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Characterization of Biharmonic Hypersurface [1]

Abstract. The main purpose of this paper is to study biharmonic hypersurface in a quasi-paraSasakian manifold $\mathbb{Q}^{2m+1}$. Biharmonic hypersurfaces are special cases of biharmonic maps and biharmonic maps are the critical points of the bienergy functional. The condition of biharmonicity for non-degenerate hypersurfaces in $\mathbb{Q}^{2m+1}$ is investigated for both cases: either the characteristic vector field of $\mathbb{Q}^{2m+1}$ is the unit normal vector field to the hypersurface or it belongs to the tangent space of the hypersurface. Some relevant examples are also illustrated.
Key words: biharmonic map, biharmonic hypersurface, quasi-paraSasakian manifold

Анотація. Основною метою цієї статті є дослідження бігармонічної гіперповерхні в квазіпараСасакієвому многовиді $\mathbb{Q}^{2m+1}$. Бігармонічні гіперповерхні є окремими випадками бігармонічних відображень, а бігармонічні відображення є критичними точками функціоналу біенергії. Умова бігармонічності для невироджених гіперповерхонь у $\mathbb{Q}^{2m+1}$ досліджується для обох випадків: або характеристичне векторне поле $\mathbb{Q}^{2m+1}$ є одиничним нормальним векторним полем до гіперповерхні, або воно належить дотичному простору гіперповерхні. Деякі відповідні приклади також проілюстровано.
Ключові слова: бігармонічне відображення, бігармонічна гіперповерхня, квазіпараСасакієвий многовид

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1. Introduction

Biharmonic hypersurfaces are biharmonic submanifolds with codimension 1. Biharmonic submanifolds are special cases of biharmonic maps. Let

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$(\mathbb{M}_i, \mathbf{g}_i)$ $(i \in \{1, 2\})$ be semi-Riemannian manifolds, $\Omega \subseteq \mathbb{M}_1$ be a compact domain, $\Gamma(\mathcal{T}\mathbb{M}_i)$ denote the sections of tangent bundle $\mathcal{T}\mathbb{M}_i$ and $\Upsilon : (\mathbb{M}_1, \mathbf{g}_1) \longrightarrow (\mathbb{M}_2, \mathbf{g}_2)$ be a smooth map.

Definition 1 (Harmonic maps)**.** A smooth map Υ is called harmonic if it is a critical point of the energy functional $E(\Upsilon)$.

$$E(\Upsilon) := \frac{1}{2} \int_{\Omega} |d\Upsilon|^2 v_{\mathbf{g}_1}. \tag{1.1}$$

Here $v_{\mathbf{g}_1}$ denotes the volume form with respect to metric $\mathbf{g}_1$ and $|d\Upsilon|$ is the Hilbert-Schmidt norm of $d\Upsilon$ [1, 2].

Definition 2 (Tension field of Υ)**.** The tension field of Υ is defined as

$$\tau(\Upsilon) = \mathrm{trace}\, \nabla \mathrm{d}\Upsilon. \tag{1.2}$$

Here, $\nabla d\Upsilon$ denotes the second fundamental form of Υ and it is given as

$$\nabla d\Upsilon(X_1, X_2) = \nabla^{\Upsilon}_{X_1} d\Upsilon(X_2) - d\Upsilon\left(\nabla^{\mathbb{M}_1}_{X_1} X_2\right), \tag{1.3}$$

for all $X_1, X_2 \in \Gamma(\mathcal{T}\mathbb{M}_1)$ and $\tau(\Upsilon)$ is a section of the pull-back bundle $\Upsilon^{-1}\mathcal{T}\mathbb{M}_2$. The map Υ is harmonic if $\tau(\Upsilon)$ vanishes [1, 2].

Definition 3 (Biharmonic maps)**.** A smooth map Υ is known as biharmonic if it is a critical point of the bienergy functional given by

$$E_2(\Upsilon) := \frac{1}{2} \int_{\Omega} |\tau(\Upsilon)|^2 v_{\mathbf{g}_1}, \tag{1.4}$$

and for bienergy, the first variation formula is

$$\frac{d}{dt} E_2(\Upsilon_t : \Omega)|_{t=0} = \int_{\Omega} \mathbf{g}_2(\tau_2(\Upsilon), w)\, v_{\mathbf{g}_1}, \tag{1.5}$$

where $\{\Upsilon_t\}$ is the variation of Υ, w is a vector field associated to it (called *variational vector field*) and the *bitension field* τ_2 of Υ is defined as follows

$$\tau_2(\Upsilon) = -J^{\Upsilon}(\tau(\Upsilon)) = -\Delta^{\Upsilon}\tau(\Upsilon) - \mathrm{trace}\, \mathrm{R}^{\mathbb{M}_2}(\mathrm{d}\Upsilon, \tau(\Upsilon))\, \mathrm{d}\Upsilon. \tag{1.6}$$

Here $R^{\mathbb{M}_2}$ is the curvature operator on $\mathbb{M}_2$, J^{Υ} is the Jacobi operator of Υ, Δ^{Υ} is the rough Laplacian on $\Gamma\left(\Upsilon^{-1}\mathcal{T}\mathbb{M}_2\right)$ and it is defined as

$$\Delta^{\Upsilon} W = -\mathrm{trace}\left\{\nabla^{\Upsilon}\nabla^{\Upsilon}\mathrm{W} - \nabla^{\Upsilon}_{\nabla}\mathrm{W}\right\}, \quad \mathrm{W} \in \Gamma\left(\Upsilon^{-1}\mathcal{T}\mathbb{M}_2\right), \tag{1.7}$$

where ∇^{Υ} is the pull-back connection on $\Upsilon^{-1}\mathcal{T}\mathbb{M}_2$ [3, 4].

In view of (1.6), it is obvious that

$$Harmonicity \quad \Rightarrow \quad Biharmonicity.$$

But converse is not necessarily true. Hence, a *proper biharmonic map* is defined as a non-harmonic biharmonic map.

Definition 4 (Biharmonic Submanifolds)**.** A submanifold $\mathbb{M}_1$ of $(\mathbb{M}, \mathbf{g})$ is said to be biharmonic if the inclusion map $\Upsilon : (\mathbb{M}_1, \Upsilon^* \mathbf{g}) \longrightarrow (\mathbb{M}, \mathbf{g})$ is a biharmonic isometric immersion.

For an isometric immersion $\Upsilon : \mathbb{M}_1 \longrightarrow \mathbb{M}$, the second fundamental form of Υ is equal to the second fundamental form of submanifold $\Upsilon(\mathbb{M}_1)$ immersed in $\mathbb{M}$. Also, an isometric immersion is harmonic if and only if it is minimal. Thus

$$Minimal\ submanifolds \quad \Rightarrow \quad Biharmonic\ submanifolds.$$

But converse is not necessarily true. A *proper biharmonic submanifold* is defined as a non-minimal biharmonic submanifold (see [5, 6, 1]).

The notion of biharmonic submanifolds initiated with the independent works of G. Y. Jiang [4, 7] and B. Y. Chen [8]. B. Y. Chen defined biharmonic submanifolds of Euclidean spaces as the manifolds with harmonic mean curvature vector field. This notion of biharmonic submanifold can be derived by applying the definition of biharmonic maps to Riemannian immersions into Euclidean space. Chen showed that the proper biharmonic surfaces in Euclidean 3-spaces does not exists. Further, he presented the classification problem: *every biharmonic submanifold of Euclidean space is minimal*, known as Chen's conjecture [8]. Dimitrić obtained various partial solutions of Chen's conjecture [9, 10].

Motivated by accomplishment of Chen's conjecture, the authors of [11] proposed generalized version of Chen's conjecture: *any biharmonic submanifold of* $(\mathbb{M}, \mathbf{g})$ *with* $Riem^{\mathbb{M}} \geq 0$ *is minimal.* Chen-Ishikawa [12] and Jiang [7] studied biharmonic submanifold in $\mathbb{R}^3$ and proved they are minimal in $\mathbb{R}^3$. After that Vlachos and Hasanis studied biharmonic hypersurface in $\mathbb{R}^4$ and proved they are minimal in $\mathbb{R}^4$ [13]. Chen's conjecture is appropriate for the hypersurfaces in pseudo-Euclidean spaces [15, 14] and Euclidean spaces [16, 17] but for submanifolds in pseudo-Euclidean spaces it is not always true [12, 18, 19, 20].

The first contribution to biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean space $\mathbb{R}^4_s$, $s = 1, 2, 3$ is made by Chen-Ishikawa [18, 12]. They proved that in pseudo-Euclidean 3-spaces, biharmonic surfaces are minimal. Biharmonic submanifolds with parallel mean curvature vector in $\mathbb{R}^4_s$ has been characterized by Yu Fu [21]. In $\mathbb{R}^4_s$ biharmonic hypersurfaces with diagonalizable Weingarten operator are minimal that have been proved by Defever et al. [14].

Biharmonic submanifolds of pseudo-Riemannian and Riemannian manifolds have been studied by several geometers [5, 23, 24, 25, 26, 11, 22, 27, 28, 32, 34, 35, 33, 40, 21, 41, 42] who mainly focused to characterize biharmonic submanifolds in Riemannian and pseudo-Riemannian manifolds. As biharmonic submanifolds have not yet been much studied in contact and paracontact manifolds except for a few works on some classifications of biharmonic submanifolds [43, 44, 45]. In the present paper, we investigate the biharmonic hypersurface in these type of manifolds particularly in quasi-paraSasakian manifold $\mathbb{Q}^{2m+1}$.

The sectional study of this paper is as follows: In second section we give some basic definitions for further use. In third section, the condition of biharmonicity for non-degenerate hypersurfaces in $\mathbb{Q}^{2m+1}$ is established for both the cases either the characteristic vector field of $\mathbb{Q}^{2m+1}$ be the unit normal vector field to the hypersurface or it belongs to the tangent space of the hypersurface. Some relevant examples are also illustrated.

2. Quasi-paraSasakian Manifolds

A smooth manifold $\mathbb{M}^{2m+1}$ is said to be an *almost paracontact metric manifold* if there exists a $1-$form η, a vector field ξ, a $(1,1)$-type tensor field φ and metric tensor $\overline{\mathbf{g}}$ on $\mathbb{M}^{2m+1}$ satisfying

$$\left.\begin{array}{r}\varphi^2 = I - \eta\otimes\xi, \quad \eta\circ\varphi = 0, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0,\\ \overline{\mathbf{g}}(\varphi X_1, \varphi X_2) = -\overline{\mathbf{g}}(X_1, X_2) + \eta(X_1)\eta(X_2).\end{array}\right\} \tag{2.1}$$

Also, the endomorphism φ induces an almost paracomplex structure P on each fiber of $2m-$dimensional horizontal distribution $D := ker\eta$. Examples of almost paracontact metric manifolds are given in [36].

Now, consider an almost paracontact manifold $\mathbb{M}^{2m+1}$. Then, on $\mathbb{M}^{2m+1}\times\mathbb{R}$ the almost paracomplex structure P is defined as

$$P\left(X_1, \ell\frac{d}{dt}\right) = \left(\varphi X_1 - \ell\xi, \eta(X_1)\frac{d}{dt}\right),$$

where $X_1\in\Gamma(\mathcal{T}\mathbb{M}^{2m+1})$, t being standard coordinate on $\mathbb{R}$ and ℓ is a smooth function on $\mathbb{M}^{2m+1}\times\mathbb{R}$. Then $\mathbb{M}^{2m+1}$ is *normal* if and only if P is integrable. Equivalently, the following condition satisfies

$$[\varphi,\varphi](X_1,X_2) + 2d\eta(X_1,X_2)\xi = 0, \tag{2.2}$$

here $[\varphi,\varphi]$ is the *Nijenhuis torsion* of φ and it is given as

$$[\varphi,\varphi](X_1,X_2) = [\varphi X_1,\varphi X_2] - \varphi[\varphi X_1, X_2] + \varphi^2[X_1,X_2] - \varphi[X_1,\varphi X_2],$$

for any tangent vector fields X_1, X_2 on $\mathbb{M}^{2m+1}$.

If $\Phi(X_1,X_2) = \overline{\mathbf{g}}(X_1,\varphi X_2) = d\eta(X_1,X_2)$, then $\left(\mathbb{M}^{2m+1};\varphi,\xi,\eta,\overline{\mathbf{g}}\right)$ is called *paracontact metric manifold*, where Φ denotes the fundamental 2-form. A paracontact metric manifold is said to be *paraSasakian* if and only if $\left(\overline{\nabla}_{X_1}\varphi\right)X_2 = -\overline{\mathbf{g}}(X_1,X_2)\xi + \eta(X_2)X_1$, where $\overline{\nabla}$ denotes the Levi-Civita connection of $\overline{\mathbf{g}}$.

Definition 5. The manifold $\left(\mathbb{M}^{2m+1}, \varphi, \xi, \eta, \overline{\mathbf{g}}\right)$ is called *quasi-paraSasakian* (denoted by $\mathbb{Q}^{2m+1}$) if the (φ, ξ, η)-structure is normal and

$$\left(\overline{\nabla}_{X_1}\varphi\right) X_2 = \overline{\mathbf{g}}\left(X_1, X_2\right)\xi - \eta\left(X_2\right) X_1, \tag{2.3}$$

for any tangent vector fields X_1, X_2 [31].

In view of (2.3), we have

$$\overline{\nabla}_{X_1}\xi = \varphi X_1.$$

2.1. Curvature Properties of $\mathbb{Q}^{2m+1}$

On a pseudo-Riemannian metric manifold $(\mathbb{M}^{2m+1}, \overline{\mathbf{g}})$, the curvature tensor $\overline{R}$ is given as

$$\overline{R}\left(X_1, X_2\right) X_3 = [\overline{\nabla}_{X_1}, \overline{\nabla}_{X_2}]X_3 - \overline{\nabla}_{[X_1, X_2]}X_3, \tag{2.4}$$

for any vector fields $X_1, X_2, X_3 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$.

From the straightforward computation, we have the following results for later use.

Proposition 1. For a quasi-paraSasakian manifold $\mathbb{Q}^{2m+1}$, we have

$$\overline{\mathbf{g}}\left(\overline{R}\left(X_1, X_2\right) X_3, \xi\right) = \overline{\mathbf{g}}\left(X_1, X_3\right)\eta\left(X_2\right) - \overline{\mathbf{g}}\left(X_2, X_3\right)\eta\left(X_1\right), \tag{2.5}$$

$$\overline{R}\left(\xi, X_1\right) X_2 = \eta\left(X_2\right) X_1 - \overline{\mathbf{g}}\left(X_1, X_2\right)\xi, \tag{2.6}$$

$$\overline{R}\left(X_1, X_2\right)\xi = \eta\left(X_1\right) X_2 - \eta\left(X_2\right) X_1, \tag{2.7}$$

$$\overline{R}\left(\xi, X_1\right)\xi = X_1 - \eta\left(X_1\right)\xi, \tag{2.8}$$

$$(\overline{\nabla}_{X_3}\overline{R})(X_1, X_2, \xi) = -\overline{R}(X_1, X_2)\varphi X_3 - \overline{\mathbf{g}}(X_2, \varphi X_3)X_1 + \overline{\mathbf{g}}(X_1, \varphi X_3)X_2, \tag{2.9}$$

where $X_1, X_2, X_3 \in \Gamma\left(\mathcal{T}\mathbb{Q}^{2m+1}\right)$.

Proposition 2. For a quasi-paraSasakian manifold $\mathbb{Q}^{2m+1}$, we have

$$\overline{S}\left(X_1, \xi\right) = -2m\eta\left(X_1\right), \tag{2.10}$$

$$\overline{S}\left(\varphi X_1, \varphi X_2\right) = -\overline{S}\left(X_1, X_2\right) - 2m\eta\left(X_1\right)\eta\left(X_2\right), \tag{2.11}$$

where $X_1, X_2 \in \Gamma\left(\mathcal{T}\mathbb{Q}^{2m+1}\right)$ and $\overline{S}$ is the Ricci tensor of $\mathbb{Q}^{2m+1}$.

3. Biharmonic hypersurfaces in quasi-paraSasakian manifold $\mathbb{Q}^{2m+1}$

Let $\mathcal{P}$ be a smooth connected $2m$-manifold and $i : \mathcal{P} \to \mathbb{Q}^{2m+1}$ be an *immersion*. Then $i(\mathcal{P})$ is known as an *immersed hypersurface* of $\mathbb{Q}^{2m+1}$. Let

i induce a symmetric tensor field $\mathbf{g}$ on $i(\mathcal{P})$ which satisfies $\mathbf{g}(X_1, X_2)|_p = \overline{\mathbf{g}}(i_* X_1, i_* X_2)|_{i(p)}$, for every $X_1, X_2 \in \Gamma(\mathcal{T}_p\mathcal{P})$, where i_* denotes the *differential map* (or *pushforward map*) of i. In view of *causal character* of vector fields of manifold, we have three types of hypersurface, specifically, *pseudo-Riemannian*, *Riemannian* and *null* (or *lightlike*). Further, the metric $\mathbf{g}$ will be *non-degenerate* on pseudo-Riemannian and Riemannian hypersurfaces and *degenerate* on the lightlike hypersurface. From hereafter, we will call both the hypersurfaces: pseudo-Riemannian and Riemannian hypersurfaces by the name *non-degenerate hypersurfaces.* Let us suppose that $\mathcal{P}$ be a non-degenerate hypersurface of $\mathbb{Q}^{2m+1}$. Then normal bundle of $\mathcal{P}$ is given by

$$\mathcal{TP}^{\perp} = \{X \in \Gamma(\mathcal{T}\mathbb{Q}^{2m+1}) | \mathbf{g}(X, Z) = 0, \text{ for each } Z \in \Gamma(\mathcal{T}\mathbb{Q}^{2m+1})\}.$$

Here $\dim(\mathcal{T}_{\mathrm{p}}\mathcal{P}^{\perp}) = 1$, due to the fact that $\mathcal{P}$ is a hypersurface. The orthogonal complementary decomposition is given by $\mathcal{T}\mathbb{Q}^{2m+1} = \mathcal{TP}^{\perp} \perp \mathcal{TP}$, $\mathcal{TP}^{\perp} \cap \mathcal{TP} = \{0\}$ (see, [29, 30]). Following Definition 4, we give:

Definition 6. The hypersurface $\mathcal{P}$ of $\mathbb{Q}^{2m+1}$ is said to be non-degenerate biharmonic if the inclusion map $\Upsilon : (\mathcal{P}, \Upsilon^* \mathbf{g}) \longrightarrow (\mathbb{Q}^{2m+1}, \mathbf{g})$ is biharmonic isometric immersion and $\mathcal{P}$ is non-degenerate.

Let us denote the non-degenerate hypersurface of a $(2m+1)$-dimensional quasi-paraSasakian manifold $\mathbb{Q}^{2m+1}$ by $\mathcal{N}$. Then, for each $X_1, X_2 \in \Gamma(\mathcal{TN})$ and $\zeta \in \Gamma(\mathcal{TN}^{\perp})$ we have the following formulas as

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + B(X_1, X_2), \quad (Gauss\ \ formula) \tag{3.1}$$

$$\overline{\nabla}_{X_1} N = -A_{\zeta} X_1 + \nabla^{\perp}_{X_1} N, \quad (Weingarten\ \ formula) \tag{3.2}$$

here ∇ is the Levi-Civita connection on $\mathcal{N}$, $\nabla^{\perp}$ is the normal connection on $\mathcal{TN}^{\perp}$, A_{ζ} is the shape operator with respect to normal vector field ζ and B is the second fundamental form of $\mathcal{N}$ and they are related as

$$\mathbf{g}(A_{\zeta} X_1, X_2) = \mathbf{g}(B(X_1, X_2), \zeta) = b(X_1, X_2)\, \mathbf{g}(\zeta, \zeta), \tag{3.3}$$

where $B(X_1, X_2) = b(X_1, X_2)\zeta$. Let us denote the mean curvature vector by $\mu_1 = H\zeta$, where H denotes the mean curvature function. Now, we characterize non-degenerate biharmonic hypersurface $\mathcal{N}$ in $\mathbb{Q}^{2m+1}$ as

(i) $\xi \in \Gamma(\mathcal{TN}^{\perp})$ and

(ii) $\xi \in \Gamma(\mathcal{TN})$.

Proposition 3. Let $\Upsilon_1 : \mathcal{N} \longrightarrow \mathbb{Q}^{2m+1}$ be an isometric immersion such that $\xi \in \Gamma(\mathcal{TN}^{\perp})$. Then, the bitension field of Υ_1 is given as

$$\tau_2(\Upsilon_1) = -2m(\Delta H)\xi - 2mH\Delta^{\Upsilon_1}\xi - 4mA_{\xi}(\mathrm{grad}H) - 4m^2\mu_1,$$

where $\mu_1 = H\xi$.

Proof. Consider a quasi-paraSasakian manifold $\mathbb{Q}^{2m+1}$ and its hypersurface $\mathcal{N}$ with unit normal vector field ξ. Also, consider an isometric immersion $\Upsilon_1 : \mathcal{N} \longrightarrow \mathbb{Q}^{2m+1}$. Now, consider a local orthonormal frame of $\mathcal{N}$ as: $\{e_i, \varphi e_i\}$, $g(e_i, e_i) = \epsilon$ and $g(\varphi e_i, \varphi e_i) = -\epsilon$, $i = 1, 2, \cdots, m$ such that adapted orthonormal frame of $\mathbb{Q}^{2m+1}$ is $\{d\Upsilon_1(e_i), d\Upsilon_1(\varphi e_i), \xi\}$, $i = 1, 2, \cdots, m$. We identify $d\Upsilon_1(X_1)$ by X_1 and $\nabla^{\Upsilon_1}_{X_1} W$ by $\nabla_{X_1} W$, for all $X_1 \in \Gamma(\mathcal{T}\mathcal{N})$, $W \in \Gamma\left(\Upsilon_1^{-1}\mathcal{T}\mathbb{Q}^{2m+1}\right)$.

The tension field of Υ_1 is:

$$\tau(\Upsilon_1) = 2m\mu_1, \qquad \mu_1 = H\xi.$$

Here H is the mean curvature function and μ_1 is the mean curvature vector field. The bitension field of an isometric immersion Υ_1 is given as follows

$$\begin{aligned}
\tau_2(\Upsilon_1) =& \sum_{i=1}^{m} \epsilon \Big\{ \nabla^{\Upsilon_1}_{e_i} \nabla^{\Upsilon_1}_{e_i} \tau(\Upsilon_1) - \nabla^{\Upsilon_1}_{\nabla_{e_i} e_i} \tau(\Upsilon_1) - \overline{R}\left(d\Upsilon_1(e_i), \tau(\Upsilon_1)\right) d\Upsilon_1(e_i) \\
& - \nabla^{\Upsilon_1}_{\varphi e_i} \nabla^{\Upsilon_1}_{\varphi e_i} \tau(\Upsilon_1) + \nabla^{\Upsilon_1}_{\nabla_{\varphi e_i} \varphi e_i} \tau(\Upsilon_1) + \overline{R}\left(d\Upsilon_1(\varphi e_i), \tau(\Upsilon_1)\right) d\Upsilon_1(\varphi e_i) \Big\} \\
=& \sum_{i=1}^{m} \epsilon \Big\{ \nabla^{\Upsilon_1}_{e_i} \nabla^{\Upsilon_1}_{e_i} (2mH\xi) - \nabla^{\Upsilon_1}_{\nabla_{e_i} e_i} (2mH\xi) - \overline{R}\left(d\Upsilon_1(e_i), 2mH\xi\right) d\Upsilon_1(e_i) \\
& - \nabla^{\Upsilon_1}_{\varphi e_i} \nabla^{\Upsilon_1}_{\varphi e_i} (2mH\xi) + \nabla^{\Upsilon_1}_{\nabla_{\varphi e_i} \varphi e_i} (2mH\xi) + \overline{R}\left(d\Upsilon_1(\varphi e_i), 2mH\xi\right) d\Upsilon_1(\varphi e_i) \Big\} \\
=& 2m \sum_{i=1}^{m} \epsilon \Big\{ \overline{\nabla}_{e_i} \left(H\overline{\nabla}_{e_i}\xi + e_i(H)\xi \right) - \nabla_{e_i} e_i(H)\xi - H\overline{\nabla}_{\nabla_{e_i} e_i}\xi \\
& - H\overline{R}\left(d\Upsilon_1(e_i), \xi\right) d\Upsilon_1(e_i) - \overline{\nabla}_{\varphi e_i} \left(\varphi e_i(H)\xi + H\overline{\nabla}_{\varphi e_i}\xi \right) \\
& + \nabla_{\varphi e_i} \varphi e_i(H)\xi + H\overline{\nabla}_{\nabla_{\varphi e_i} \varphi e_i}\xi + H\overline{R}\left(d\Upsilon_1(\varphi e_i), \xi\right) d\Upsilon_1(\varphi e_i) \Big\} \\
=& 2m \sum_{i=1}^{m} \epsilon \left\{ 2e_i(H)\overline{\nabla}_{e_i}\xi + e_i(e_i(H))\xi + H\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi - (\nabla_{e_i} e_i)(H)\xi \right. \\
& - H\overline{\nabla}_{\nabla_{e_i} e_i}\xi - H\overline{R}\left(d\Upsilon_1(e_i), \xi\right) d\Upsilon_1(e_i) - \varphi e_i(\varphi e_i(H))\xi \\
& - 2\varphi e_i(H)\overline{\nabla}_{\varphi e_i}\xi - H\overline{\nabla}_{\varphi e_i}\overline{\nabla}_{\varphi e_i}\xi + (\nabla_{\varphi e_i}\varphi e_i)(H)\xi + H\overline{\nabla}_{\nabla_{\varphi e_i}\varphi e_i}\xi \\
& \left. + H\overline{R}\left(d\Upsilon_1(\varphi e_i), \xi\right) d\Upsilon_1(\varphi e_i) \right\}
\end{aligned}$$

With the help of equation (1.7) and using the definition of ΔH and $\mathrm{grad} H$, we get

$$\begin{aligned}
\tau_2(\Upsilon_1) =& -2mH \sum_{i=1}^{m} \epsilon \left\{ \overline{R}\left(d\Upsilon_1(e_i), \xi\right) d\Upsilon_1(e_i) - \overline{R}\left(d\Upsilon_1(\varphi e_i), \xi\right) d\Upsilon_1(\varphi e_i) \right\} \\
& - 2mH\Delta^{\Upsilon_1}\xi - 2m(\Delta H)\xi - 4mA_\xi(\mathrm{grad} H).
\end{aligned}$$

Now by using equation (2.6), we get the required result.

Theorem 1. *Let $\Upsilon_1 : \mathcal{N} \longrightarrow \mathbb{Q}^{2m+1}$ be an isometric immersion such that $\xi \in \Gamma\left(\mathcal{T}\mathcal{N}^{\perp}\right)$. Then, the non-degenerate hypersurface $\mathcal{N}$ is biharmonic if and only if*

$$\left.\begin{aligned} m\,\mathrm{grad}H^2 + 2A_\xi\left(\mathrm{grad}H\right) = 0,\\ 2m\left(\Delta H\right)\xi = 0.\end{aligned}\right\} \tag{3.4}$$

Proof. From Definition 4 it is clear that $\mathcal{N}$ is biharmonic if and only if bi-tension field of Υ_1 vanishes, *i.e.*, $\tau_2\left(\Upsilon_1\right) = 0$. In light of above proposition, we have

$$2m\left(\Delta H\right)\xi + 2mH\Delta^{\Upsilon_1}\xi + 4mA_\xi\left(\mathrm{grad}H\right) + 4m^2H\xi = 0. \tag{3.5}$$

Here, it is sufficient to compute the normal and tangential part of $\Delta^{\Upsilon_1}\xi$:

(i) The tangential part of $\Delta^{\Upsilon_1}\xi$ is given by

$$\begin{aligned}
\left(\Delta^{\Upsilon_1}\xi\right)^{\top} &= -\sum_{i,j=1}^{m}\Big\{\overline{\mathbf{g}}\left(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi - \overline{\nabla}_{\nabla_{e_i}e_i}\xi + \overline{\nabla}_{\varphi e_i}\overline{\nabla}_{\varphi e_i}\xi - \overline{\nabla}_{\nabla_{\varphi e_i}\varphi e_i}\xi, e_j\right)e_j\\
&\qquad + \overline{\mathbf{g}}\left(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi - \overline{\nabla}_{\nabla_{e_i}e_i}\xi + \overline{\nabla}_{\varphi e_i}\overline{\nabla}_{\varphi e_i}\xi - \overline{\nabla}_{\nabla_{\varphi e_i}\varphi e_i}\xi, \varphi e_j\right)\varphi e_j\Big\}\\
&= \sum_{i,j=1}^{m}\left\{\overline{\mathbf{g}}\left(\overline{\nabla}_{e_i}A_\xi e_i - A_\xi\left(\nabla_{e_i}e_i\right) + \overline{\nabla}_{\varphi e_i}A_\xi\left(\varphi e_i\right) - A_\xi\left(\nabla_{\varphi e_i}\varphi e_i\right), e_j\right)e_j\right.\\
&\qquad \left. + \overline{\mathbf{g}}\left(\overline{\nabla}_{e_i}A_\xi e_i - A_\xi\left(\nabla_{e_i}e_i\right) + \overline{\nabla}_{\varphi e_i}A_\xi\left(\varphi e_i\right) - A_\xi\left(\nabla_{\varphi e_i}\varphi e_i\right), \varphi e_j\right)\varphi e_j\right\}\\
&= \sum_{i,j=1}^{m}\left\{e_i\overline{\mathbf{g}}\left(A_\xi e_i, e_j\right) - \overline{\mathbf{g}}\left(A_\xi e_i, \nabla_{e_i}e_j\right) - \overline{\mathbf{g}}\left(A_\xi\left(\nabla_{e_i}e_i\right), e_j\right)\right.\\
&\qquad + \varphi e_i\overline{\mathbf{g}}\left(A_\xi\varphi e_i, e_j\right) - \overline{\mathbf{g}}\left(A_\xi\varphi e_i, \nabla_{\varphi e_i}e_j\right) - \overline{\mathbf{g}}\left(A_\xi\left(\nabla_{\varphi e_i}\varphi e_i\right), e_j\right)\\
&\qquad + e_i\overline{\mathbf{g}}\left(A_\xi e_i, \varphi e_j\right) - \overline{\mathbf{g}}\left(A_\xi e_i, \nabla_{e_i}\varphi e_j\right) - \overline{\mathbf{g}}\left(A_\xi\left(\nabla_{e_i}e_i\right), \varphi e_j\right)\\
&\qquad \left. + \varphi e_i\overline{\mathbf{g}}\left(A_\xi\varphi e_i, \varphi e_j\right) - \overline{\mathbf{g}}\left(A_\xi\varphi e_i, \nabla_{\varphi e_i}\varphi e_j\right) - \overline{\mathbf{g}}\left(A_\xi\left(\nabla_{\varphi e_i}\varphi e_i\right), \varphi e_j\right)\right\}\\
&= \sum_{i,j=1}^{m}\left\{e_ib\left(e_i, e_j\right) - b\left(e_i, \nabla_{e_i}e_j\right) - b\left(\nabla_{e_i}e_i, e_j\right)\right.\\
&\qquad + \varphi e_ib\left(\varphi e_i, e_j\right) - b\left(\varphi e_i, \nabla_{\varphi e_i}e_j\right) - b\left(\nabla_{\varphi e_i}\varphi e_i, e_j\right)\\
&\qquad + e_ib\left(e_i, \varphi e_j\right) - b\left(e_i, \nabla_{e_i}\varphi e_j\right) - \left(\nabla_{e_i}e_i, \varphi e_j\right)\\
&\qquad \left. + \varphi e_ib\left(\varphi e_i, \varphi e_j\right) - b\left(\varphi e_i, \nabla_{\varphi e_i}\varphi e_j\right) - b\left(\nabla_{\varphi e_i}\varphi e_i, \varphi e_j\right)\right\}\\
&= \sum_{i,j=1}^{m}\left\{\left\{\nabla_{e_i}b\left(e_j, e_i\right)\right\}e_j + \left\{\nabla_{\varphi e_i}b\left(e_j, \varphi e_i\right)\right\}e_j\right.\\
&\qquad \left. + \left\{\nabla_{e_i}b\left(\varphi e_j, e_i\right)\right\}\varphi e_j + \left\{\nabla_{\varphi e_i}b\left(\varphi e_j, \varphi e_i\right)\right\}\varphi e_j\right\}.
\end{aligned}$$

Using Codazzi-Mainardi equation [46], we get

$$\sum_{i=1}^{m}\left(\nabla_{e_i} b\left(e_j, e_i\right)-\nabla_{e_j} b\left(e_i, e_i\right)\right)=\sum_{i=1}^{m} \overline{\mathbf{g}}\left(\overline{R}\left(e_i, e_j\right) e_i, \xi\right)$$

$$=-\overline{S}\left(\xi, e_j\right).$$

Since $\overline{S}\left(\xi, e_j\right)=0$ above equation becomes

$$\sum_{i=1}^{m}\left(\nabla_{e_i} b\left(e_j, e_i\right)-\nabla_{e_j} b\left(e_i, e_i\right)\right)=0.$$

Hence,

$$\begin{aligned}\left(\Delta^{\Upsilon_1} \xi\right)^{\top}=\sum_{i, j=1}^{m} & \left\{\left\{\nabla_{e_j}\left(b\left(e_i, e_i\right)+b\left(\varphi e_i, \varphi e_i\right)\right)\right\} e_j\right. \\ & \left.+\left\{\nabla_{e_j}\left(b\left(e_i, e_i\right)+b\left(\varphi e_i, \varphi e_i\right)\right)\right\} \varphi e_j\right\}=2 m \operatorname{grad} H. \end{aligned} \tag{3.6}$$

(3.7)

(ii) The normal part of $\Delta^{\Upsilon_1} \xi$ is given by

$$\left(\Delta^{\Upsilon_1} \xi\right)^{\perp}=\overline{\mathbf{g}}\left(\Delta^{\Upsilon_1} \xi, \xi\right) \xi,$$

$$\begin{aligned}\overline{\mathbf{g}}\left(\Delta^{\Upsilon_1} \xi, \xi\right) & =-\sum_{i=1}^{m} \epsilon\left\{\overline{\mathbf{g}}\left(\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} \xi-\overline{\nabla}_{\nabla_{e_i} e_i} \xi, \xi\right)\right. \\ & \left.\quad-\overline{\mathbf{g}}\left(\overline{\nabla}_{\varphi e_i} \overline{\nabla}_{\varphi e_i} \xi-\overline{\nabla}_{\nabla_{\varphi e_i} \varphi e_i} \xi, \xi\right)\right\} \\ & =-\sum_{i=1}^{m} \epsilon\left\{\overline{\mathbf{g}}\left(\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} \xi, \xi\right)-\overline{\mathbf{g}}\left(\overline{\nabla}_{\varphi e_i} \overline{\nabla}_{\varphi e_i} \xi, \xi\right)\right\} \\ & =\sum_{i=1}^{m} \epsilon\left\{\overline{\mathbf{g}}\left(\overline{\nabla}_{e_i} \xi, \overline{\nabla}_{e_i} \xi\right)-\overline{\mathbf{g}}\left(\overline{\nabla}_{\varphi e_i} \xi, \overline{\nabla}_{\varphi e_i} \xi\right)\right\}.\end{aligned}$$

Now using equations (2.1) and (2.3), we get

$$\left(\Delta^{\Upsilon_1} \xi\right)^{\perp}=\overline{\mathbf{g}}\left(\Delta^{\Upsilon_1} \xi, \xi\right) \xi=-2 m \xi. \tag{3.8}$$

Using equations (3.6) and (3.8) in (3.5), we have

$$\left.\begin{aligned}\left(\tau_2\left(\Upsilon_1\right)\right)^{\top} & =-4 m A_{\xi}(\operatorname{grad} H)-2 m^2 \operatorname{grad} H^2, \\ \left(\tau_2\left(\Upsilon_1\right)\right)^{\perp} & =-2 m(\Delta H) \xi.\end{aligned}\right\} \tag{3.9}$$

Corollary 1. *Let $\mathcal{N}$ be a non-degenerate hypersurface of $\mathbb{Q}^{2m+1}$ with harmonic mean curvature vector field and $\xi \in \Gamma\left(\mathcal{T}\mathcal{N}^{\perp}\right)$. Then $\mathcal{N}$ is biharmonic if and only if one of the following two conditions holds:*

(i) $\mathcal{N}$ is minimal.

(ii) $A_{\xi}\left(\mathrm{grad}H\right) = -\frac{m}{2}\left(\mathrm{grad}H^2\right)$ and $\Delta H = 0$.

Proof. (i) For minimal hypersurface we have $\mu_1 = H\xi = 0$, then by using (3.9) we get $\mathcal{N}$ is a biharmonic hypersurface.

(ii) Let $A_{\xi}\left(\mathrm{grad}H\right) = -\frac{m}{2}\left(\mathrm{grad}H^2\right)$ and $\Delta H = 0$, then by using (3.9) we find that $\mathcal{N}$ is a biharmonic hypersurface.

Theorem 2. *Let $\mathcal{N}$ be a totally umbilical non-degenerate biharmonic hypersurface of a $(2m+1)$-dimensional quasi-paraSasakian manifold $\mathbb{Q}^{2m+1}$. Then $\mathcal{N}$ is minimal.*

Proof. Consider an isometric immersion $\Upsilon_1 : \mathcal{N} \longrightarrow \mathbb{Q}^{2m+1}$, a local orthonormal frame of $\mathcal{N}$ as: $\{e_i, \varphi e_i\}$; $g\left(e_i, e_i\right) = \epsilon$ and $g\left(\varphi e_i, \varphi e_i\right) = -\epsilon$, $i = 1, 2, \cdots, m$ such that adapted orthonormal frame of $\mathbb{Q}^{2m+1}$ is $\{d\Upsilon_1\left(e_i\right), d\Upsilon_1\left(\varphi e_i\right), \xi\}$, $i = 1, 2, \cdots, m$. We identify $d\Upsilon_1\left(X_1\right)$ by X_1 for all $X_1 \in \Gamma\left(\mathcal{T}\mathcal{N}\right)$.
Now, we have an orthonormal frame of the ambient manifold $\mathbb{Q}^{2m+1}$ as: $\{e_i, \varphi e_i, \xi\}$, $i = 1, 2, \cdots, m$ such that $A_{\xi} e_i = \lambda_i e_i.$, where λ_i is the principal curvature in the direction of e_i. We know for the totally umbilical manifold $\mathbb{Q}^{2m+1}$ all the principal curvatures are equal to the same number $\lambda\left(q\right)$ at any point q of $\mathbb{Q}^{2m+1}$. From (3.3) we have

$$
\begin{aligned}
H &= \frac{1}{2m}\sum_{i=1}^{m}\epsilon\left\{\overline{\mathbf{g}}\left(B\left(e_i, e_i\right), \xi\right) - \overline{\mathbf{g}}\left(B\left(\varphi e_i, \varphi e_i\right), \xi\right)\right\} \\
&= \frac{1}{2m}\sum_{i=1}^{m}\epsilon\left\{\overline{\mathbf{g}}\left(A_{\xi} e_i, e_i\right) - \overline{\mathbf{g}}\left(A_{\xi}\varphi e_i, \varphi e_i\right)\right\} \\
&= \frac{1}{2m}\sum_{i=1}^{m}\epsilon\left\{\overline{\mathbf{g}}\left(\lambda e_i, e_i\right) - \overline{\mathbf{g}}\left(\lambda\varphi e_i, \varphi e_i\right)\right\} = \lambda. \qquad (3.10)
\end{aligned}
$$

On the other hand by using equation (3.10), we get

$$2A_{\xi}\left(\mathrm{grad}H\right) = \mathrm{grad}\lambda^2. \qquad (3.11)$$

Since $\mathcal{N}$ is a biharmonic hypersurface of $\mathbb{Q}^{2m+1}$ and using (3.4), (3.10) and (3.11), we get $\Delta\lambda = 0$ and $(m-1)\,\mathrm{grad}\lambda^2 = 0$. This completes the proof.

Example 1. Let $\mathcal{Q}^3 := \mathbb{R}^3_1\left(\varphi, \xi, \eta\right)$, where $\xi = e_3$ and given the 1-form η and endomorphism φ as: $\varphi e_1 = e_2$, $\varphi e_2 = e_1 - 2ye_3$, $\varphi e_3 = 0$, $\eta = 2ydx + dz$ ((x, y, z) being the Cartesian coordinates and $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z}$) and

the metric tensor $\overline{\mathbf{g}} = dx^2 - dy^2 + \eta \otimes \eta$. Then by straightforward computations, we see that $\mathcal{Q}^3$ equipped with $(\varphi, \xi, \eta, \overline{\mathbf{g}})$-structure is a quasi-paraSasakian 3-manifold.
Consider a hypersurface $\mathcal{N} = \{(x, y, z) \mid z = c, c > 0\}$ of $\mathcal{Q}^3$ and an isometric immersion $\Upsilon_1 : \mathcal{N} \longrightarrow \mathcal{Q}^3$ defined by $\Upsilon_1(x, y) = (x, y, c)$. Now, we find an orthonormal frame on $\mathcal{Q}^3$ adapted to the surface $\mathcal{N}$ as

$$F_1 = e_1 - 2ye_3, \quad F_2 = e_2, \quad F_3 = e_3.$$

Thus, $\mathcal{N}$ becomes the hypersurface of quasi-paraSasakian manifold $\mathcal{Q}^3$. By further computation, we get

$$[F_1, F_2] = 2e_3, \quad [F_1, F_3] = 0, \quad [F_2, F_3] = 0.$$

$$\begin{aligned} &\overline{\nabla}_{F_1} F_1 = 0, \quad \overline{\nabla}_{F_2} F_2 = 0, \quad \overline{\nabla}_{F_3} F_3 = 0, \quad \overline{\nabla}_{F_1} F_2 = -\overline{\nabla}_{F_2} F_1 = F_3, \\ &\overline{\nabla}_{F_1} F_3 = -F_2, \quad \overline{\nabla}_{F_3} F_1 = -F_3, \quad \overline{\nabla}_{F_2} F_3 = F_1, \quad \overline{\nabla}_{F_3} F_2 = F_3. \end{aligned} \tag{3.12}$$

Next, the components of the second fundamental form can be computed as

$$\left.\begin{aligned} b(F_1, F_1) &= \overline{\mathbf{g}}\left(\overline{\nabla}_{F_1} F_1, \xi\right) = 0, \\ b(F_1, F_2) &= \overline{\mathbf{g}}\left(\overline{\nabla}_{F_1} F_2, \xi\right) = 1, \\ b(F_2, F_1) &= \overline{\mathbf{g}}\left(\overline{\nabla}_{F_2} F_1, \xi\right) = -1, \\ b(F_2, F_2) &= \overline{\mathbf{g}}\left(\overline{\nabla}_{F_2} F_2, \xi\right) = 0. \end{aligned}\right\} \tag{3.13}$$

Using equation (3.13), we get the mean curvature of the isometric immersion Υ_1 as

$$H = \frac{1}{2}\left[b(F_1, F_1) + b(F_2, F_2) + b(F_1, F_2) + b(F_2, F_1)\right] = 0.$$

Then $\mathcal{N}$ becomes minimal hypersurface of $\mathcal{Q}^3$ with a unit normal vector field ξ, so it will be biharmonic also.

Proposition 4. Let $\mathbb{Q}^{2m+1}$ be a quasi-paraSasakian manifold and $\mathcal{N}$ be its non-degenerate hypersurface with $\xi \in \Gamma(\mathcal{TN})$ and an isometric immersion $\Upsilon_2 : \mathcal{N} \longrightarrow \mathbb{Q}^{2m+1}$. Then

(i) the tension field of Υ_2 is given as

$$\tau(\Upsilon_2) = 2mHN, \qquad N = e_m \text{ or } \varphi \mathrm{e_m},$$

(ii) the bitension field of Υ_2 is given as

$$\begin{aligned} \tau_2(\Upsilon_2) = -2\epsilon mH &\left\{ \sum_{i=1}^{m-1} \overline{R}\left(d\Upsilon_2(e_i), e_m\right) d\Upsilon_2(e_i) \right. \\ &\left. - \sum_{i=1}^{m} \overline{R}\left(d\Upsilon_2(\varphi e_i), e_m\right) d\Upsilon_2(\varphi e_i) \right\} \\ &- 2mH\Delta^{\Upsilon_2} e_m - 4mA_N(\mathrm{grad}H) - 2m(\Delta H)e_m - 2mHe_m, \end{aligned}$$

or

$$\tau_2(\Upsilon_2) = -2\epsilon mH \left\{ \sum_{i=1}^{m} \overline{R}\left(d\Upsilon_2(e_i), \varphi e_m\right) d\Upsilon_2(e_i) - \sum_{i=1}^{m-1} \overline{R}\left(d\Upsilon_2(\varphi e_i), \varphi e_m\right) d\Upsilon_2(\varphi e_i) \right\} - 2mH\Delta^{\Upsilon_2}\varphi e_m - 4mA_N(\mathrm{grad}H) - 2m(\Delta H)\varphi e_m - 2mH\varphi e_m,$$

where A_N is the shape operator of $\mathcal{N}$ with respect to unit normal vector field N and the mean curvature vector is He_m (or $H\varphi e_m$).

Proof. (i) Consider a quasi-paraSasakian manifold $\mathbb{Q}^{2m+1}(\varphi, \xi, \eta, \overline{\mathbf{g}})$ and its hypersurface $\mathcal{N}$ with unit normal vector field N. Also, consider an isometric immersion $\Upsilon_2 : \mathcal{N} \longrightarrow \mathbb{Q}^{2m+1}$.

Case I : $N = e_m$.
Now, consider a local orthonormal frame of $\mathcal{N}$ as: $\{e_{i-1}, \varphi e_i, \xi\}$, $i = 1, 2, \cdots, m$ such that adapted orthonormal frame of $\mathbb{Q}^{2m+1}$ is $\{d\Upsilon_2(e_{i-1}), d\Upsilon_2(\varphi e_i), d\Upsilon_2(\xi), e_m\}$. We identify $d\Upsilon_2(X_1)$ by X_1 and $\nabla^{\Upsilon_2}_{X_1}W$ by $\nabla_{X_1}W$, for all $X_1 \in \Gamma(\mathcal{TN}), W \in \Gamma\left(\Upsilon_2^{-1}\mathcal{T}\mathbb{Q}^{2m+1}\right)$. The tension field of Υ_2 is given by

$$\tau(\Upsilon_2) = 2m\mu_2, \qquad \mu_2 = He_m,$$

here H is the mean curvature function and μ_2 is the mean curvature vector field.

Case II : $N = \varphi e_m$.
Now, consider a local orthonormal frame of $\{e_i, \varphi e_{i-1}, \xi\}$, $i = 1, 2, \cdots, m$ such that adapted orthonormal frame of $\mathbb{Q}^{2m+1}$ is $\{d\Upsilon_2(e_i), d\Upsilon_2(\varphi e_{i-1}), d\Upsilon_2(\xi), \varphi e_m\}$, $i = 1, 2, \cdots, m$. We identify $d\Upsilon_2(X_1)$ by X_1 and $\nabla^{\Upsilon_2}_{X_1}W$ by $\nabla_{X_1}W$, for all $X_1 \in \Gamma(\mathcal{TN}), W \in \Gamma\left(\Upsilon_2^{-1}\mathcal{T}\mathbb{Q}^{2m+1}\right)$. The tension field of Υ_2 is given by $\tau(\Upsilon_2) = 2m\mu_3$, $\mu_3 = H\varphi e_m$, here H is the mean curvature function and μ_3 is the mean curvature vector field.

(ii) The bitension field of an isometric immersion $\Upsilon_2 : \mathcal{N} \longrightarrow \mathbb{Q}^{2m+1}$ is given as follows

$$\tau_2(\Upsilon_2) = \sum_{i=1}^{m} \epsilon \Big\{ \nabla^{\Upsilon_2}_{e_i}\nabla^{\Upsilon_2}_{e_i}\tau(\Upsilon_2) - \nabla^{\Upsilon_2}_{\nabla_{e_i}e_i}\tau(\Upsilon_2) - \overline{R}\left(d\Upsilon_2(e_i), \tau(\Upsilon_2)\right) d\Upsilon_2(e_i) - \nabla^{\Upsilon_2}_{\varphi e_i}\nabla^{\Upsilon_2}_{\varphi e_i}\tau(\Upsilon_2) + \nabla^{\Upsilon_2}_{\nabla_{\varphi e_i}\varphi e_i}\tau(\Upsilon_2) + \overline{R}\left(d\Upsilon_2(\varphi e_i), \tau(\Upsilon_2)\right) d\Upsilon_2(\varphi e_i) \Big\}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \epsilon \Big\{ \nabla^{\Upsilon_2}_{e_i} \nabla^{\Upsilon_2}_{e_i} (2mHN) - \nabla^{\Upsilon_2}_{\nabla_{e_i} e_i} (2mHN) \\
&\quad - \overline{R}(d\Upsilon_2(e_i), 2mHN)\, d\Upsilon_2(e_i) \\
&\quad - \nabla^{\Upsilon_2}_{\varphi e_i} \nabla^{\Upsilon_2}_{\varphi e_i} (2mHN) + \nabla^{\Upsilon_2}_{\nabla_{\varphi e_i} \varphi e_i} (2mHN) \\
&\quad + \overline{R}(d\Upsilon_2(\varphi e_i), 2mHN)\, d\Upsilon_2(\varphi e_i) \Big\} \\
&= 2m \sum_{i=1}^{m} \epsilon \Big\{ \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} (HN) - \overline{\nabla}_{\nabla_{e_i} e_i} (HN) - \overline{R}(d\Upsilon_2(e_i), HN)\, d\Upsilon_2(e_i) \\
&\quad - \overline{\nabla}_{\varphi e_i} \overline{\nabla}_{\varphi e_i} (HN) + \overline{\nabla}_{\nabla_{\varphi e_i} \varphi e_i} (HN) + \overline{R}(d\Upsilon_2(\varphi e_i), HN)\, d\Upsilon_2(\varphi e_i) \Big\} \\
&= 2m \sum_{i=1}^{m} \epsilon \Big\{ \overline{\nabla}_{e_i} \big(e_i(H) N + H \overline{\nabla}_{e_i} N \big) - \nabla_{e_i} e_i (H) N - H \overline{\nabla}_{\nabla_{e_i} e_i} N \\
&\quad - H \overline{R}(d\Upsilon_2(e_i), N)\, d\Upsilon_2(e_i) - \overline{\nabla}_{\varphi e_i} \big(\varphi e_i(H) N + H \overline{\nabla}_{\varphi e_i} N \big) \\
&\quad + \nabla_{\varphi e_i} \varphi e_i (H) N + H \overline{\nabla}_{\nabla_{\varphi e_i} \varphi e_i} N + H \overline{R}(d\Upsilon_2(\varphi e_i), N)\, d\Upsilon_2(\varphi e_i) \Big\} \\
&= 2m \sum_{i=1}^{m} \epsilon \big\{ e_i(e_i(H)) N + 2 e_i(H) \overline{\nabla}_{e_i} N + H \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} N \\
&\quad - \nabla_{e_i} e_i (H) N - H \overline{\nabla}_{\nabla_{e_i} e_i} N - H \overline{R}(d\Upsilon_2(e_i), N)\, d\Upsilon_2(e_i) - \varphi e_i(\varphi e_i(H)) N \\
&\quad - 2 \varphi e_i(H) \overline{\nabla}_{\varphi e_i} N - H \overline{\nabla}_{\varphi e_i} \overline{\nabla}_{\varphi e_i} N + \nabla_{\varphi e_i} \varphi e_i (H) N + H \overline{\nabla}_{\nabla_{\varphi e_i} \varphi e_i} N \\
&\quad + H \overline{R}(d\Upsilon_2(\varphi e_i), N)\, d\Upsilon_2(\varphi e_i) \big\}
\end{aligned}
$$

With the help of equation (1.7) and using the definition of ΔH and grad$H$, we get

$$
\begin{aligned}
\tau_2(\Upsilon_1) = &- 2mH \sum_{i=1}^{m} \epsilon \big\{ \overline{R}(d\Upsilon_2(e_i), N)\, d\Upsilon_2(e_i) - \overline{R}(d\Upsilon_2(\varphi e_i), N)\, d\Upsilon_2(\varphi e_i) \big\} \\
&- 2mH \Delta^{\Upsilon_2} N - 2m(\Delta H) N - 4m A_N(\mathrm{grad} H).
\end{aligned}
$$

Now by taking $N = e_m$ (or $N = \varphi e_m$) and using equation (2.8), we get the required result.

Theorem 3. *Let $\Upsilon_2 : \mathcal{N} \longrightarrow \mathbb{Q}^{2m+1}$ be an isometric immersion such that $\xi \in \Gamma(\mathcal{TN})$. Then, the non-degenerate hypersurface $\mathcal{N}$ is biharmonic if and only if*

$$
\left.
\begin{aligned}
2A_N(\mathrm{grad} H) - 2H\overline{Q}(N) + m\mathbf{g}(N,N)\big(\mathrm{grad} H^2\big) = 0, \\
(\Delta H) - H\overline{S}(N,N) - \epsilon H |A_N|^2 + (1 - \mathbf{g}(N,N)) H = 0.
\end{aligned}
\right\} \tag{3.14}
$$

Proof. From Definition 4 it is clear that for biharmonic hypersurface, isometric immersion $\Upsilon_2 : \mathcal{N} \longrightarrow \mathbb{Q}^{2m+1}$ has to be biharmonic *i.e.* bitension field of Υ_2 has to be zero, *i.e.*,

$$
\begin{aligned}
\tau_2(\Upsilon_2) = &- 2mH \sum_{i=1}^{m} \epsilon \big\{ \overline{R}(d\Upsilon_2(e_i), N)\, d\Upsilon_2(e_i) - \overline{R}(d\Upsilon_2(\varphi e_i), N)\, d\Upsilon_2(\varphi e_i) \big\} \\
&- 2mH \Delta^{\Upsilon_2} N - 2m(\Delta H) N - 4m A_N(\mathrm{grad} H) = 0.
\end{aligned} \tag{3.15}
$$

Now, computing the normal part of $\Delta^{\Upsilon_2} e_m$ and $\Delta^{\Upsilon_2} \varphi e_m$ as

(i) The normal part of $\Delta^{\Upsilon_2} N$ is given by

$$\left(\Delta^{\Upsilon_2} N\right)^{\perp} = \overline{\mathbf{g}}\left(\Delta^{\Upsilon_2} N, N\right) N$$

Now,

$$\begin{aligned}
\overline{\mathbf{g}}\left(\Delta^{\Upsilon_2} N, N\right) &= -\sum_{i=1}^{m} \epsilon \left\{\overline{\mathbf{g}}\left(\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} N, N\right) - \overline{\mathbf{g}}\left(\overline{\nabla}_{\varphi e_i} \overline{\nabla}_{\varphi e_i} N, N\right)\right\} \\
&= \sum_{i=1}^{m} \epsilon \left\{\overline{\mathbf{g}}\left(\overline{\nabla}_{e_i} N, \overline{\nabla}_{e_i} N\right) - \overline{\mathbf{g}}\left(\overline{\nabla}_{\varphi e_i} N, \overline{\nabla}_{\varphi e_i} N\right)\right\} \\
&= \sum_{i=1}^{m} \epsilon \left\{\overline{\mathbf{g}}\left(A_N e_i, A_N e_i\right) - \overline{\mathbf{g}}\left(A_N\left(\varphi e_i\right), A_N\left(\varphi e_i\right)\right)\right\}.
\end{aligned}$$

This implies that

$$\left(\Delta^{\Upsilon_2} N\right)^{\perp} = \overline{\mathbf{g}}\left(N, N\right) \overline{\mathbf{g}}\left(\Delta^{\Upsilon_2} N, N\right) N = \overline{\mathbf{g}}\left(N, N\right) |A_N|^2 N. \tag{3.16}$$

For $N = e_m$, we get

$$\left(\Delta^{\Upsilon_2} e_m\right)^{\perp} = \overline{\mathbf{g}}\left(\Delta^{\Upsilon_2} e_m, e_m\right) e_m = \epsilon |A_N|^2 e_m. \tag{3.17}$$

$N = \varphi e_m$, we get

$$\left(\Delta^{\Upsilon_2} \varphi e_m\right)^{\perp} = -\overline{\mathbf{g}}\left(\Delta^{\Upsilon_2} \varphi e_m, \varphi e_m\right) \varphi e_m = -\epsilon |A|^2 \varphi e_m. \tag{3.18}$$

Now, computing the tangential parts of $\Delta^{\Upsilon_2} e_m$ and $\Delta^{\Upsilon_2} \varphi e_m$ as

(ii) The tangential part of $\Delta^{\Upsilon_2} N$ is given by

$$\begin{aligned}
&\left(\Delta^{\Upsilon_2} N\right)^{\top} = \\
&= -\sum_{i,j=1}^{m} \Big\{\overline{\mathbf{g}}\Big(\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} N - \overline{\nabla}_{\nabla_{e_i} e_i} N + \overline{\nabla}_{\varphi e_i} \overline{\nabla}_{\varphi e_i} N - \overline{\nabla}_{\nabla_{\varphi e_i} \varphi e_i} N, e_j\Big) e_j \\
&\quad - \overline{\mathbf{g}}\Big(\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} N - \overline{\nabla}_{\nabla_{e_i} e_i} N + \overline{\nabla}_{\varphi e_i} \overline{\nabla}_{\varphi e_i} N - \overline{\nabla}_{\nabla_{\varphi e_i} \varphi e_i} e_1, \varphi e_j\Big) \varphi e_j\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\left(\Delta^{\Upsilon_2} N\right)^{\top} = \\
&= \sum_{i,j=1}^{m} \left\{ \overline{\mathbf{g}} \left(\overline{\nabla}_{e_i} A_N e_i - A_N \left(\nabla_{e_i} e_i\right) + \overline{\nabla}_{\varphi e_i} A_N \left(\varphi e_i\right) - A_N \left(\nabla_{\varphi e_i} \varphi e_i\right), e_j \right) e_j \right. \\
&\quad \left. - \overline{\mathbf{g}} \left(\overline{\nabla}_{e_i} A_N e_i - A_N \left(\nabla_{e_i} e_i\right) + \overline{\nabla}_{\varphi e_i} A_N \left(\varphi e_i\right) - A_N \left(\nabla_{\varphi e_i} \varphi e_i\right), \varphi e_j \right) \varphi e_j \right\} \\
&= \sum_{i,j=1}^{m} \left\{ e_i \overline{\mathbf{g}} \left(A_N e_i, e_j\right) - \overline{\mathbf{g}} \left(A_N e_i, \nabla_{e_i} e_j\right) - \overline{\mathbf{g}} \left(A_N \left(\nabla_{e_i} e_i\right), e_j\right) \right. \\
&\quad + \varphi e_i \overline{\mathbf{g}} \left(A_N \varphi e_i, e_j\right) - \overline{\mathbf{g}} \left(A_N \varphi e_i, \nabla_{\varphi e_i} e_j\right) - \overline{\mathbf{g}} \left(A_N \left(\nabla_{\varphi e_i} \varphi e_i\right), e_j\right) \\
&\quad + e_i \overline{\mathbf{g}} \left(A_N e_i, \varphi e_j\right) - \overline{\mathbf{g}} \left(A_N e_i, \nabla_{e_i} \varphi e_j\right) - \overline{\mathbf{g}} \left(A_N \left(\nabla_{e_i} e_i\right), \varphi e_j\right) \\
&\quad \left. + \varphi e_i \overline{\mathbf{g}} \left(A_N \varphi e_i, \varphi e_j\right) - \overline{\mathbf{g}} \left(A_N \varphi e_i, \nabla_{\varphi e_i} \varphi e_j\right) - \overline{\mathbf{g}} \left(A_N \left(\nabla_{\varphi e_i} \varphi e_i\right), \varphi e_j\right) \right\} \\
&= \overline{\mathbf{g}} (N, N) \sum_{i,j=1}^{m} \left\{ e_i b \left(e_i, e_j\right) - b \left(e_i, \nabla_{e_i} e_j\right) - b \left(\nabla_{e_i} e_i, e_j\right) \right. \\
&\quad + \varphi e_i b \left(\varphi e_i, e_j\right) - b \left(\varphi e_i, \nabla_{\varphi e_i} e_j\right) - b \left(\nabla_{\varphi e_i} \varphi e_i, e_j\right) \\
&\quad + e_i b \left(e_i, \varphi e_j\right) - b \left(e_i, \nabla_{e_i} \varphi e_j\right) - \left(\nabla_{e_i} e_i, \varphi e_j\right) \\
&\quad \left. + \varphi e_i b \left(\varphi e_i, \varphi e_j\right) - b \left(\varphi e_i, \nabla_{\varphi e_i} \varphi e_j\right) - b \left(\nabla_{\varphi e_i} \varphi e_i, \varphi e_j\right) \right\}
\end{aligned}
$$

$$
\begin{aligned}
\left(\Delta^{\Upsilon_2} e_1\right)^{\top} = \overline{\mathbf{g}} (N, N) \sum_{i,j=1}^{m} & \left[\left\{ \nabla_{e_i} b \left(e_j, e_i\right) \right\} e_j + \left\{ \nabla_{\varphi e_i} b \left(e_j, \varphi e_i\right) \right\} e_j \right. \\
& \left. + \left\{ \nabla_{e_i} b \left(\varphi e_j, e_i\right) \right\} \varphi e_j + \left\{ \nabla_{\varphi e_i} b \left(\varphi e_j, \varphi e_i\right) \right\} \varphi e_j \right].
\end{aligned}
$$

Using Codazzi-Mainardi equation [46], we get

$$
\begin{aligned}
\sum_{i=1}^{m} & \left\{ \nabla_{e_i} b \left(e_j, e_i\right) + \nabla_{\varphi e_i} b \left(e_j, \varphi e_i\right) \right\} - \left\{ \nabla_{e_j} b \left(e_i, e_i\right) + \nabla_{e_j} b \left(\varphi e_i, \varphi e_i\right) \right\} \\
&= \overline{\mathbf{g}} (N, N) \sum_{i=1}^{m} \left\{ \overline{\mathbf{g}} \left(\overline{R} \left(e_i, e_j\right) e_i, N \right) + \overline{\mathbf{g}} \left(\overline{R} \left(\varphi e_i, e_j\right) \varphi e_i, N \right) \right\} \\
&= -\overline{\mathbf{g}} (N, N) \overline{S} \left(N, e_j\right).
\end{aligned}
$$

Therefore,

$$
\sum_{i=1}^{m} \left\{ \nabla_{e_i} b \left(e_j, e_i\right) + \nabla_{\varphi e_i} b \left(e_j, \varphi e_i\right) \right\} = m \operatorname{grad} H - \overline{\mathbf{g}} (N, N) \overline{S} \left(N, e_j\right).
$$

Hence,

$$
\begin{aligned}
\left(\Delta^{\Upsilon_2} N\right)^{\top} = \overline{\mathbf{g}} (N, N) \sum_{i,j=1}^{m} & \left[\left\{ \nabla_{e_j} \left(b \left(e_i, e_i\right) + b \left(\varphi e_i, \varphi e_i\right) \right) - \overline{\mathbf{g}} (N, N) \overline{S} \left(N, e_j\right) \right\} e_j \right. \\
& \left. + \left\{ \nabla_{\varphi e_j} \left(b \left(e_i, e_i\right) + b \left(\varphi e_i, \varphi e_i\right) \right) - \overline{\mathbf{g}} (N, N) \overline{S} \left(N, \varphi e_j\right) \right\} \varphi e_j \right] \\
= 2\, \overline{\mathbf{g}} (N, N)\, m \operatorname{grad} H - \overline{Q} (N). & \qquad (3.19)
\end{aligned}
$$

For $N = e_m$ or $N = \varphi e_m$, we get

$$\left(\Delta^{\Upsilon_2} e_m\right)^{\top} = 2\epsilon m \,\mathrm{grad} H - \overline{Q}\,(N)\,, \tag{3.20}$$

$$\left(\Delta^{\Upsilon_2} \varphi e_m\right)^{\top} = -\,2\epsilon m \,\mathrm{grad} H - \overline{Q}\,(N)\,. \tag{3.21}$$

Next, we compute the normal part of the curvature term that appears in equation (3.15) as

$$\sum_{i=1}^{m} \epsilon \left\{\mathbf{g}\left(\overline{R}\,(d\Upsilon_2\,(e_i)\,, N)\, d\Upsilon_2\,(e_i) - \overline{R}\,(d\Upsilon_2\,(\varphi e_i)\,, N)\, d\Upsilon_2\,(\varphi e_i)\,, N\right) N\right\} =$$

$$= -\overline{S}\,(N, N)\, N.$$

For $N = e_m$, we have

$$\begin{aligned}\sum_{i=1}^{m-1} \epsilon \mathbf{g}\left(\overline{R}\,(d\Upsilon_2\,(e_i)\,, e_m)\, d\Upsilon_2\,(e_i)\,, e_m\right) e_m \\ + \sum_{i=1}^{m} (-\epsilon)\, \mathbf{g}\left(\overline{R}\,(d\Upsilon_2\,(\varphi e_i)\,, e_m)\, d\Upsilon_2\,(\varphi e_i)\,, e_m\right) e_m \\ = -\overline{S}\,(e_m, e_m)\, e_m - \epsilon e_m.\end{aligned} \tag{3.22}$$

For $N = \varphi e_m$, we have

$$\begin{aligned}\sum_{i=1}^{m} \epsilon\, \mathbf{g}\left(\overline{R}\,(d\Upsilon_2\,(e_i)\,, \varphi e_m)\, d\Upsilon_2\,(e_i)\,, \varphi e_m\right) \varphi e_m \\ + \sum_{i=1}^{m-1} (-\epsilon)\, \mathbf{g}\left(\overline{R}\,(d\Upsilon_2\,(\varphi e_i)\,, \varphi e_m)\, d\Upsilon_2\,(\varphi e_i)\,, \varphi e_m\right) \varphi e_m \\ = -\overline{S}\,(\varphi e_m, \varphi e_m)\, \varphi e_m + \epsilon \varphi e_m.\end{aligned} \tag{3.23}$$

Also, the tangential part of the curvature term appearing in equation (3.15) is

$$\begin{aligned}\sum_{i=1}^{m} \epsilon \left\{\mathbf{g}\left(\overline{R}\,(d\Upsilon_2\,(e_i)\,, N)\, d\Upsilon_2\,(e_i) - \overline{R}\,(d\Upsilon_2\,(\varphi e_i)\,, N)\, d\Upsilon_2\,(\varphi e_i)\,, e_j\right) e_j\right\} \\ + \sum_{i=1}^{m} \epsilon \left\{\mathbf{g}\left(\overline{R}\,(d\Upsilon_2\,(e_i)\,, N)\, d\Upsilon_2\,(e_i) - \overline{R}\,(d\Upsilon_2\,(\varphi e_i)\,, N)\, d\Upsilon_2\,(\varphi e_i)\,, \varphi e_j\right) \varphi e_j\right\} \\ = -\sum_{j=1}^{m} \left\{\overline{S}\,(N, e_j)\, e_j + \overline{S}\,(N, \varphi e_j)\, \varphi e_j\right\}.\end{aligned}$$

For $N = e_m$, we have

$$\sum_{i=1}^{m-1} \epsilon \left\{ \mathbf{\overline{g}} \left(\overline{R} \left(d\Upsilon_2 \left(e_i \right), e_m \right) d\Upsilon_2 \left(e_i \right) - \overline{R} \left(d\Upsilon_2 \left(\varphi e_i \right), e_m \right) d\Upsilon_2 \left(\varphi e_i \right), e_j \right) e_j \right\}$$
$$+ \sum_{i=1}^{m} \epsilon \left\{ \mathbf{\overline{g}} \left(\overline{R} \left(d\Upsilon_2 \left(e_i \right), e_m \right) d\Upsilon_2 \left(e_i \right) - \overline{R} \left(d\Upsilon_2 \left(\varphi e_i \right), e_m \right) d\Upsilon_2 \left(\varphi e_i \right), \varphi e_j \right) \varphi e_j \right\}$$
$$= - \sum_{j=1}^{m} \left\{ \overline{S} \left(e_m, e_j \right) e_j + \overline{S} \left(e_m, \varphi e_j \right) \varphi e_j \right\}. \tag{3.24}$$

For $N = \varphi e_m$, we have

$$\sum_{i=1}^{m} \epsilon \left\{ \mathbf{\overline{g}} \left(\overline{R} \left(d\Upsilon_2 \left(e_i \right), \varphi e_m \right) d\Upsilon_2 \left(e_i \right) - \overline{R} \left(d\Upsilon_2 \left(\varphi e_i \right), \varphi e_m \right) d\Upsilon_2 \left(\varphi e_i \right), e_j \right) e_j \right\}$$
$$+ \sum_{i=1}^{m-1} \epsilon \left\{ \mathbf{\overline{g}} \left(\overline{R} \left(d\Upsilon_2 \left(e_i \right), \varphi e_m \right) d\Upsilon_2 \left(e_i \right) - \overline{R} \left(d\Upsilon_2 \left(\varphi e_i \right), \varphi e_m \right) d\Upsilon_2 \left(\varphi e_i \right), \varphi e_j \right) \varphi e_j \right\}$$
$$= - \sum_{j=1}^{m} \left\{ \overline{S} \left(\varphi e_m, e_j \right) e_j + \overline{S} \left(\varphi e_m, \varphi e_j \right) \varphi e_j \right\}. \tag{3.25}$$

Using equations (3.17)-(3.25) in (3.15), we get

$$\left.\begin{aligned} \left(\tau_2 \left(\Upsilon_2 \right) \right)^{\top} &= - 2m \left\{ 2A_N \left(\mathrm{grad} H \right) + \mathbf{\overline{g}} \left(N, N \right) m \,\mathrm{grad} \left(H^2 \right) - 2H \, \overline{Q} \left(N \right) \right\}, \\ \left(\tau_2 \left(\Upsilon_2 \right) \right)^{\perp} &= - 2m \left\{ \left(\Delta H \right) - H \overline{S} \left(N, N \right) - \mathbf{\overline{g}} \left(N, N \right) H |A_N|^2 + \left(1 - \mathbf{\overline{g}} \left(N, N \right) \right) H \right\} N. \end{aligned}\right\} \tag{3.26}$$

So, we get the desired result.

In view of above theorem, we can give the following results as corollary:

Corollary 2. *Consider a quasi-paraSasakian manifold $\mathbb{Q}^{2m+1}$ with constant mean curvature and its hypersurface $\mathcal{N}$ with spacelike normal vector field N. Then the non-degenerate hypersurface $\mathcal{N}$ is biharmonic if and only if either it is minimal or*

$$\overline{S} \left(N, N \right) = - |A_N|^2 \quad \text{and} \quad \overline{\mathrm{Q}} \left(\mathrm{N} \right) = 0. \tag{3.27}$$

Corollary 3. *Consider a quasi-paraSasakian manifold $\mathbb{Q}^{2m+1}$ with constant mean curvature and its hypersurface $\mathcal{N}$ with timelike normal vector field N. Then the non-degenerate hypersurface $\mathcal{N}$ is biharmonic if and only if it is minimal.*

Example 2. Let $\mathcal{Q}^3 := \mathbb{R}^3 \left(\varphi, \xi, \eta, \mathbf{\overline{g}} \right)$ be a quasi-paraSasakian manifold, here φ, ξ, η and $\mathbf{\overline{g}}$ are as given in Example 1.
Consider a hypersurface $\mathcal{N} = \left\{ (x, y, z) \mid y = f(x) \right\}$ of $\mathcal{Q}^3$ and an isometric

immersion $\Upsilon_2 : \mathcal{N} \longrightarrow \mathcal{Q}^3$ defined by $\Upsilon_2(x, z) = (x, f(x), z)$.

Now, we find an orthonormal frame on $\mathcal{N}$ as

$$\left\{ V_1 = \left(\frac{1}{\sqrt{1 - f'^2}}, \frac{f'}{\sqrt{1 - f'^2}}, 0 \right), \quad V_2 = \xi = (0, 0, 1) \right\}.$$

Also, the unit normal vector field of $\mathcal{N}$ is

$$N = \varphi V_1 = \left(\frac{f'}{\sqrt{1 - f'^2}}, \frac{1}{\sqrt{1 - f'^2}}, 0 \right).$$

By further computation we have the following coefficients of Levi-Civita connection

$$\begin{aligned} &\overline{\nabla}_{e_1} e_1 = 4y e_2,\ \overline{\nabla}_{e_1} e_2 = \overline{\nabla}_{e_2} e_1 = 2y e_1 + \left(-4y^2 + 1\right) e_3,\ \overline{\nabla}_{e_2} e_2 = 0, \\ &\overline{\nabla}_{e_1} e_3 = \overline{\nabla}_{e_3} e_1 = e_2,\ \overline{\nabla}_{e_2} e_3 = \overline{\nabla}_{e_3} e_2 = e_1 - 2y e_3,\ \overline{\nabla}_{e_3} e_3 = 0. \end{aligned} \tag{3.28}$$

Next, the components of the second fundamental form can be computed as

$$\left.\begin{aligned} b(V_1, V_1) &= -\mathbf{g}\left(\overline{\nabla}_{V_1} V_1, N\right) = \frac{f''}{(1 - f'^2)^{\frac{3}{2}}}, \\ b(V_1, V_2) &= -\mathbf{g}\left(\overline{\nabla}_{V_1} V_2, N\right) = 1, \\ b(V_2, V_1) &= -\mathbf{g}\left(\overline{\nabla}_{V_2} V_1, N\right) = 1, \\ b(V_2, V_2) &= -\mathbf{g}\left(\overline{\nabla}_{V_2} V_2, N\right) = 0. \end{aligned}\right\} \tag{3.29}$$

Using equation (3.29), we get the mean curvature of the isometric immersion Υ_2 as

$$H = \frac{f''}{\left(1 - (f')^2\right)^{\frac{3}{2}}} + 2.$$

Then $\mathcal{N}$ becomes minimal and so biharmonic hypersurface of $\mathcal{Q}^3$ with a unit normal vector field φV_1 if and only if f satisfies

$$f'' + 2\left(1 - \left(f'\right)^2\right)^{\frac{3}{2}} = 0.$$

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