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A parametric type of Bernoulli polynomials with higher level

Abstract. In this paper, we introduce a parametric type of Bernoulli polynomials with higher level and study their characteristic and combinatorial properties. We also give determinant expressions of a parametric type of Bernoulli polynomials with higher level. The results are generalizations of those with level 2 by Masjed-Jamei, Beyki and Koepf and with level 3 by the author.

Key words: Bernoulli polynomials and numbers, recurrence relations, determinants

Анотація. У цій роботі ми вводимо параметричний тип поліномів Бернуллі вищого рівня та вивчаємо їх характеристики та комбінаторні властивості. Наведено також вирази визначника параметричного типу поліномів Бернуллі вищого рівня. Результати є узагальненнями результатів рівня 2 за Масджедом-Джамеї, Бейкі та Кепфом й рівня 3 за автором.

Ключові слова: Поліноми та числа Бернуллі, рекурентні співвідношення, визначники

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1. Introduction

Let $r \geq 1$. For real numbers p, q , define *bivariate Bernoulli polynomials* $B_n^{(r,i)}(p, q)$ with higher level ($i = 0, 1, \dots, r - 1$) as

$$\frac{te^{pt}f_q^{(r,i)}(t)}{e^t - 1} = \sum_{n=0}^{\infty} B_n^{(r,i)}(p, q) \frac{t^n}{n!}, \tag{1}$$

where

$$f_q^{(r,i)}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(qt)^{rn+i}}{(rn+i)!}. \tag{2}$$

Their complementary polynomials $\widehat{B}_n^{(r,i)}(p, q)$ ($i = 0, 1, \dots, r - 1$) are defined as

$$\frac{te^{pt} \widehat{f}_q^{(r,i)}(t)}{e^t - 1} = \sum_{n=0}^{\infty} \widehat{B}_n^{(r,i)}(p, q) \frac{t^n}{n!}, \quad (3)$$

where

$$\widehat{f}_q^{(r,i)}(t) = \sum_{n=0}^{\infty} \frac{(qt)^{rn+i}}{(rn+i)!}. \quad (4)$$

When $r = 1$, $f_q^{(1,0)} = f_q^{(1,0)}(t) = e^{-qt}$ and $\widehat{f}_q^{(1,0)} = e^{-qt}$. When $r = 2$, $f_q^{(2,0)} = \cos qt$ and $\widehat{f}_q^{(2,0)} = \cosh qt$, $f_q^{(2,1)} = \sin qt$ and $\widehat{f}_q^{(2,1)} = \sinh qt$ ([8, 9]). When $r = 3$,

$$\begin{aligned} f_q^{(3,0)} &= \frac{e^{-qt} + e^{-q\omega t} + e^{-q\omega^2 t}}{3}, & \widehat{f}_q^{(3,0)} &= \frac{e^{qt} + e^{q\omega t} + e^{q\omega^2 t}}{3}, \\ f_q^{(3,1)} &= \frac{e^{-qt} + \omega^2 e^{-q\omega t} + \omega e^{-q\omega^2 t}}{3}, & \widehat{f}_q^{(3,1)} &= -\frac{e^{qt} + \omega^2 e^{q\omega t} + \omega e^{q\omega^2 t}}{3}, \\ f_q^{(3,2)} &= \frac{e^{-qt} + \omega e^{-q\omega t} + \omega^2 e^{-q\omega^2 t}}{3}, & \widehat{f}_q^{(3,2)} &= \frac{e^{qt} + \omega e^{q\omega t} + \omega^2 e^{q\omega^2 t}}{3}, \end{aligned}$$

where $\omega = \frac{-1+\sqrt{-3}}{2}$ and $\omega^2 = \bar{\omega} = \frac{-1-\sqrt{-3}}{2}$ are the primitive cube roots of unity ([4]).

By using the Taylor expansion of the two functions $e^{pt} \cos qt$ and $e^{pt} \sin qt$ in [10], a parametric type of Bernoulli polynomials is introduced and their basic properties are presented ([9]). Precisely, two kinds of bivariate Bernoulli polynomials are introduced as

$$\frac{te^{pt}}{e^t - 1} f_q^{(2,0)} = \sum_{n=0}^{\infty} B_n^{(2,0)}(p, q) \frac{t^n}{n!} \quad (5)$$

and

$$\frac{te^{pt}}{e^t - 1} f_q^{(2,1)} = \sum_{n=0}^{\infty} B_n^{(2,1)}(p, q) \frac{t^n}{n!}. \quad (6)$$

In [8], by defining two specific exponential generating functions, a kind of Euler polynomials is introduced and its basic properties are studied in detail. In [12], a kind of parametric Fubini-type polynomials is defined and some fundamental properties of these parametric-kind Fubini-type polynomials are studied. In [13], a type of generalized parametric Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials is introduced and systematically their basic properties are studied. Then, in [14], it is shown that the real and imaginary parts of a general set of complex Appell polynomials can be represented in terms of the Chebyshev polynomials of the first and second kind. In [4], a parametric type of Bernoulli polynomials with level 3 is introduced and their characteristic and combinatorial properties are studied. In particular, we show

some determinant expressions of these polynomials. In this sense, polynomials in [9] can be called parametric type of Bernoulli polynomials with level 2. In this paper, as a generalization of level 2 and 3 to any level $r \geq 1$, we introduce a parametric type of Bernoulli polynomials with level r and study their characteristic and combinatorial properties. It is important to see that several properties and behaviors differ when r is odd and even.

In 1935, D. H. Lehmer [7] introduced and investigated generalized Euler numbers W_n , defined by the generating function

$$\frac{1}{f_1^{(3,0)}} = \sum_{n=0}^{\infty} W_n \frac{t^n}{n!}.$$

Notice that $W_n = 0$ unless $n \equiv 0 \pmod{3}$. More general Lehmer's type of Euler numbers were considered in [1]. Cauchy type numbers which are similar but not included in Lehmer's type were considered in [5].

Since for $i = 0, 1, \dots, r - 1$

$$(-1)^{rn+i} = \begin{cases} (-1)^{n+i} & \text{if } r \text{ is odd;} \\ (-1)^i & \text{if } r \text{ is even} \end{cases}$$

we see that

$$f_q^{(r,i)}(-t) = \begin{cases} (-1)^i \widehat{f}_q^{(r,i)}(t) & \text{if } r \text{ is odd;} \\ (-1)^i f_q^{(r,i)}(t) & \text{if } r \text{ is even,} \end{cases} \quad (7)$$

$$\widehat{f}_q^{(r,i)}(-t) = \begin{cases} (-1)^i f_q^{(r,i)}(t) & \text{if } r \text{ is odd;} \\ (-1)^i \widehat{f}_q^{(r,i)}(t) & \text{if } r \text{ is even.} \end{cases} \quad (8)$$

Since

$$\sum_{i=0}^{r-1} \widehat{f}_q^{(r,i)} = e^{qt},$$

we get

$$\sum_{i=0}^{r-1} \widehat{B}_n^{(r,i)}(p, q) = B_n(p + q).$$

When r is odd, since

$$\sum_{i=0}^{r-1} (-1)^i f_q^{(r,i)} = e^{-qt},$$

we get

$$\sum_{i=0}^{r-1} (-1)^i B_n^{(r,i)}(p, q) = B_n(p - q).$$

When r is even, since

$$\sum_{i=0}^{r-1} (-1)^i \widehat{f}_q^{(r,i)} = e^{-qt},$$

we get

$$\sum_{i=0}^{r-1} (-1)^i \widehat{B}_n^{(r,i)}(p, q) = B_n(p - q).$$

Here $B_n(x)$ is the Bernoulli polynomial, defined by

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \quad (9)$$

It is trivial to see the following.

Proposition 1. For $n \geq 0$ and $i, j = 0, 1, 2$,

$$\begin{aligned} B_n^{(r,i)}(p, \zeta_r^j q) &= \zeta_r^{ij} B_n^{(r,i)}(p, q), \\ \widehat{B}_n^{(r,i)}(p, \zeta_r^j q) &= \zeta_r^{ij} \widehat{B}_n^{(r,i)}(p, q), \end{aligned}$$

where ζ_r ($r \geq 2$) is a primitive r -th root of unity with $\zeta_1 = 1$.

In the next section, we show several properties of bivariate Bernoulli polynomials with higher level. In particular, Theorem 2 entails fundamental recurrence formulas. By using these formulas, we give determinant expressions of bivariate Bernoulli polynomials with higher level. In special cases, we can get determinant expressions of the classical Bernoulli polynomials and numbers.

2. Basic properties

In this section, we show several properties of bivariate Bernoulli polynomials with higher level.

Proposition 2. For $n \geq 0$ and $i = 0, 1, 2$,

$$B_n^{(r,i)}(1 - p, q) = \begin{cases} (-1)^{n+i} \widehat{B}_n^{(r,i)}(p, q) & \text{if } r \text{ is odd;} \\ (-1)^{n+i} B_n^{(r,i)}(p, q) & \text{if } r \text{ is even,} \end{cases} \quad (10)$$

$$\widehat{B}_n^{(r,i)}(1 - p, q) = \begin{cases} (-1)^{n+i} B_n^{(r,i)}(p, q) & \text{if } r \text{ is odd;} \\ (-1)^{n+i} \widehat{B}_n^{(r,i)}(p, q) & \text{if } r \text{ is even.} \end{cases} \quad (11)$$

Proof. If r is odd, by (7)

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^{(r,i)}(p, q) \frac{(-t)^n}{n!} &= \frac{(-t)e^{-pt} f_q^{(r,i)}(-t)}{e^{-t} - 1} \\ &= \frac{te^{(1-p)t} (-1)^i \widehat{f}_q^{(r,i)}(t)}{e^t - 1} \\ &= (-1)^i \sum_{n=0}^{\infty} \widehat{B}_n^{(r,i)}(1 - p, q) \frac{t^n}{n!}. \end{aligned}$$

Hence, we get the first identity of (11). It is similar when r is even. By (8), the identity (10) is similarly proved.

We introduce auxiliary polynomials $F_n^{(r,i)}(p, q)$ and $\widehat{F}_n^{(r,i)}(p, q)$ as

$$e^{pt} f_q^{(3,i)}(t) = \sum_{n=0}^{\infty} F_n^{(3,i)}(p, q) \frac{t^n}{n!}, \quad (12)$$

$$e^{pt} \widehat{f}_q^{(3,i)}(t) = \sum_{n=0}^{\infty} \widehat{F}_n^{(3,i)}(p, q) \frac{t^n}{n!}. \quad (13)$$

respectively.

Proposition 3. For $n \geq 0$ and $i = 0, 1, \dots, r-1$,

$$F_n^{(r,i)}(p, q) = \sum_{k=0}^{\lfloor \frac{n-i}{r} \rfloor} (-1)^k \binom{n}{rk+i} p^{n-rk-i} q^{rk+i}, \quad (14)$$

$$\widehat{F}_n^{(r,i)}(p, q) = \sum_{k=0}^{\lfloor \frac{n-i}{r} \rfloor} \binom{n}{rk+i} p^{n-rk-i} q^{rk+i}. \quad (15)$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} F_n^{(r,i)}(p, q) \frac{t^n}{n!} &= e^{pt} f_q^{(r,i)}(t) \\ &= \left(\sum_{l=0}^{\infty} \frac{(pt)^l}{l!} \right) \left(\sum_{k=0}^{\infty} (-1)^k \frac{(qt)^{rk+i}}{(rk+i)!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n-i}{r} \rfloor} (-1)^k \binom{n}{rk+i} p^{n-rk-i} q^{rk+i} \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients on both sides, we get the identity (14). The identity (15) is similarly proved.

$B_n^{(r,i)}(p, q)$ (respectively, $\widehat{B}_n^{(r,i)}(p, q)$) can be written in terms of $F_n^{(r,i)}(p, q)$ (respectively, $\widehat{F}_n^{(r,i)}(p, q)$).

Proposition 4. For $n \geq 0$ and $i = 0, 1, \dots, r-1$,

$$B_n^{(r,i)}(p, q) = \sum_{k=0}^n \binom{n}{k} B_{n-k} F_k^{(r,i)}(p, q), \quad (16)$$

$$\widehat{B}_n^{(r,i)}(p, q) = \sum_{k=0}^n \binom{n}{k} B_{n-k} \widehat{F}_k^{(r,i)}(p, q). \quad (17)$$

Proof.

$$\begin{aligned}
 \sum_{n=0}^{\infty} \widehat{B}_n^{(r,i)}(p, q) \frac{t^n}{n!} &= \frac{t}{e^t - 1} \cdot e^{pt} \widehat{f}_q^{(r,i)}(t) \\
 &= \left(\sum_{l=0}^{\infty} B_l \frac{t^l}{l!} \right) \left(\sum_{k=0}^{\infty} \widehat{F}_k^{(r,i)}(p, q) \frac{t^k}{k!} \right) \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} B_{n-k} \widehat{F}_k^{(r,i)}(p, q) \frac{t^n}{n!}.
 \end{aligned}$$

Comparing the coefficients on both sides, we get the identity (17). The identity (16) is similarly proved.

$B_n^{(r,i)}(p, q)$ and $\widehat{B}_n^{(r,i)}(p, q)$ can be also written in terms of the Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$, which generating function is given by

$$\frac{(e^t - 1)^k}{k!} = \sum_{n=k}^{\infty} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \frac{t^n}{n!}.$$

Theorem 1. For $n \geq 0$ and $i = 0, 1, \dots, r - 1$

$$B_n^{(r,i)}(p, q) = \sum_{k=0}^n \sum_{\ell=0}^k \binom{n}{k} \left\{ \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right\} B_{n-k}^{(r,i)}(0, q) (p)_{\ell}, \quad (18)$$

$$\widehat{B}_n^{(r,i)}(p, q) = \sum_{k=0}^n \sum_{\ell=0}^k \binom{n}{k} \left\{ \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right\} \widehat{B}_{n-k}^{(r,i)}(0, q) (p)_{\ell}, \quad (19)$$

where $(x)_{\ell} = x(x - 1) \cdots (x - \ell + 1)$ ($\ell \geq 1$) denotes the falling factorial with $(x)_0 = 1$.

Proof.

$$\begin{aligned}
 \sum_{n=0}^{\infty} B_n^{(r,i)}(p, q) \frac{t^n}{n!} &= \frac{t f_q^{(r,i)}(t)}{e^t - 1} \cdot ((e^t - 1) + 1)^p \\
 &= \frac{t f_q^{(r,i)}(t)}{e^t - 1} \sum_{\ell=0}^p \binom{p}{\ell} \ell! \sum_{n=\ell}^{\infty} \left\{ \begin{smallmatrix} n \\ \ell \end{smallmatrix} \right\} \frac{t^n}{n!} \\
 &= \left(\sum_{n=0}^{\infty} B_n^{(r,i)}(0, q) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \sum_{\ell=0}^n \left\{ \begin{smallmatrix} n \\ \ell \end{smallmatrix} \right\} (p)_{\ell} \frac{t^n}{n!} \right) \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} B_{n-k}^{(r,i)}(0, q) \left(\sum_{\ell=0}^k \left\{ \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right\} (p)_{\ell} \right) \frac{t^n}{n!} \cdots
 \end{aligned}$$

Comparing the coefficients on both sides, we get the identity (18). The identity (19) is similarly proved.

On the contrary to Proposition 4, $F_n^{(r,i)}(p, q)$ (respectively, $\widehat{F}_n^{(r,i)}(p, q)$) can be written in terms of $B_n^{(r,i)}(p, q)$ (respectively, $\widehat{B}_n^{(r,i)}(p, q)$).

Proposition 5. For $n \geq 0$ and $i = 0, 1, \dots, r - 1$,

$$F_n^{(r,i)}(p, q) = \sum_{k=0}^n \frac{1}{n-k+1} \binom{n}{k} B_k^{(r,i)}(p, q), \quad (20)$$

$$\widehat{F}_n^{(r,i)}(p, q) = \sum_{k=0}^n \frac{1}{n-k+1} \binom{n}{k} \widehat{B}_k^{(r,i)}(p, q). \quad (21)$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} F_n^{(r,i)}(p, q) \frac{t^n}{n!} &= \frac{e^t - 1}{t} \frac{te^{pt} f_q^{(r,i)}(t)}{e^t - 1} \\ &= \left(\sum_{l=0}^{\infty} \frac{t^l}{(l+1)!} \right) \left(\sum_{k=0}^{\infty} B_k^{(r,i)}(p, q) \frac{t^k}{k!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{n-k+1} \binom{n}{k} B_k^{(r,i)}(p, q) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients on both sides, we get the identity (20). The identity (21) is similarly proved.

We have a summation formula for $B_n^{(r,i)}(p, q)$ (respectively, $\widehat{B}_n^{(r,i)}(p, q)$).

Theorem 2. For $n \geq 0$ and $i = 0, 1, \dots, r - 1$,

$$\begin{aligned} &\sum_{k=0}^n \binom{n+1}{k} ((1-p)^{n-k+1} - (-p)^{n-k+1}) B_k^{(r,i)}(p, q) \\ &= \begin{cases} (-1)^{(n-i)/r} (n+1)q^n & \text{if } n \equiv i \pmod{r}; \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (22)$$

$$\begin{aligned} &\sum_{k=0}^n \binom{n+1}{k} ((1-p)^{n-k+1} - (-p)^{n-k+1}) \widehat{B}_k^{(r,i)}(p, q) \\ &= \begin{cases} (n+1)q^n & \text{if } n \equiv i \pmod{r}; \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (23)$$

Proof. From the definition in (1) and the proof of Proposition 5, we have

$$f_q^{(r,i)}(t) = e^{-pt} \cdot \frac{e^t - 1}{t} \sum_{n=0}^{\infty} B_n^{(r,i)}(p, q) \frac{t^n}{n!}$$

$$\begin{aligned}
&= \left(\sum_{m=0}^{\infty} \frac{(-pt)^m}{m!} \right) \left(\sum_{l=0}^{\infty} \sum_{k=0}^l \frac{1}{l-k+1} \binom{l}{k} B_k^{(r,i)}(p, q) \frac{t^l}{l!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} (-p)^{n-l} \sum_{k=0}^l \frac{1}{l-k+1} \binom{l}{k} B_k^{(r,i)}(p, q) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n B_k^{(r,i)}(p, q) \sum_{l=k}^n \binom{n}{l} \binom{l}{k} \frac{(-p)^{n-l}}{(l-k+1)} \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{1}{n+1} \binom{n+1}{k} B_k^{(r,i)}(p, q) \sum_{l=k}^n \binom{n-k+1}{n-l} (-p)^{n-l} \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{1}{n+1} \binom{n+1}{k} B_k^{(r,i)}(p, q) ((1-p)^{n-k+1} - (-p)^{n-k+1}) \right) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients with

$$f_q^{(r,i)}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(qt)^{rn+i}}{(rn+i)!}$$

in (2), we get the identity (22). The identity (23) is similarly proved.

From Theorem 2, we have the recurrence relations

$$\begin{aligned}
B_n^{(r,i)}(p, q) &= -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} ((1-p)^{n-k+1} - (-p)^{n-k+1}) B_k^{(r,i)}(p, q) \\
&\quad + \begin{cases} (-1)^{(n-i)/r} q^n & \text{if } n \equiv i \pmod{r}; \\ 0 & \text{otherwise,} \end{cases} \tag{24}
\end{aligned}$$

$$\begin{aligned}
\widehat{B}_n^{(r,i)}(p, q) &= -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} ((1-p)^{n-k+1} - (-p)^{n-k+1}) \widehat{B}_k^{(r,i)}(p, q) \\
&\quad + \begin{cases} q^n & \text{if } n \equiv i \pmod{r}; \\ 0 & \text{otherwise} \end{cases} \tag{25}
\end{aligned}$$

with

$$B_n^{(r,i)}(p, q) = \widehat{B}_n^{(r,i)}(p, q) = \begin{cases} 0 & \text{if } n = 0, 1, \dots, i-1; \\ q^n & \text{if } n = i. \end{cases}$$

Using recurrence relations, we can obtain the exact values of $B_n^{(r,i)}(p, q)$ and $\widehat{B}_n^{(r,i)}(p, q)$ for small n .

Theorem 3. For $n \geq 0$ and $i = 0, 1, \dots, r-1$,

$$B_n^{(r,i)}(p+1, q) - B_n^{(r,i)}(p, q) = nF_{n-1}^{(r,i)}(p, q), \tag{26}$$

$$\widehat{B}_n^{(r,i)}(p+1, q) - \widehat{B}_n^{(r,i)}(p, q) = n\widehat{F}_{n-1}^{(r,i)}(p, q). \tag{27}$$

Proof.

$$\begin{aligned}
 & \sum_{n=0}^{\infty} B_n^{(r,i)}(p+1, q) \frac{t^n}{n!} - \sum_{n=0}^{\infty} B_n^{(r,i)}(p, q) \frac{t^n}{n!} \\
 &= \frac{te^{(p+1)t} f_q^{(r,i)}(t)}{e^t - 1} - \frac{te^{pt} f_q^{(r,i)}(t)}{e^t - 1} \\
 &= te^{pt} f_q^{(r,i)}(t) = t \sum_{n=0}^{\infty} F_n^{(r,i)}(p, q) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} n F_{n-1}^{(r,i)}(p, q) \frac{t^n}{n!}.
 \end{aligned}$$

Comparing the coefficients on both sides, we get the identity (26). The identity (27) is similarly proved.

As a special case in Theorem 3, where $p = 1$ and n is replaced by $rn + j$, we have the following.

Corollary 1. For $n \geq 0$, $i = 0, 1, \dots, r - 1$ and $j = 1, 2, \dots, r$,

$$B_{rn+j}^{(r,i)}(1, q) - B_{rn+j}^{(r,i)}(0, q) = \begin{cases} (rn+j)(-1)^n q^{rn+j-1} & \text{if } i \equiv j \pmod{r}; \\ 0 & \text{otherwise,} \end{cases} \quad (28)$$

$$\widehat{B}_{rn+j}^{(r,i)}(1, q) - \widehat{B}_{rn+j}^{(r,i)}(0, q) = \begin{cases} (rn+j)q^{rn+j-1} & \text{if } i \equiv j \pmod{r}; \\ 0 & \text{otherwise.} \end{cases} \quad (29)$$

Proof. Notice that for $n \geq 0$, $i = 0, 1, \dots, r - 1$ and $j = 0, 1, \dots, r - 1$,

$$F_{rn+j}^{(r,i)}(0, q) = \begin{cases} (-1)^n q^{rn+j} & \text{if } i \equiv j \pmod{r}; \\ 0 & \text{otherwise} \end{cases}$$

and

$$\widehat{F}_{rn+j}^{(r,i)}(0, q) = \begin{cases} q^{rn+j} & \text{if } i \equiv j \pmod{r}; \\ 0 & \text{otherwise.} \end{cases}$$

Putting $p = 0$ in Theorem 3, we get the results.

We have recurrence relations of $B_n^{(r,i)}(p, q)$ and $\widehat{B}_n^{(r,i)}(p, q)$ on p by using the binomial coefficients. They remind us the translation identities of Bernoulli polynomials

$$B_n(x+y) = \sum_{k=0}^n \binom{n}{k} B_k(x) y^{n-k}.$$

Theorem 4. For $n \geq 0$ and $i = 0, 1, \dots, r-1$,

$$B_n^{(r,i)}(p+s, q) = \sum_{k=0}^n \binom{n}{k} B_k^{(r,i)}(p, q) s^{n-k}, \quad (30)$$

$$\widehat{B}_n^{(r,i)}(p+s, q) = \sum_{k=0}^n \binom{n}{k} \widehat{B}_k^{(r,i)}(p, q) s^{n-k}. \quad (31)$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^{(r,i)}(p+s, q) \frac{t^n}{n!} &= \frac{te^{(p+s)t} f_q^{(r,i)}(t)}{e^t - 1} \\ &= \left(\sum_{n=0}^{\infty} B_n^{(r,i)}(p, q) \frac{t^n}{n!} \right) \left(\sum_{l=0}^{\infty} \frac{(st)^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} B_k^{(r,i)}(p, q) s^{n-k} \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients on both sides, we get the identity (30). The identity (31) is similarly proved.

It is well-known that Bernoulli numbers satisfy the relations

$$\begin{aligned} \sum_{k=0}^n \binom{n+1}{k} B_k &= \sum_{k=0}^n (-1)^k \binom{n+1}{k} \mathbb{B}_k = 0 \quad (n \geq 1), \\ \sum_{k=0}^n \binom{n+1}{k} \mathbb{B}_k &= n+1 \quad (n \geq 0). \end{aligned}$$

We can get kinds of generalized relations.

Theorem 5. For $n \geq 0$ and $i = 0, 1, \dots, r-1$,

$$\sum_{k=0}^n \binom{n+1}{k} B_k^{(r,i)}(p, q) = (n+1) F_n^{(r,i)}(p, q), \quad (32)$$

$$\sum_{k=0}^n \binom{n+1}{k} \widehat{B}_k^{(r,i)}(p, q) = (n+1) \widehat{F}_n^{(r,i)}(p, q). \quad (33)$$

Proof. By using Theorem 4 (30),

$$B_{n+1}^{(r,i)}(p+1, q) - B_{n+1}^{(r,i)}(p, q) = \sum_{k=0}^n \binom{n+1}{k} B_k^{(r,i)}(p, q).$$

Together with Theorem 3 (26), we get the identity (32). The identity (33) is similarly proved.

The following recurrence relations are direct results from Theorem 2 when $p = 0$. Nevertheless, they are also special cases of Theorem 5.

Corollary 2. For $n \geq 0$ and $i = 0, 1, \dots, r - 1$,

$$\sum_{k=0}^n \binom{n+1}{k} B_k^{(r,i)}(0, q) = \begin{cases} (-1)^{(n-i)/r} (n+1)q^n & \text{if } n \equiv i \pmod{r}; \\ 0 & \text{otherwise,} \end{cases} \quad (34)$$

$$\sum_{k=0}^n \binom{n+1}{k} \widehat{B}_k^{(r,i)}(0, q) = \begin{cases} (n+1)q^n & \text{if } n \equiv i \pmod{r}; \\ 0 & \text{otherwise.} \end{cases} \quad (35)$$

Proof. Since

$$F_n^{(r,i)}(0, q) = \begin{cases} (-1)^{(n-i)/r} q^n & \text{if } n \equiv i \pmod{r}; \\ 0 & \text{otherwise,} \end{cases}$$

we get the identity (34). Since

$$\widehat{F}_n^{(r,i)}(0, q) = \begin{cases} q^n & \text{if } n \equiv i \pmod{r}; \\ 0 & \text{otherwise,} \end{cases}$$

we get the identity (35).

It is well-known that Bernoulli polynomials are Appell:

$$\frac{d}{dx} B_n(x) = nB_{n-1}(x).$$

We have generalized relations about $B_n^{(r,i)}(p, q)$ and $\widehat{B}_n^{(r,i)}(p, q)$.

Theorem 6. For $n \geq 1$,

$$\frac{\partial}{\partial p} B_n^{(r,i)}(p, q) = nB_{n-1}^{(r,i)}(p, q) \quad (i = 0, 1, \dots, r - 1), \quad (36)$$

$$\frac{\partial}{\partial q} B_n^{(r,i)}(p, q) = \begin{cases} -nB_{n-1}^{(r,2)}(p, q) & \text{if } i = 0; \\ nB_{n-1}^{(r,i-1)}(p, q) & \text{if } i = 1, 2, \dots, r - 1, \end{cases} \quad (37)$$

$$\frac{\partial}{\partial p} \widehat{B}_n^{(r,i)}(p, q) = n\widehat{B}_{n-1}^{(r,i)}(p, q) \quad (i = 0, 1, \dots, r - 1), \quad (38)$$

$$\frac{\partial}{\partial q} \widehat{B}_n^{(r,i)}(p, q) = \begin{cases} n\widehat{B}_{n-1}^{(r,2)}(p, q) & \text{if } i = 0; \\ n\widehat{B}_{n-1}^{(r,i-1)}(p, q) & \text{if } i = 1, 2, \dots, r - 1. \end{cases} \quad (39)$$

Proof.

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial p} B_n^{(r,i)}(p, q) \frac{t^n}{n!} = \frac{t^2 e^{pt} f_q^{(r,i)}(t)}{e^t - 1} = \sum_{n=0}^{\infty} B_n^{(r,i)}(p, q) \frac{t^{n+1}}{n!}$$

$$= \sum_{n=1}^{\infty} n B_{n-1}^{(r,i)}(p, q) \frac{t^n}{n!},$$

yielding (36). Since

$$\begin{aligned} \frac{\partial}{\partial q} f_q^{(r,0)}(t) &= t \sum_{n=1}^{\infty} \frac{(-1)^n (qt)^{rn-1}}{(rn-1)!} \\ &= -t f_q^{(r,r-1)}(t) \end{aligned}$$

and

$$\frac{\partial}{\partial q} f_q^{(r,i)}(t) = t f_q^{(r,i-1)}(t) \quad (i = 1, 2, \dots, r-1),$$

we get (37). The identities (38) and (39) are similarly proved.

From Theorem 6, we can get integral relations.

Corollary 3. For $n \geq 0$ and $i, j = 0, 1, \dots, r-1$,

$$\int_0^1 B_{rn+j}^{(r,i)}(p, q) dp = \begin{cases} (-1)^n q^{rn+j} & \text{if } i \equiv j \pmod{r}; \\ 0 & \text{otherwise,} \end{cases} \quad (40)$$

$$\int_0^1 \widehat{B}_{rn+j}^{(r,i)}(p, q) dp = \begin{cases} q^{rn+j} & \text{if } i \equiv j \pmod{r}; \\ 0 & \text{otherwise.} \end{cases} \quad (41)$$

Proof. From Corollary 1 (28) and Theorem 6 (36), we have

$$\begin{aligned} \int_0^1 B_{rn+j}^{(r,i)}(p, q) dp &= \frac{1}{rn+j+1} \int_0^1 \frac{\partial}{\partial p} B_{rn+j+1}^{(r,i)}(p, q) dp \\ &= \frac{1}{rn+1} (B_{rn+j+1}^{(r,i)}(1, q) - B_{rn+j+1}^{(r,i)}(0, q)) \\ &= \begin{cases} (-1)^n q^{rn+j} & \text{if } i \equiv j \pmod{r}; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly, from Corollary 1 (29) and Theorem 6 (38), we have the identity (41).

3. Determinants

Bivariate Bernoulli polynomials $B_n^{(r,i)}(p, q)$ with higher level and their complementary polynomials $\widehat{B}_n^{(r,i)}(p, q)$ have determinant expressions.

Theorem 7. For $n \geq i + 1$,

$$B_n^{(r,i)}(p, q) = \frac{n!q^i}{i!} \begin{vmatrix} d_p(2) & 1 & 0 & \cdots & 0 \\ d_p(3) & d_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ d_p(n-i) & d_p(n-i-1) & \cdots & d_p(2) & 1 \\ d_p^*(n-i+1) & d_p^*(n-i) & \cdots & d_p^*(3) & d_p^*(2) \end{vmatrix}$$

and

$$\widehat{B}_n^{(r,i)}(p, q) = \frac{n!q^i}{i!} \begin{vmatrix} d_p(2) & 1 & 0 & \cdots & 0 \\ d_p(3) & d_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ d_p(n-i) & d_p(n-i-1) & \cdots & d_p(2) & 1 \\ \widehat{d}_p(n-i+1) & \widehat{d}_p(n-i) & \cdots & \widehat{d}_p(3) & \widehat{d}_p(2) \end{vmatrix},$$

where

$$d_p(n) = \frac{p^n - (p-1)^n}{n!}$$

with

$$d_p^*(n) = d_p(n) - \begin{cases} \frac{(-1)^{n+(n-1)/r} i! q^{n-1}}{(n+i-1)!} & \text{if } n \equiv 1 \pmod{r}; \\ 0 & \text{otherwise} \end{cases} \quad (n \geq 2)$$

and

$$\widehat{d}_p(n) = d_p(n) + \begin{cases} \frac{(-1)^n i! q^{n-1}}{(n+i-1)!} & \text{if } n \equiv 1 \pmod{r}; \\ 0 & \text{otherwise} \end{cases} \quad (n \geq 2).$$

Proof. By the recurrence relation in (24) from Theorem 2, for $n \geq i$ we have

$$\beta_n = \sum_{k=i}^{n-1} (-1)^{n-k+1} d_p(n-k+1) \beta_k + \begin{cases} \frac{(-1)^{(n-i)/r} i! q^{n-i}}{n!} & \text{if } n \equiv i \pmod{r}; \\ 0 & \text{otherwise,} \end{cases} \quad (42)$$

where

$$\beta_n := \frac{i!}{q^i} \frac{B_n^{(r,i)}(p, q)}{n!}.$$

Notice that $\beta_0 = \cdots = \beta_{i-1} = 0$ and $\beta_i = 1$ since $B_0^{(r,i)}(p, q) = \cdots = B_{i-1}^{(r,i)}(p, q) = 0$ and $B_i^{(r,i)}(p, q) = q^i$. By induction, we shall prove that

$$\beta_n = \begin{vmatrix} d_p(2) & 1 & 0 & \cdots & 0 \\ d_p(3) & d_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ d_p(n-i) & d_p(n-i-1) & \cdots & d_p(2) & 1 \\ d_p^*(n-i+1) & d_p^*(n-i) & \cdots & d_p^*(3) & d_p^*(2) \end{vmatrix}. \quad (43)$$

For $n = i + 1$, by the recurrence relation (42), we get

$$\beta_{i+1} = d_p(2) = |d_p(2)|.$$

Assume that the determinant expression of (43) is valid up to $n - 1$. Then by expanding the right-hand side of (43) along the first row repeatedly, we have

$$\begin{aligned} & d_p(2)\beta_{n-1} - \begin{vmatrix} d_p(3) & 1 & 0 & \cdots & 0 \\ d_p(4) & d_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ d_p(n-i) & d_p(n-i-2) & \cdots & d_p(2) & 1 \\ d_p^*(n-i+1) & d_p^*(n-i-1) & \cdots & d_p^*(3) & d_p^*(2) \end{vmatrix} \\ &= d_p(2)\beta_{n-1} - d_p(3)\beta_{n-2} + \cdots + (-1)^{n-i-1}d_p(n-i-1)\beta_{i+2} \\ & \quad + (-1)^{n-i} \begin{vmatrix} d_p(n-i) & 1 \\ d_p^*(n-i+1) & d_p^*(2) \end{vmatrix} \\ &= d_p(2)\beta_{n-1} - d_p(3)\beta_{n-2} + \cdots + (-1)^{n-i-1}d_p(n-i-1)\beta_{i+2} \\ & \quad + (-1)^{n-i}d_p(n-i)\beta_{i+1} + (-1)^{n-i+1}d_p^*(n-i+1)\beta_i \\ &= \beta_n. \end{aligned}$$

The last identity is entailed from the recurrence relation (42) and

$$d_p^*(n-i+1) = d_p(n-i+1) + (-1)^{n-i+1} \frac{(-1)^{(n-i)/r} q^{n-i}}{n!}$$

for $n \equiv i \pmod{r}$. By putting $m = n - i + 1$, we find that

$$d_p^*(m) = d_p(m) - \frac{(-1)^{m+(m-1)/r} i! q^{m-1}}{(m+i-1)!}$$

for $m \equiv 1 \pmod{r}$. Another identity can be yielded similarly from the recurrence relation

$$\widehat{\beta}_n = \sum_{k=i}^{n-1} (-1)^{n-k+1} d_p(n-k+1) \widehat{\beta}_k + \begin{cases} \frac{i! q^{n-i}}{n!} & \text{if } n \equiv i \pmod{r}; \\ 0 & \text{otherwise,} \end{cases}$$

with

$$\widehat{\beta}_n := \frac{i! \widehat{B}_n^{(r,i)}(p, q)}{q^i n!}.$$

When $q = i = 0$, bivariate Bernoulli polynomials $B_n(p) := B_n^{(r,0)}(p, 0)$ with higher level and their complementary polynomials $\widehat{B}_n(p) := \widehat{B}_n^{(r,0)}(p, 0)$ have simpler determinant expressions. In fact, $B_n(p)$ (or $\widehat{B}_n(p)$) is the Bernoulli polynomial in (9).

For simplification of determinant expressions, we use the Jordan matrix

$$J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

J^0 is identity matrix and J^T is the transpose matrix of J .

Corollary 4. For $n \geq 1$,

$$B_n(p) = \widehat{B}_n(p) = n! \left| J^T + \sum_{k=1}^n d_p(k+1) J^{k-1} \right|.$$

We need the following equivalent relations (see, e.g., [6]).

Lemma 1.

$$\sum_{k=0}^n (-1)^{n-k} x_{n-k} z_k = 0 \quad \text{with} \quad x_0 = z_0 = 1$$

$$\iff x_n = \left| J^T + \sum_{k=1}^n z_k J^{k-1} \right| \iff z_n = \left| J^T + \sum_{k=1}^n x_k J^{k-1} \right|.$$

When $p = 1$ in Corollary 4, we have a determinant expression of Bernoulli numbers $\mathbb{B}_n = B_n^{(r,0)}(1,0)$ with $\mathbb{B}_1 = 1/2$.

Corollary 5. For $n \geq 1$,

$$\mathbb{B}_n = n! \left| J^T + \sum_{k=1}^n \frac{1}{(k+1)!} J^{k-1} \right|.$$

Remark 1. A determinant expression of Bernoulli numbers B_n with $B_1 = -1/2$ is given as follows [3, p. 53]:

$$B_n = (-1)^n n! \left| J^T + \sum_{k=1}^n \frac{1}{(k+1)!} J^{k-1} \right|.$$

By using Lemma 1 again, we have the inversion relation of Corollary 4.

Corollary 6. For $n \geq 1$,

$$d_p(n+1) = \left| J^T + \sum_{k=1}^n \frac{B_k(p)}{k!} J^{k-1} \right|.$$

We shall use Trudi's formula to obtain different explicit expressions and inversion relations for the numbers $B_n^{(r,i)}(p, 0)$.

Lemma 2. For $n \geq 1$, we have

$$\begin{aligned} & \left| a_0 J^T + \sum_{k=1}^n a_k J^{k-1} \right| \\ &= \sum_{t_1+2t_2+\dots+nt_n=n} \binom{t_1+\dots+t_n}{t_1, \dots, t_n} (-a_0)^{n-t_1-\dots-t_n} a_1^{t_1} a_2^{t_2} \dots a_n^{t_n}, \end{aligned}$$

where $\binom{t_1+\dots+t_n}{t_1, \dots, t_n} = \frac{(t_1+\dots+t_n)!}{t_1! \dots t_n!}$ are the multinomial coefficients.

This relation is known as Trudi's formula [11, Vol.3, p.214],[15] and the case $a_0 = 1$ of this formula is known as Brioschi's formula [2],[11, Vol.3, pp.208–209].

By Corollary 4 and Corollary 6 with Lemma 2, we get different expressions of $B_n(p)$ and $d_p(n)$.

Corollary 7. For $n \geq 1$,

$$\begin{aligned} B_n(p) &= \sum_{t_1+2t_2+\dots+nt_n=n} \binom{t_1+\dots+t_n}{t_1, \dots, t_n} \\ &\quad \times (-1)^{n-t_1-\dots-t_n} (d_p(2))^{t_1} (d_p(3))^{t_2} \dots (d_p(n+1))^{t_n}, \\ d_p(n+1) &= \sum_{t_1+2t_2+\dots+nt_n=n} \binom{t_1+\dots+t_n}{t_1, \dots, t_n} \\ &\quad \times (-1)^{n-t_1-\dots-t_n} (B_1(p))^{t_1} \left(\frac{B_2(p)}{2!}\right)^{t_2} \dots \left(\frac{B_n(p)}{n!}\right)^{t_n}. \end{aligned}$$

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