2m+n-1}(b)_n}{(c)_m(d)_n} (a-a-2m-n) \frac{z_1^m}{m!} \frac{z_2^n}{n!} \\ &= -\sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b)_n}{(d)_n} n \frac{z_2^n}{n!} - \sum_{m\geq 1,\, n\geq 0} \frac{(a+1)_{2m+n-1}(b)_n}{(c)_m(d)_n} (2m+n) \frac{z_1^m}{m!} \frac{z_2^n}{n!} \\ &= -\frac{b}{d} z_2 \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b+1)_{n-1}}{(d+1)_{n-1}} \frac{z_2^{n-1}}{(n-1)!} \\ &\quad - \sum_{m\geq 1,\, n\geq 0} \frac{(a+1)_{2m+n-1}(b)_n}{(c+1)_{m-1}(d)_n} \frac{2}{c} \frac{z_1^m}{(m-1)!} \frac{z_2^n}{n!} \\ &\quad - \sum_{m\geq 1,\, n\geq 1} \frac{(a+1)_{2m+n-1}(b+1)_{n-1}}{(c)_m(d+1)_{n-1}} \frac{b}{d} \frac{z_1^m}{m!} \frac{z_2^n}{(n-1)!} \\ &= -\frac{b}{d} z_2 \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(d+1)_n} \frac{z_2^n}{n!} \\ &\quad - \frac{2(a+1)}{c} z_1 \sum_{m\geq 1,\, n\geq 0} \frac{(a+2)_{2m+n-2}(b)_n}{(c+1)_{m-1}(d)_n} \frac{z_1^{m-1}}{(m-1)!} \frac{z_2^n}{n!} \end{aligned}$$

$$
\begin{aligned}
&-\frac{b}{d}z_2\sum_{m\geq 1,\, n\geq 1}\frac{(a+1)_{2m+n-1}(b+1)_{n-1}}{(c)_m(d+1)_{n-1}}\frac{z_1^m}{m!}\frac{z_2^{n-1}}{(n-1)!}\\
=&-\frac{2(a+1)}{c}z_1\sum_{m\geq 0,\, n\geq 0}\frac{(a+2)_{2m+n}(b)_n}{(c+1)_m(d)_n}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\\
&-\frac{b}{d}z_2\sum_{m\geq 0,\, n\geq 0}\frac{(a+1)_{2m+n}(b+1)_n}{(c)_m(d+1)_n}\frac{z_1^m}{m!}\frac{z_2^n}{n!}.
\end{aligned}
$$

Finally, using the formula (1), we obtain the recurrent relation (7).

References

1. *Antonova T., Dmytryshyn R., Kravtsiv V.*: Branched continued fraction expansions of Horn's hypergeometric function H_3 ratios. Mathematics **9(2)** (2021), 148. doi:10.3390/math9020148
2. *Antonova T., Dmytryshyn R., Sharyn S.*: Generalized hypergeometric function ${}_3F_2$ ratios and branched continued fraction expansions. Axioms **10(4)** (2021), 310. doi:10.3390/axioms10040310
3. *Antonova T.M*: On convergence of branched continued fraction expansions of Horn's hypergeometric function ratios. Carpathian Math. Publ. **13(3)** (2021), 642–650. doi:10.15330/cmp.13.3.642-650
4. *Bodnar D.I., Dmytryshyn R.I.*: Multidimensional associated fractions with independent variables and multiple power series. Ukr. Math. J. **71(3)** (2019), 370–386. doi:10.1007/s11253-019-01652-5
5. *Bodnar D.I., Hoyenko N.P.*: Approximation of the ratio of Lauricella functions by a branched continued fraction. Mat. Stud. **20(2)** (2003), 210–214. (in Ukrainian)
6. *Bodnar D.I., Manzii O.S.*: Expansion of the ratio of Appel hypergeometric functions F_3 into a branching continued fraction and its limit behavior. J. Math. Sci. **107(1)** (2001), 3550–3554. doi:10.1023/A:1011977720316
7. *Brychkov Yu.A., Savischenko N.V.*: On some formulas for the Horn functions $H_4(a,b;c,c';w,z)$ and $H_7^{(c)}(a;c,c';w,z)$. Integral Transforms Spec. Funct. **32(2)** (2021), 969–987. doi:10.1080/10652469.2021.1878356
8. *Dmytryshyn R.I.*: Associated branched continued fractions with two independent variables. Ukr. Math. J. **66(9)**, (2015), 1312–1323. doi:10.1007/s11253-015-1011-6
9. *Dmytryshyn R.I.*: Multidimensional regular C-fraction with independent variables corresponding to formal multiple power series. Proc. Roy. Soc. Edinburgh Sect. A **150(4)** (2020), 1853–1870. doi:10.1017/prm.2019.2
10. *Dmytryshyn R.I.*: On the expansion of some functions in a two-dimensional g-fraction with independent variables. J. Math. Sci. **181(3)** (2012), 320–327. doi:10.1007/s10958-012-0687-5
11. *Dmytryshyn R.I., Sharyn S.V.*: Approximation of functions of several variables by multidimensional S-fractions with independent variables. Carpathian Math. Publ. **13(3)** (2021), 592–607. doi:10.15330/cmp.13.3.592-607
12. *Dmytryshyn R.I.*: The multidimensional generalization of g-fractions and their application. J. Comp. and Appl. Math. **164–165** (2004), 265–284. doi:10.1016/S0377-0427(03)00642-3

13. *Dmytryshyn R.I.*: The two-dimensional g-fraction with independent variables for double power series. J. Approx. Theory **164(12)** (2012), 1520–1539. doi:10.1016/j.jat.2012.09.002
14. *Dmytryshyn R.I.*: Two-dimensional generalization of the Rutishauser qd-algorithm. J. Math. Sci. **208(3)** (2015), 301–309. doi:10.1007/s10958-015-2447-9
15. *Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G.*: Higher transcendental functions. Volume 1. McGraw-Hill Book Co., 1953.
16. *Horn J.*: Hypergeometrische funktionen zweier veränderlichen. Math. Ann. **105(1)** (1931), 381–407. doi:10.1007/BF01455825
17. Hoyenko N., Hladun V., Manzij O.: On the infinite remains of the Nórlund branched continued fraction for Appell hypergeometric functions. Carpathian Math. Publ. **6(1)** (2014), 11–25. doi:10.15330/cmp.6.1.11-25 (in Ukrainian)
18. *Petreolle M., Sokal A.D.*: Lattice paths and branched continued fractions II. Multivariate Lah polynomials and Lah symmetric functions. Eur. J. Combin. **92** (2021), 103235. doi:10.1016/j.ejc.2020.103235
19. *Shehata A.*: On basic Horn hypergeometric functions H_3 and H_4. Adv. Differ. Equ. **2020** (2020), 595. doi:10.1186/s13662-020-03056-3

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