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Application of spectral decomposition to establish inequalities for operators

Abstract. We give specific examples of the spectral decomposition of self-adjoint operators in application to establish sharp inequalities for their powers.
Key words: resolution of the identity, spectral decomposition, inequalities for operators, differentiation operator, Laplace operator

Анотація. У цій статті ми наводимо конкретні приклади застосування елементів спектрального розкладу самоспряжених операторів для встановлення точних нерівностей для їх степенів.
Ключові слова: розклад одиниці, спектральний розклад, нерівності для операторів, оператор диференціювання, оператор Лапласа

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1. Introduction

For any natural r let $L^r_{2,2}(\mathbb{R})$, denote the space of all functions whose $x^{(r-1)}$ is locally absolutely continuous and $x^{(r)} \in L_2(\mathbb{R})$. Let also for $\alpha > 0$

$$\beta_{r,\alpha} = \left\{ \left(\frac{r}{\alpha}\right)^{\frac{\alpha}{r}} \left(\frac{r}{r-\alpha}\right)^{\frac{r-\alpha}{r}} \right\}^{\frac{1}{2}}.$$

In 1968, L. Taikov [9] established the sharp inequality for the functions $x \in L^r_{2,2}(\mathbb{R})$ and $0 \le k < r$

$$\left\|x^{(k)}\right\|_\infty \le \left\{2r \sin\pi\frac{2k+1}{2r}\right\}^{-\frac{1}{2}} \beta_{r,k+1/2} \|x\|_2^{\frac{r-k-1/2}{r}} \left\|x^{(r)}\right\|_2^{\frac{k+1/2}{r}}. \qquad (1.1)$$

In 1990, A. Shadrin [8] obtained an analogue of Taikov inequality for the functions $x \in L^r_{2,2}(\mathbb{T})$ given on the unit circle:

$$\left\|x^{(k)}\right\|_\infty \le V_{r,k}\pi^{-1/2}\beta_{r,k+1/2} \|x\|_2^{\frac{r-k-1/2}{r}} \left\|x^{(r)}\right\|_2^{\frac{k+1/2}{r}}, \qquad (1.2)$$

$$V_{r,k}=\sup_{t>0}\left\{\sum_{n=1}^{\infty}\frac{t^{2k+1}n^{2k}}{1+t^{2r}n^{2r}}\right\}^{1/2}.$$

The the analogues of Taikov-Shadrin inequalities for the norms of derivatives in spaces $L_{2,r;\alpha,\beta}((-1;1))$ and $L_{2;e^{-t^2}}(\mathbb{R})$ can be found in [4].

In 2010, V. Babenko and R. Bilichenko [2] generalized above results for arbitrary powers of self-adjoint operator acting in Hilbert space:

Let H be a Hilbert space, A be a linear, unbounded, self-adjoint operator in H, and let $D(A)$ be its domain of definition, f be an arbitrary linear functional, $r,k\in\mathbb{N}, k<r$. For all $x\in D\left(A^r\right)$ and any $\tau>0$ sharp inequality holds

$$\left|(A^k x,f)\right|\leq\left\{\int\limits_{-\infty}^{\infty}\frac{t^{2k}}{1+\tau t^{2r}}\,d\left(E_t f,f\right)\right\}^{\frac{1}{2}}\left\{\|x\|^2+\tau\left\|A^r x\right\|^2\right\}^{\frac{1}{2}}. \tag{1.3}$$

E_t is the resolution of the identity corresponding to the operator A. This will be written later.

The multiplicative inequality is proposed in the same paper. If for $\alpha,z>0$ put

$$V_{r,k,\alpha}(f,z)=z^{\alpha}\left\{\int\limits_{-\infty}^{\infty}\frac{t^{2k}}{1+z^{2r}t^{2r}}\,d\left(E_t f,f\right)\right\}^{\frac{1}{2}},\quad V_{r,k,\alpha}(f)=\sup_{z>0}V_{r,k,\alpha}(f,z)$$

then the following sharp inequality in multiplicative form holds

$$\left|(A^k x,f)\right|\leq V_{r,k,\alpha}(f)\beta_{r,\alpha}\left\|x\right\|^{\frac{r-\alpha}{r}}\left\|A^r x\right\|^{\frac{\alpha}{r}}, \tag{1.4}$$

where $V_{r,k,\alpha}(f)$ converges.

In 2011, V. Babenko and R. Bilichenko [3] obtained results for arbitrary powers of normal operator acting in Hilbert space.

2. Information on the spectral theory of operators

The following supporting information can be found in more detail, for example, in [1], [6].

A resolution of the identity is a one-parameter family of projection operators E_t defined on a finite or an infinite segment $[\alpha,\beta]$ and satisfying the following conditions:

1. $E_u E_v=E_s\ \forall u,v\in[\alpha,\beta],\ \ s=\min\{u,v\}$;

2. in the sence of stronge convergence, one has $E_{t-0}=E_t$;

3. $E_\alpha=0, E_\beta=I$.

According to the spectral theorem, every self-adjoint operator A is associated with resolution of the identity E_t, $t \in \mathbb{R}$. The equality holds for all $x \in D(A)$

$$Ax = \int\limits_{-\infty}^{\infty} t\, dE_t x,$$

where the integral is an operator Stieltjes integral. And

$$\|Ax\|^2 = \int\limits_{-\infty}^{\infty} t^2\, d\,(E_t x, x) < \infty.$$

For $x \in D(A^k)$, $k \in \mathbb{N}$,

$$A^k x = \int\limits_{-\infty}^{\infty} t^k\, dE_t x \tag{2.1}$$

and

$$\left\|A^k x\right\|^2 = \int\limits_{-\infty}^{\infty} t^{2k}\, d\,(E_t x, x)\,.$$

Using the spectral decomposition (2.1) for $x \in D\,(A^r)$ and we will have the functional f

$$\left(A^k x, f\right) = \left(\int\limits_{-\infty}^{\infty} t^k\, dE_t x, f\right) = \int\limits_{-\infty}^{\infty} t^k\, d\,(E_t x, f)\,.$$

The purpose of our publication is to find out the resolution of the identity for specific examples of self-adjoint operators. In order to obtain new inequalities for operators based on the above results.

3. Spectral decomposition and inequalities for degrees of specific self-adjoint operators

Differentiation operator in $L^r_{2,2}(\mathbb{R})$. As the operator present in inequalities (1.3) and (1.4), we consider the operator $Ax(t) = ix'(t)$. For $s < t$ the resolution of the identity corresponding to the operator A can be represented in the form (see [1]):

$$(E_t - E_s)x(u) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{e^{it(z-u)} - e^{is(z-u)}}{i(z-u)} x(z)\, dz.$$

We choose for all $\varepsilon > 0$ $f_\varepsilon(t) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{t^2}{2\varepsilon}}$ as the functional in inequality (1.4). The following estimate is true:

$$V_{r,k,k+1/2}(f_\varepsilon, z) = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{u^{2k}}{1+u^{2r}} e^{-\varepsilon\left(\frac{u}{z}\right)^2} du \right)^{\frac{1}{2}} \leq \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{u^{2k}}{1+u^{2r}} du \right)^{\frac{1}{2}}.$$

Substituting this into inequality (1.4), we get

$$\left| \int_{-\infty}^{\infty} x^{(k)}(t) f_\varepsilon(t) dt \right| \leq$$

$$\leq \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{u^{2k}}{1+u^{2r}} du \right)^{\frac{1}{2}} \beta_{r,k+1/2} \left\| x \right\|_{L_2(\mathbb{R})}^{\frac{r-k-1/2}{r}} \left\| x^{(r)} \right\|_{L_2(\mathbb{R})}^{\frac{k+1/2}{r}}. \tag{3.1}$$

Given that $\int\limits_{-\infty}^{\infty} x^{(k)(t)} f(t) dt \to x^{(k)}(0)$, inequality (3.1) will lead to inequality (1.1).

Differentiation operator in $L^r_{2,2}(\mathbb{T})$. Now consider the differentiation operator $Ax(t) = ix'(t)$ acting in $L_2(\mathbb{T})$. The corresponding resolution of the identity is determined by equality:

$$(E_t x)(u) = \sum_{\substack{n \in \mathbb{Z} \\ n < t}} \hat{x}(n) e^{int}.$$

$\hat{x}(n)$ is the n-th Fourier coefficient for x:

$$\hat{x}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) e^{-int} dt.$$

We choose for $m \in \mathbb{N}$ $f_m(t) = \sum\limits_{n=-m}^{m} e^{int}$, that is, in fact, the Dirichlet kernel $D_m(t)$. Equality holds for such a functional:

$$\frac{1}{2\pi} \int_{0}^{2\pi} f_m(t) x(t) dt = S_m(x, 0),$$

in which $S_m(x, 0)$ is the n-th partial sum of the Fourier series of the function x. So

$$(A^k x, f_m) = x^{(k)}(0). \tag{3.2}$$

In addition

$$V_{r,k,k+1/2}(f_m, z) = \left\{ \frac{1}{2\pi} \sum_{n=-m}^{m} \frac{z^{2k+1} n^{2k}}{1 + z^{2r} n^{2r}} \right\}^{1/2} \leq \pi^{-\frac{1}{2}} V_{r,k}. \tag{3.3}$$

Substituting (3.2), (3.3) into inequality (1.4) written for the functional f_m, we obtain

$$S_m(x^{(k)},0) \leq \pi^{-\frac{1}{2}} V_{r,k}\beta_{r,k+1/2}\|x\|_{L_2(\mathbb{T})}^{\frac{r-k-1/2}{r}} \|x^{(r)}\|_{L_2(\mathbb{T})}^{\frac{k+1/2}{r}}.$$

As $m \to \infty$, since $S_m(x^{(k)},0) \to x^{(k)}(0)$, we get an inequality (1.2).

Laplace operator in $L_2(\mathbb{R}^{>})$. Let us also consider the Laplace operator: $\Delta \in L_2(\mathbb{R}^{>}) to \mathbb{R}$: for $x = (x_1, x_2, ..., x_m) \in \mathbb{R}^{>}$

$$\Delta x = \frac{\partial^2 x}{\partial x_1^2} + ... + \frac{\partial^2 x}{\partial x_m^2} \;\; (m \in \mathbb{N}).$$

It is known (see,e.g., [7]) the resolution of the identity corresponding to this operator is characterized by the relation:

$$(E_t x)(u) = \frac{t^{m/4}}{(2\pi)^{m/2}} \int\limits_{\mathbb{R}^m} \frac{J_{m/2}(\sqrt{t}|u-z|)}{|u-z|^{m/2}} x(z) dz, \tag{3.4}$$

where $J_{m/2}$ – Bessel function (see, e.g., [5], [10]):

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu}, \tag{3.5}$$

(the series on the right side is absolutely convergent).

For $\varepsilon > 0$ define a functional $f_\varepsilon(t) = \frac{1}{(2\pi\varepsilon)^{m/2}} e^{-\frac{|z|^2}{2\varepsilon}}, \quad z \in \mathbb{R}^m$. Given the (3.4), equality

$$\frac{d}{dt}\{t^\nu J_\nu(t)\} = t^\nu J_{\nu-1}(t), \;\; t \in \mathbb{R}$$

estimates for the Bessel function, one can show that for all $z > 0$

$$V_{r,k,k+\frac{m+1}{4}}(f_\varepsilon, z) \leq \left(\int\limits_{-\infty}^{\infty} \frac{u^{2k+(m-1)/2}}{1+u^{2r}} du \right)^{\frac{1}{2}}.$$

By substituting this estimates into inequality (1.4) written for the functional f_ε, we obtain for $x \in L_{2,2}^r(\mathbb{R}^m)$ and $\varepsilon > 0$

$$\left| \int\limits_{-\infty}^{\infty} \Delta^k x(t) f_\varepsilon(t) dt \right| \leq$$

$$\leq V_{r,k,k+\frac{m+1}{4}} \beta_{r,k+(m+1)/4} \|x\|_{L_2(\mathbb{R}^m)}^{\frac{r-k-(m+1)/4}{r}} \|\Delta^r x\|_{L_2(\mathbb{R}^m)}^{\frac{k+(m+1)/4}{r}}. \tag{3.6}$$

There $\Delta^n x(t) = \Delta\left(\Delta^{n-1} x(t)\right)$ for $x \in L_{2,2}^r(\mathbb{R}^m)$.

Given that

$$\int_{-\infty}^{\infty} \Delta^k x(t) f_\varepsilon(t) dt \to \Delta^k x(0),$$

as $\varepsilon \to 0$, from (3.6) we get inequality for estimating the norm of the Laplace operator

$$\left\|\Delta^k x\right\|_\infty \leq V_{r,k,k+\frac{m+1}{4}} \beta_{r,k+(m+1)/4} \left\|x\right\|_{L_2(\mathbb{R}^m)}^{\frac{r-k-(m+1)/4}{r}} \left\|\Delta^r x\right\|_{L_2(\mathbb{R}^m)}^{\frac{k+(m+1)/4}{r}}.$$

4. Conclusion

In the present paper, we consider the main examples of spectral decompositions for various examples of self-adjoint operators. On the basis of previously obtained inequalities for intermediate degrees of operators, we established already proven and new inequalities for specific operators acting in Hilbert space.

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