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The homology groups $H_{n+1}(\mathbb{C}\Omega_n)$

Abstract. The topic of the paper is the investigation of the homology groups of the $(2n+1)$-dimensional CW-complex $\mathbb{C}\Omega_n$. The spaces $\mathbb{C}\Omega_n$ consist of complex-valued functions and are the analogue of the spaces Ω_n, widely known in the approximation theory. The spaces $\mathbb{C}\Omega_n$ have been introduced in 2015 by A.M. Pasko who has built the CW-structure of the spaces $\mathbb{C}\Omega_n$ and using this CW-structure established that the spaces $\mathbb{C}\Omega_n$ are simply connected. Note that the mentioned CW-structure of the spaces $\mathbb{C}\Omega_n$ is the analogue of the CW-structure of the spaces Ω_n constructed by V.I. Ruban. Further A.M. Pasko found the homology groups of the space $\mathbb{C}\Omega_n$ in the dimensionalities $0, 1, \ldots, n, 2n-1, 2n, 2n+1$. The goal of the present paper is to find the homology group $H_{n+1}(\mathbb{C}\Omega_n)$. It is proved that $H_{n+1}(\mathbb{C}\Omega_n) = \mathbb{Z}^{\frac{n+1}{2}}$ if n is odd and $H_{n+1}(\mathbb{C}\Omega_n) = \mathbb{Z}^{\frac{n+2}{2}}$ if n is even.
Key words: homology group, spline, CW-complex

Анотація. Стаття присвячена дослідженню гомологічних груп $(2n+1)$-вимірного клітинного простору $\mathbb{C}\Omega_n$. Простори $\mathbb{C}\Omega_n$ складаються з комплекснозначних функцій і є аналогами широко відомих у теорії апроксимації просторів Ω_n. Простори $\mathbb{C}\Omega_n$ введено А.М. Паськом у роботі 2015 р., у якій автор побудував клітинну структуру просторів $\mathbb{C}\Omega_n$, з допомогою якої довів їх однозв'язність. Зазначимо, що згадана клітинна структура аналогічна побудованій В.І. Рубаном клітинній структурі простору Ω_n. В подальшому А.М. Пасько знайшов гомологічні групи простору $\mathbb{C}\Omega_n$ у вимірностях $0, 1, \ldots, n, 2n-1, 2n, 2n+1$. Метою статті є обчислення гомологічних груп $H_{n+1}(\mathbb{C}\Omega_n)$. У роботі доведено, що $H_{n+1}(\mathbb{C}\Omega_n) = \mathbb{Z}^{\frac{n+1}{2}}$ для непарного n, і $H_{n+1}(\mathbb{C}\Omega_n) = \mathbb{Z}^{\frac{n+2}{2}}$ для n парного.
Ключові слова: група гомологій, сплайн, клітинний простір

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Let $\omega(t), t \geq 0$, be the non-negative, continuous increasing function, $\omega(0) = 0$, and $n \in \mathbb{N}$. Consider integer $q \geq 0$ and the system of the knots

$$0 = \eta_0 < \eta_1 < \ldots < \eta_q < \eta_{q+1} = 1.$$

For each $q \leq n$ and $0 \leq k \leq q$ we consider the function (ω-spline)

$$F(\eta_k, \eta_{k+1}, s_k, t) = s_k \cdot \min\{\omega(t-\eta_k), \omega(\eta_{k+1}-t)\}, \quad for \quad t \in [\eta_k, \eta_{k+1}], \quad (1)$$

with $s_k \in \mathbb{C}, |s_k| = 1$, and the subspace $\mathbb{C}\Omega_n$ of the space $L_1[0,1]$ that consists of the splines of the form (1) for $q \leq n$.

The $\mathbb{C}\Omega_n$ spaces were introduced in [2] and are analogue of the spaces Ω_n investigated in [1], [5], [6]. The homology groups of the spaces $\mathbb{C}\Omega_n$ has been the topic of the papers [3], [4], the result of this investigation can be written as

$$H_k(\mathbb{C}\Omega_n) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, k = 2n+1; \\ 0, & \text{if } 1 \leq k < n; \\ 0, & \text{if } k = n, n \text{ is even}; \\ \mathbb{Z}, & \text{if } k = n, n \text{ is odd}; \\ \mathbb{Z}^{n+\frac{(n-1)(n-2)}{2}}, & \text{if } k = 2n-1; \\ \mathbb{Z}^n, & \text{if } k = 2n. \end{cases}$$

In [2] the CW-structure of the space $\mathbb{C}\Omega_n$ has been built:

$$\mathbb{C}\Omega_n = \bigcup_{k=0}^{2n+1} \bigcup_{q+h(u)=k} c^q(u), \tag{2}$$

where each k-cell $c^q(u)$ is defined by the sequence (word) $u = u_0 u_1 \dots u_q$ of the elements (symbols) $u_i \in \{1, e\}$, $h = h(u)$ is the amount of u_i such that $u_i = e$.

Note that the CW-structure (2) of the space $\mathbb{C}\Omega_n$ is the analogue of the CW-structure of the space Ω_n introduced by Ruban [5], [6].

It is also known from [2] that the boundary of the $(q + h(u))$-cell $c^q(u)$ of the space $\mathbb{C}\Omega_n$ equals

$$\partial c^q(u_0 \dots u_q) = \sum_{k:\ u_k=1} (-1)^k c^{q-1}(u_0 \dots \hat{u}_k \dots u_q), \tag{3}$$

where the word $u_0 \dots \hat{u}_k \dots u_q$ is $u_0 \dots u_{k-1} u_{k+1} \dots u_q$.

The goal of this paper is to find the homology groups $H_{n+1}(\mathbb{C}\Omega_n), n \geq 2$. The main result of the paper is the following theorem.

Theorem 1. *For each $n \geq 2$ the homology group*

$$H_{n+1}(\mathbb{C}\Omega_n) = \begin{cases} \mathbb{Z}^{\frac{n+1}{2}}, & n \text{ is odd}; \\ \mathbb{Z}^{\frac{n+2}{2}}, & n \text{ is even}. \end{cases} \tag{4}$$

Proof of Theorem 1. Consider the integer $n \geq 2$. It follows from [4] that for any k the next short exact sequence holds

$$0 \to H_{k+1}(\mathbb{C}\Omega_n) \xrightarrow{j_*} H_{k+1}(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1}) \xrightarrow{\partial} H_k(\mathbb{C}\Omega_{n-1}) \to 0,$$

in particular, when $k = n$,

$$0 \to H_{n+1}(\mathbb{C}\Omega_n) \xrightarrow{j_*} H_{n+1}(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1}) \xrightarrow{\partial} H_n(\mathbb{C}\Omega_{n-1}) \to 0. \tag{5}$$

The exact sequence (5) implies

$$\operatorname{Ker}\partial = \operatorname{Im} j_* = H_{n+1}(\mathbb{C}\Omega_n). \tag{6}$$

Thus in order to find $H_{n+1}(\mathbb{C}\Omega_n)$ we must investigate the kernel of the operator

$$\partial : H_{n+1}(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1}) \to H_n(\mathbb{C}\Omega_{n-1}).$$

It follows from the proof of Lemma 1 in [4] that $H_{n+1}(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1})$ is free abelian group with the generators defined by the (n+1)-chains $c^n(e11\ldots1)$, $c^n(1e1\ldots1)$, $c^n(11e\ldots1)$, …, $c^n(111\ldots e)$. Each of these chains can be written in the form

$$c^n(\underbrace{1\ldots1}_{i_1}e\underbrace{1\ldots1}_{i_2}), i_1+i_2=n. \tag{7}$$

The equality (3) implies that the boundary of the chain (7) equals

$$\partial c^n(\underbrace{1\ldots1}_{i_1}e\underbrace{1\ldots1}_{i_2}) =$$

$$= \begin{cases} 0, & i_1, i_2 \text{ both are even;} \\ c^{n-1}(\underbrace{1\ldots1}_{i_1-1}e\underbrace{1\ldots1}_{i_2}), & i_1 \text{ is odd}, i_2 \text{ is even;} \\ (-1)^{i_1+1}c^{n-1}(\underbrace{1\ldots1}_{i_1}e\underbrace{1\ldots1}_{i_2-1}), & i_1 \text{ is even}, i_2 \text{ is odd;} \\ c^{n-1}(\underbrace{1\ldots1}_{i_1-1}e\underbrace{1\ldots1}_{i_2})+ & \\ +(-1)^{i_1+1}c^{n-1}(\underbrace{1\ldots1}_{i_1}e\underbrace{1\ldots1}_{i_2-1}), & i_1, i_2 \text{ both are odd.} \end{cases} \tag{8}$$

Obviously the elements of the relative homology group $H_{n+1}(\mathbb{C}\Omega_n, \mathbb{C}\Omega_{n-1})$ can be defined by the (n+1)-chains

$$A = \sum_{i=0}^{n} k_i c^n(\underbrace{1\ldots1}_{i}e\underbrace{1\ldots1}_{n-i}). \tag{9}$$

Consider n is odd, $n = 2m+1$. Accordingly to (8) the boundary of the chain (9) equals

$$\partial A = \sum_{\substack{i=0,\\ i \text{ is odd}}}^{n} k_i c^{n-1}(\underbrace{1\ldots1}_{i-1}e\underbrace{1\ldots1}_{n-i}) + \sum_{\substack{i=0,\\ i \text{ is even}}}^{n} k_i (-1)^{i+1} c^{n-1}(\underbrace{1\ldots1}_{i}e\underbrace{1\ldots1}_{n-i-1}) =$$

$$= \sum_{\substack{i=1,\\ i \text{ is odd}}}^{n} k_i c^{n-1}(\underbrace{1\ldots1}_{i-1}e\underbrace{1\ldots1}_{n-i}) - \sum_{\substack{i=0,\\ i \text{ is even}}}^{n-1} k_i c^{n-1}(\underbrace{1\ldots1}_{i}e\underbrace{1\ldots1}_{n-i-1}).$$

Replacing $i-1$ by i in the first sum, we get

$$\partial A = \sum_{\substack{i=0,\\ i \text{ is even}}}^{n-1} k_{i+1} c^{n-1}(\underbrace{1\dots1}_{i} e \underbrace{1\dots1}_{n-i-1}) - \sum_{\substack{i=0,\\ i \text{ is even}}}^{n-1} k_i c^{n-1}(\underbrace{1\dots1}_{i} e \underbrace{1\dots1}_{n-i-1}),$$

or

$$\partial A = \sum_{\substack{i=0,\\ i \text{ is even}}}^{n-1} (k_{i+1}-k_i) c^{n-1}(\underbrace{1\dots1}_{i} e \underbrace{1\dots1}_{n-i-1}).$$

This equality implies that $\partial A = 0$ if and only if $k_i = k_{i+1}$ for all even i, thus only the coefficients $k_1, k_3, \dots, k_{2m+1}$ are “free”. Therefore

$$\operatorname{Ker}\partial = \mathbb{Z}^{m+1} = \mathbb{Z}^{\frac{n+1}{2}}.$$

By the virtue of (6) the equality (4) holds when n is odd.

Consider n is even, $n = 2m$. Accordingly to (8) the boundary of the chain (9) equals

$$\partial A = \sum_{\substack{i=1,\\ i \text{ is odd}}}^{n} (k_i c^{n-1}(\underbrace{1\dots1}_{i-1} e \underbrace{1\dots1}_{n-i}) + k_i c^{n-1}(\underbrace{1\dots1}_{i} e \underbrace{1\dots1}_{n-i-1})).$$

The boundary $\partial A = 0$ if and only if $k_i = 0$ for all odd i, thus an arbitrary element $A \in \operatorname{Ker}\partial$ is defined by integers $k_0, k_2, \dots, k_{2m}$. Therefore

$$\operatorname{Ker}\partial = \mathbb{Z}^{m+1} = \mathbb{Z}^{\frac{n+2}{2}}.$$

By the virtue of (6) the equality (4) holds when n is even. The proof of the theorem is completed.

References

1. *Koshcheev V.A.*: The fundamental groups of the spaces of generalized perfect splines. Proceedings of the Institute of Mathematics and Mechanics UrO RAN, 2009; 15(1): pp. 159–165.
2. *Pasko A.M.*: Simple connectedness of one space of complex-valued functions. Res. Math., 2015; 20: pp. 70–74. doi:10.15421/241508
3. *Pasko A.M.*: The homology groups of the space $\mathbb{C}\Omega_n$ for certain dimensionalities. Res. Math., 2016; 21: pp. 71–76. doi:10.15421/241612
4. *Pasko A.M.*: On the homology groups $H_k(\mathbb{C}\Omega_n)$, $k = 1, ..., n$. Res. Math., 2021; 29(1): pp. 24–30. doi:10.15421/242103
5. *Ruban V.I.*: The CW-structure of the spaces of Ω-splines. The Investigations on the Modern Problems of Summation and Approximation of Functions and their Applications, 1985: pp. 39–40.
6. *Ruban V.I.*: The CW-structure and the cohomology of the spaces of generalized perfect splines. The Bulletin of Dnipropetrovsk University. Ser. Mat., 1999; 4: pp. 85–90.

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