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On the structure of some nilpotent braces [1]

Abstract. We prove a criteria for nilpotency of left braces in terms of the $\star$-central series and also discuss Noetherian braces, obtaining some of their elementary properties. We also show that if a finitely generated brace A is Smoktunowicz-nilpotent, then the additive and multiplicative groups of A are likewise finitely generated.

Key words: brace, nilpotent brace, soluble brace, Noetherian brace

Анотація. Доведено критерій нільпотентності лівих брейсів у термінах $\star$-центрального ряду, а також досліджено нетерові брейси, отримано деякі з їхніх елементарних властивостей. Також показано, що якщо скінченно породжений брейс A є нільпотентним у сенсі Смоктунович, то адитивна та мультиплікативна групи A також є скінченно породженими.

Ключові слова: брейс, нільпотентний брейс, розв'язний брейс, нетеровий брейс

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1. Introduction.

A *left brace* is a set A together with two binary operations, addition denoted by $+$ and multiplication denoted by $\cdot$ (which is often omitted), satisfying the following conditions:

LB1 A is an abelian group under addition;

LB2 A is a group under multiplication;

LB3 $a(b+c) = ab + ac - a$ for all $a, b, c \in A$.

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It is easy to see from this definition that the additive and multiplicative identities of A coincide and we denote this common identity by 0. We shall denote the additive group of A by $(A, +)$ and the multiplicative group of A by $(A, \cdot)$.

Braces were first introduced by W. Rump in [8] as a generalization of Jacobson radical rings in order to help study involutive set-theoretic solutions of the Yang-Baxter equation. Since then braces have been solely studied for their algebraic properties and have been linked with other research areas. The reader is referred to [2] for some elementary properties.

If A is a left brace, then a subset S of A is called a *(left) subbrace* if S is closed under addition and multiplication and is itself a left brace by restriction of these operations to S. Thus S is a subbrace of A if and only if $(S, +)$ is a subgroup of $(A, +)$ and $(S, \cdot)$ is a subgroup of $(A, \cdot)$.

For the left brace A and for each element $a \in A$ we define a map $\lambda_a : A \longrightarrow A$ by $\lambda_a(x) = ax - a$ for all $x \in A$. Furthermore we define $a \star b = ab - a - b$ for all $a, b \in A$. It is easy to see that $a \star b = \lambda_a(b) - b$. The operation $\star$ plays a very important role in the study of left braces.

A left brace A is called *trivial* or *abelian* if $a \star b = 0$ for all $a, b \in A$, or equivalently, $a + b = ab$. In this case the groups $(A, +)$ and $(A, \cdot)$ coincide.

As with other algebraic structures it is possible to study the concept of nilpotency. However, in the theory of braces there are several different approaches to this concept and we refer the reader to the papers [2, 3, 4, 7, 10]. Our approach to nilpotency is based on the following concept.

Let A be a left brace. The set

$$\begin{aligned}\zeta(\star, A) &= \{a \in A | a \star x = x \star a = 0 \text{ for all } x \in A\} \\ &= \{a \in A | ax = a + x = xa \text{ for all } x \in A\} \\ &= \{a \in A | \lambda_a(x) = x \text{ and } \lambda_x(a) = a \text{ for all } x \in A\}\end{aligned}$$

is called the *$\star$-center* of A. We show in Proposition 2 that $\zeta(\star, A)$ is an ideal of A which clearly contains the center, $\zeta(A)$, of $(A, \cdot)$.

By analogy with groups we construct the *upper $\star$-central series*

$$0 = \zeta_0(\star, A) \leq \zeta_1(\star, A) \leq \dots \zeta_\alpha(\star, A) \leq \zeta_{\alpha+1}(\star, A) \leq \dots \zeta_\gamma(\star, A)$$

as follows. We let

$$\begin{gathered}\zeta_1(\star, A) = \zeta(\star, A) \text{ and} \\ \zeta_{\alpha+1}(\star, A)/\zeta_\alpha(\star, A) = \zeta(\star, A/\zeta_\alpha(\star, A))\end{gathered}$$

for all ordinals α; as usual for limit ordinals λ we set $\zeta_\lambda(\star, A) = \cup_{\mu<\lambda}\zeta_\mu(\star, A)$.

The definition implies that each term of this series is an ideal of A. The last term $\zeta_\infty(\star, A) = \zeta_\gamma(\star, A)$ of this series is called the *upper $\star$-hypercenter* of A and we denote the length of the upper $\star$-central series of A by $\mathrm{zl}(A)$. Furthermore, if $A = \zeta_\infty(\star, A)$, then A is called a *$\star$-hypercentral* brace.

When A is a left brace and B, C are ideals of A with $B \leq C$, then the factor C/B is called *$\star$-central* if $A \star C, C \star A \leq B$. Thus for all $a \in A, c \in C$ we have $ac - a - c, ca - a - c \in B$ in this case.

Let

$$0 = C_0 \leq C_1 \leq \dots C_\alpha \leq C_{\alpha+1} \leq \dots C_\gamma$$

be an ascending series of ideals of the brace A. This series is called *$\star$-central* if every factor of this series is $\star$-central; in other words $A \star C_{\alpha+1}, C_{\alpha+1} \star A \leq C_\alpha$ for all $\alpha < \gamma$. It is clear that the upper $\star$-central series is $\star$-central.

If A is a left brace and K, L are subbraces, then we let $K \star L$ denote the subgroup of $(A, +)$ generated by the elements $x \star y$, where $x \in K, y \in L$.

This allows us to define two further canonical series of the left brace A. First, let $A^{(1)} = A$ and recursively define $A^{(\alpha+1)} = A^{(\alpha)} \star A$ for all ordinals α and as usual set $A^{(\lambda)} = \cap_{\mu<\lambda} A^{(\mu)}$ for all limit ordinals λ. As in [3] A is called *right nilpotent* if $A^{(n)} = 0$ for some natural number n.

Similarly, let $A^1 = A$, $A^{\alpha+1} = A \star A^\alpha$, for all ordinals α and $A^\lambda = \cap_{\mu<\lambda} A^\mu$ for all limit ordinals μ. As in [3] A is *left nilpotent* if $A^m = 0$ for some natural number m. We note that a left nilpotent brace need not be right nilpotent and vice-versa (see [1] and [2], for example). Indeed Rump [9] gave an example of a left brace of cardinality 6 such that $A^{(3)} = 0$, but $A^m \neq 0$ for all m.

We shall say that A is *Smoktunowicz-nilpotent or S-nilpotent* if there are natural numbers n, k such that $A^{(n)} = A^k$. Such braces were introduced by Smoktunowicz in the paper [10].

The first main result of our paper is the following which we prove in Section 2 along with a series of general results leading to its proof.

Theorem A. Let A be a left brace. Then A has a finite $\star$-central series if and only if A is Smoktunowicz-nilpotent.

In Section 3 we discuss finitely generated braces which we define as follows. Let A be a left brace and let $\mathcal{S}$ be a family of subbraces of A. Then $\cap\{S : S \in \mathcal{S}\}$ is a subbrace of A. Furthermore, if M is a subset of A and $\mathcal{M}$ is the family of all subbraces of A containing M, then $\cap\{M | M \in \mathcal{M}\}$ is the smallest subbrace of A containing M which we call the *subbrace of A generated by M*, denoted by $\langle M \rangle$.

A subbrace S of a brace A is called *finitely generated* if there is a finite subset M of S such that $S = \langle M \rangle$. Evidently, if either of $(A, +)$ or $(A, \cdot)$ is finitely generated, then A is a finitely generated brace. The converse of this fails (our thanks to Agata Smoktunowicz for pointing this out): if A is a finitely generated infinite nil algebra over a finite field, then A can be made into a left brace by defining the brace operation $\cdot$ as $a \cdot b = ab + a + b$ (where ab is the multiplication operation in the ring A). Of course such an A exists by the result of Golod (see [6] or [5]).

Our second main result, which we prove in Section 3, is connected to this concept and we also obtain results concerning Noetherian braces which we define later.

Theorem B. Let A be a left brace and suppose that A is Smoktunowicz-nilpotent. If A is finitely generated, then $(A, +)$ and $(A, \cdot)$ are finitely generated.

2. Smoktunowicz-nilpotency

We recall that a subbrace L of a left brace A is an ideal if $a \star z, z \star a \in L$ for all $a \in A, z \in L$. We prove the following criteria for a subset of A to be an ideal, which is well-known (indeed some authors, see [2] for example, use it as the definition of ideal).

Proposition 1. Let A be a left brace and let L be an ideal of A. Then the following conditions hold:

I1 If $x, y \in L$, then $xy^{-1} \in L$;

I2 if $a \in A$ and $z \in L$, then $a^{-1}za \in L$;

I3 if $a \in A$ and $z \in L$, then $\lambda_a(z) \in L$.

Conversely, if L is a non-empty subset of A satisfying (I1)-(I3), then L is an ideal of A.

Proof. Let L be an ideal of A. Then L is a subbrace and hence $(L, \cdot)$ is a subgroup of $(A, \cdot)$ so that L satisfies (I1) trivially. Next, if $a \in A, z \in L$, then $\lambda_a(z) = az - a - z + z = a \star z + z \in L$. Hence L satisfies condition (I3). Furthermore,

$$\begin{aligned} aza^{-1} &= a(za^{-1} - z - a^{-1} + z + a^{-1}) = a(z \star a^{-1} + z + a^{-1}) \\ &= a(z \star a^{-1}) + az + aa^{-1} - 2a \\ &= a(z \star a^{-1}) + az - a - z + z - a = a(z \star a^{-1}) + a \star z + z - a, \end{aligned}$$

whereas

$$\begin{aligned} & a \star (z \star a^{-1}) + z \star a^{-1} + a \star z + z \\ &= a(z \star a^{-1}) - z \star a^{-1} - a + z \star a^{-1} + a \star z + z \\ &= a(z \star a^{-1}) - a + a \star z + z. \end{aligned}$$

Comparing these we see that $aza^{-1} = a \star (z \star a^{-1}) + z \star a^{-1} + a \star z + z$, which is an element of L. It follows that L satisfies condition (I2).

Conversely, let L satisfy conditions (I1)-(I3). Conditions (I1) and (I2) together show that $(L, \cdot)$ is a normal subgroup of $(A, \cdot)$. Let $x, y \in L$. Then $x - y = y(y^{-1}x) - y = \lambda_y(y^{-1}x) \in L$, using (I3) and (I1). Hence $(L, +)$ is a subgroup of $(A, +)$ so that L is a subbrace of A. Moreover, (I3) shows that $\lambda_a(z) = az - a \in L$ for all $a \in A, z \in L$. Thus

$$a \star z = az - a - z = \lambda_a(z) - z \in L.$$

Furthermore,

$$z \star a = za - z - a = (aa^{-1}za - a) - z = \lambda_a(a^{-1}za) - z.$$

By condition (I2) we have $a^{-1}za \in L$ so that $\lambda_a(a^{-1}za) \in L$. It follows that $z \star a \in L$ as required.

We shall also need the following properties of $\star$ and λ_a whose proofs can be found in [2] or [7].

Lemma 1. *Let A be a left brace. Then*

(i) $a \star (b + c) = a \star b + a \star c$;

(ii) $(ab) \star c = a \star (b \star c) + b \star c + a \star c$;

(iii) $(a + b) \star c = a \star (\lambda_{a^{-1}}(b) \star c) + (\lambda_{a^{-1}}(b) \star c) + a \star c$;

(iv) $\lambda_y(b \star a) = yby^{-1} \star \lambda_y(a)$;

(v) $yby^{-1} = \lambda_y(\lambda_b(y^{-1}) - y^{-1} + b) = \lambda_y(b \star y^{-1} + b)$

for all elements $a, b, c, y \in A$.

Proposition 2. Let A be a left brace. The $\star$-center of A is an ideal of A.

Proof. We use Proposition 1 Let $a \in \zeta(\star, A)$ and let $x \in A$. Then $a \in \zeta(A)$ so $a^{-1} \in \zeta(A)$ and $a^{-1}x = xa^{-1}$. Furthermore, we have

$$x = a^{-1}(ax) = a^{-1}(a + x) = a^{-1}a + a^{-1}x - a^{-1} = a^{-1}x - a^{-1}.$$

It follows that $xa^{-1} = a^{-1}x = a^{-1} + x$ and hence $a^{-1} \in \zeta(\star, A)$.

Let also $b \in \zeta(\star, A)$. Then $ab \in \zeta(A)$ so

$$(ab)x = x(ab) = x(a + b) = xa + xb - x = x + a + b = x + ab$$

so that $ab \in \zeta(\star, A)$. Thus $\zeta(\star, A)$ satisfies condition (I1) of Proposition 1.

For each $a \in \zeta(\star, A)$ and $x \in A$ we have $\lambda_x(a) = a$ so that $\zeta(\star, A)$ satisfies condition (I3) of Proposition 1.

Finally $a \in \zeta(A)$ so $x^{-1}ax = a \in \zeta(\star, A)$.

Next we give an important property of the upper $\star$-hypercenter, which is analogous to the well-known result in group theory.

Proposition 3. Let A be a left brace and suppose that K is a non-trivial ideal of A such that $K \leq \zeta_\infty(\star, A)$. Then $K \cap \zeta(\star, A)$ is non-trivial.

Proof. Let

$$0=\zeta_0(\star,A)\le\zeta_1(\star,A)\le\dots\zeta_\alpha(\star,A)\le\dots\zeta_\gamma(\star,A)=\zeta_\infty(\star,A)$$

be the upper $\star$-central series of A. Since $K\le\zeta_\infty(\star,A)$, there is an ordinal α such that $K\cap\zeta_\alpha(\star,A)\ne 0$ and we let β be the least such ordinal with this property. Clearly β is not a limit ordinal and the definition of β gives $K\cap\zeta_{\beta-1}(\star,A)=0$. Since $S=K\cap\zeta_\beta(\star,A)\le\zeta_\beta(\star,A)$ it follows that $S\star A, A\star S\le\zeta_{\beta-1}(\star,A)$. On the other hand, K is an ideal of A so $S\star A, A\star S\le K$. It follows that $S\star A, A\star S\le\zeta_{\beta-1}(\star,A)\cap K=0$. Thus $S\le\zeta(\star,A)$ giving the required result.

If A is a left brace, then a subbrace L of A is called a *left ideal* of A if $a\star b\in L$ for all elements $a\in A$ and $b\in L$. We note the following results whose proofs can be found in [2].

Proposition 4. Let A be a left brace. If L is a left ideal of A, then L satisfies the following conditions':

(LI1) If $x,y\in L$, then $x-y\in L$;

(LI2) if $a\in A$ and $z\in L$, then $\lambda_a(z)\in L$.

Conversely, if L is a non-empty subset of A satisfying the conditions (LI1) and (LI2), then L is a left ideal of A.

Proposition 5. Let A be a left brace and let L be a left ideal of A. Then $L\star A$ and $A\star L$ are left ideals of A. Moreover, if L is an ideal of A, then $L\star A$ is an ideal of A.

Evidently $A\star L$ need not be an ideal of A in general, but in some situations it is an ideal as we observe next.

Let A be a left brace. The set

$$\begin{aligned}\mathrm{Soc}(A)&=\{a\in A|a\star x=0\text{ for all }x\in A\}\\&=\{a\in A|ax=a+x\text{ for all }x\in A\}\end{aligned}$$

is called the *socle* of A. In [2] it is shown that $\mathrm{Soc}(A)$ is an ideal of A, a result attributable to Rump. Clearly $\mathrm{Soc}(A)$ contains $\zeta(\star,A)$.

Proposition 6. Let A be a left brace and let L be an ideal of A. If $\mathrm{Soc}(A)$ contains L, then $A\star L$ is an ideal of A.

Proof. Let $x\in L, a,b\in A$. Lemma 1 shows that $\lambda_a(b\star x)=aba^{-1}\star\lambda_a(x)$ and since L is an ideal of A, $\lambda_a(x)\in L$. It follows that $\lambda_a(b\star x)\in A\star L$. Hence for all $y\in A\star L$ we have $\lambda_a(y)\in A\star L$.

Since $a \star x \in L$ and $L \leq \mathrm{Soc}(A)$ we have $(a \star x)b = (a \star x) + b$ and we now have

$$b^{-1}(a \star x)b = b^{-1}((a \star x) + b) = b^{-1}(a \star x) + b^{-1}b - b^{-1}$$
$$= b^{-1}(a \star x) - b^{-1} = \lambda_{b^{-1}}(a \star x).$$

From what we proved above, we have $\lambda_{b^{-1}}(a \star x) \in A \star L$ and hence $b^{-1}(a \star x)b \in A \star L$.

Suppose that $u, v \in A \star L$. Since $uv^{-1} = -\lambda_{uv^{-1}}(v) + u$ and $\lambda_{uv^{-1}}(v) \in A \star L$ we have $uv^{-1} \in A \star L$. Hence $A \star L$ is a subgroup of the multiplicative group of A. Also, $v^{-1}u \in A \star L$ and $u - v = v(v^{-1}u) - v = \lambda_v(v^{-1}u \in A \star L$. The result follows from Proposition 1.

It is shown in [2] and indeed this follows from Proposition 5 that A^n is always a left ideal of A and $A^{(n)}$ is an ideal of A. Using Proposition 5 we may generalize this to:

Proposition 7. Let A be a left brace. Then A^α is a left ideal for each ordinal α and $A^{(\alpha)}$ is an ideal for each ordinal α.

Next we give the relationship between $A^{(j)}, A^j$ and $\star$-central series.

Proposition 8. Let A be a left brace and let

$$0 = C_0 \leq C_1 \leq \cdots \leq C_n = A$$

be a finite $\star$-central series of A. Then

(i) $A^{(j)}, A^j \leq C_{n-j+1}$ and hence $A^{(n+1)} = 0 = A^{n+1}$;

(ii) $C_j \leq \zeta_j(\star, A)$ and hence $\zeta_n(\star, A) = A$.

Proof. (i) We use induction on j. For $j = 2$ we have $A^2 = A^{(2)} = A \star A = C_n \star A \leq C_{n-1}$ so the result holds for $j = 2$.

Suppose now that $j > 2$ and that we have already proved $A^{(m)}, A^m \leq C_{n-m+1}$ for all $m < j$. Then $A^{(j)} = A^{(j-1)} \star A \leq C_{n-j+1+1} \star A \leq C_{n-j+1}$ and similarly $A^j = A \star A^{j-1} \leq A \star C_{n-j+1+1} \leq C_{n-j+1}$.

(ii) We again use induction on j. For $j = 1$ we have $C_1 \star A = 0 = A \star C_1$ and hence $C_1 \leq \zeta_1(\star, A)$. Suppose that $j > 2$ and we have already proved that $C_m \leq \zeta_m(\star, A)$ for all $m < j$. Since C_j/C_{j-1} is $\star$-central we have $C_j \star A, A \star C_j \leq C_{j-1} \leq \zeta_{j-1}(\star, A)$, by the induction hypothesis. Let $a \in A, c \in C_j$. Then $ca - c - a = c \star a \in \zeta_{j-1}(\star, A)$ so that

$$\zeta_{j-1}(\star, A) = (ca - c - a + \zeta_{j-1}(\star, A)) \quad = (c + \zeta_{j-1}(\star, A)) \star (a + \zeta_{j-1}(\star, A))$$

In a similar way we also obtain $(a + \zeta_{j-1}(\star, A)) \star (c + \zeta_{j-1}(\star, A)) = \zeta_{j-1}(\star, A)$. This means that

$$(C_j + \zeta_{j-1}(\star, A))/\zeta_{j-1}(\star, A) \leq \zeta(\star, A/\zeta_{j-1}(A)) \leq \zeta_j(\star, A)/\zeta_{j-1}(\star, A)$$

so that $C_j \leq \zeta_j(\star, A)$ as required.

Proposition 9. Let A be a Smoktunowicz-nilpotent left brace. Then A has a finite $\star$-central series.

Proof. Let n and k denote the least natural numbers such that $A^{(n)} = 0 = A^k$. We use induction on n. If $n = 2$, then A is abelian and the result clearly holds, so suppose that $n > 2$. Let $S = A^{(n-1)}$. Then S is a non-zero ideal of A such that $S \star A = 0$ and it follows that $S \leq \mathrm{Soc}(A)$. Let $S_1 = S$ and $S_{j+1} = A \star S_j$ for $j \geq 2$. Proposition 6 shows that S_j is an ideal for all j. Since $S_j \leq A^j$, it follows that $S_k = 0$ and we therefore obtain a finite series of ideals

$$0 = S_k \leq S_{k-1} \leq \cdots \leq S_2 \leq S_1 = S. \tag{2.1}$$

Then $A \star S_j = S_{j+1}$ by definition and $S_j \star A = 0 \leq S_{j+1}$, since $S \leq \mathrm{Soc}(A)$ which shows that every factor of the series (2.1) is $\star$-central. The equality $(a + S) \star (b + S) = a \star b + S$ and the fact that $S \leq A^{(j)}$ for $j \leq n - 1$ show, using a simple induction, that

$$(A/S)^{(j)} = A^{(j)}/S, \text{ for all } j \leq n - 1.$$

Similarly, $(A/S)^j = (A^j + S)/S$ for all natural numbers j. Since $S = A^{(n-1)}$ we have $(A/S)^{(n-1)} = 0$. Also $(A/S)^k = 0$ and we may apply the induction hypothesis. Thus the factor brace A/S has a finite $\star$-central series of ideals

$$0 = Z_0/S \leq Z_1/S \leq \cdots \leq Z_t/S = A/S.$$

Then

$$0 = S_k \leq S_{k-1} \leq \cdots \leq S_2 \leq S_1 = S = Z_0 \leq Z_1 \leq \cdots \leq Z_t = A$$

is a finite $\star$-central series of ideals of the brace A, completing the proof.

Using Propositions 8 and 9 we have:

Theorem A. Let A be a left brace. Then A has a finite $\star$-central series if and only if A is Smoktunowicz-nilpotent.

We may easily establish the following natural corollary.

Corollary 1. *Let A be a Smoktunowicz-nilpotent left brace.*

(i) If S is a subbrace of A, then S is Smoktunowicz-nilpotent and $zl(S) \leq zl(A)$;

(ii) if L is an ideal of A, then A/L is Smoktunowicz-nilpotent and $zl(A/L) \leq zl(A)$.

Proof. (i) Let

$$0 = Z_0 \leq Z_1 \leq Z_2 \leq \cdots \leq Z_{n-1} \leq Z_n = A$$

be the upper $\star$-central series of A and set $C_j = Z_j \cap S$ for $0 \leq j \leq n$. Then, for each j, C_j is an ideal of S. We have $S \star C_{j+1} \leq S \star Z_{j+1} \leq Z_j$ and $C_{j+1} \star S \leq Z_{j+1} \star S \leq Z_j$ for $0 \leq j \leq n-1$. On the other hand, since S is a subbrace we have $S \star C_{j+1} \leq S$ and $C_{j+1} \star S \leq S$ so that

$$S \star C_{j+1} \leq S \cap Z_j = C_j, C_{j+1} \star S \leq S \cap Z_j = C_j$$

for $0 \leq j \leq n-1$. These inclusions show that the series

$$0 = C_0 \leq C_1 \leq C_2 \leq \cdots \leq C_{n-1} \leq C_n = S$$

is a $\star$-central series of ideals of S. Hence S is Smoktunowicz-nilpotent and Proposition 8 shows that $\mathrm{zl}(S) \leq n = \mathrm{zl}(A)$.

(ii) We note that $Z_j + L$ is an ideal of A and $(Z_j + L)/L$ is an ideal of A/L for $0 \leq j \leq n$. Since $(a + L) \star (b + L) = (a \star b) + L$ we have

$$(A/L) \star (Z_{j+1} + L)/L = (A \star Z_{j+1} + L)/L \leq (Z_j + L)/L,$$
$$(Z_{j+1} + L)/L \star (A/L) = (Z_{j+1} \star A + L)/L \leq (Z_j + L)/L$$

for $0 \leq j \leq n$. In turn, these inclusions show that the series

$$0 = (Z_0 + L)/L \leq (Z_1 + L)/L \leq \cdots \leq (Z_{n-1} + L)/L \leq (Z_n + L)/L = A/L$$

is a $\star$-central series of ideals for A/L. Thus A/L is Smoktunowicz-nilpotent and Proposition 8 shows that $\mathrm{zl}(A/L) \leq n = \mathrm{zl}(A)$ as required.

Let $\mathfrak{N}$ denote the class of left braces which are Smoktunowicz-nilpotent. More precisely, we let $\mathfrak{N}_{n,k}$ denote the class of left braces A such that $A^{(n)} = 0 = A_k$ and n, k are least with these properties.

Let A be a left brace and let $\mathfrak{X}$ be a class of left braces. Let

$$A^{\mathfrak{X}} = \cap\{H | H \text{ is an ideal of } A \text{ and } A/H \in \mathfrak{X}\},$$

an ideal which is called, as usual, the $\mathfrak{X}$-*residual* of A.

There are two cases to consider, when $A/A^{\mathfrak{X}} \in \mathfrak{X}$ and $A/A^{\mathfrak{X}} \notin \mathfrak{X}$.

In the case when $A/A^{\mathfrak{X}} \in \mathfrak{X}$, it follows that $A^{\mathfrak{X}}$ is the smallest ideal of A such that $A/A^{\mathfrak{X}}$ is an $\mathfrak{X}$-brace.

Remak's theorem gives us the following very useful result.

Corollary 2. *Let A be a left brace and let $\mathfrak{X}$ be a class of left braces closed with respect to taking subbraces and Cartesian products. Then $A/A^{\mathfrak{X}} \in \mathfrak{X}$.*

In particular, if $\mathfrak{X} = \mathfrak{A}$ the class of all abelian braces, then the $\mathfrak{A}-$*residual* $A^{\mathfrak{A}}$ is exactly the derived subbrace $A^{(2)}$. In this case we have $A/A^{\mathfrak{A}} \in \mathfrak{A}$.

Proposition 10. Let n, k be fixed natural numbers and let $\{A_\lambda | \lambda \in \Lambda\}$ be a family of left braces such that $A_\lambda \in \mathfrak{N}_{n,k}$ for each $\lambda \in \Lambda$. Then $C = \underset{\lambda \in \Lambda}{\mathrm{Cr}}\, A_\lambda \in \mathfrak{N}_{n,k}$.

Proof. Clearly $C^{(j)} = \underset{\lambda\in\Lambda}{\mathrm{Cr}}\, A_\lambda^{(j)}$ and $C^j = \underset{\lambda\in\Lambda}{\mathrm{Cr}}\, A_\lambda^j$ for all natural numbers j. It follows that $C^{(n)} = 0 = C^k$, so that $C \in \mathfrak{N}_{n,k}$.

Corollary 3. *The class $\mathfrak{N}_{n,k}$ is a variety of braces for all natural numbers n, k.*

3. Finitely generated nilpotent braces

As usual, a brace will be called *Noetherian* if the family $\mathcal{L}(A)$ of all subbraces, ordered by inclusion, satisfies the maximal condition. Thus $\mathcal{L}(A)$ has no infinite ascending chains of subbraces. As with other algebraic structures, the following results are true.

Proposition 11. Let A be a left brace. If D is an ideal of A such that D and A/D are Noetherian, then A is also Noetherian.

Corollary 4. *Let A be a left brace and suppose that A has a finite series of ideals,*

$$0 = A_0 \leq A_1 \leq \cdots \leq A_{n-1} \leq A_n = A,$$

whose factors A_j/A_{j-1} are Noetherian for $1 \leq j \leq n$. Then A is also Noetherian.

Proposition 12. Let A be a left brace. Then A is Noetherian if and only if every subbrace of A is finitely generated. In particular, every subbrace and every factor brace of A is Noetherian.

A brace A is called *soluble* if A has a finite series of ideals,

$$0 = A_0 \leq A_1 \leq \cdots \leq A_{n-1} \leq A_n = A,$$

whose factors A_j/A_{j-1} are abelian for $1 \leq j \leq n$.

Proposition 13. Let A be a soluble left brace. Then A is Noetherian if and only if A has a finite series of ideals

$$0 = A_0 \leq A_1 \leq \cdots \leq A_{n-1} \leq A_n = A,$$

in which the factors A_j/A_{j-1} are abelian and $(A_j/A_{j-1}, +)$ is finitely generated for $1 \leq j \leq n$.

Proof. Suppose first that A is Noetherian. Since A is soluble there is a finite series of ideals

$$0 = A_0 \leq A_1 \leq \cdots \leq A_{n-1} \leq A_n = A,$$

whose factors A_j/A_{j-1} are abelian for $1 \leq j \leq n$. By Proposition 12, each ideal A_j is finitely generated as a subbrace and hence A_j/A_{j-1} is finitely generated as a brace. On the other hand the factor A_j/A_{j-1} is abelian, so its additive and multiplicative groups coincide. It follows that $(A_j/A_{j-1},+)$ is finitely generated for $1 \leq j \leq n$.

Conversely, since A_j/A_{j-1} is abelian, its additive and multiplicative groups coincide, so A_j/A_{j-1} is finitely generated as a brace. Also if B/A_{j-1} is a subbrace of A_j/A_{j-1}, then $(B/A_{j-1},+)$ is also finitely generated. Thus every subbrace of A_j/A_{j-1} is finitely generated. Hence A_j/A_{j-1} is Noetherian as a brace and Corollary 4 implies that A is a Noetherian brace.

Corollary 5. *Let A be a soluble left brace. Then A is Noetherian if and only if $(A,+)$ is finitely generated.*

Proof. Suppose that A is Noetherian as a brace. Since A is soluble there is a finite series of ideals

$$0 = A_0 \leq A_1 \leq \cdots \leq A_{n-1} \leq A_n = A,$$

whose factors A_j/A_{j-1} are abelian for $1 \leq j \leq n$. By Proposition 13, $(A_j/A_{j-1},+)$ is finitely generated for each j. Hence $(A,+)$ is finitely generated.

Conversely, suppose $(A,+)$ is a finitely generated abelian group. Then each of the factors $(A_j/A_{j-1},+)$ is finitely generated. By Proposition 13 the brace A is Noetherian.

Let $\mathfrak{X}, \mathfrak{Y}$ be classes of braces. Then, as usual, we say that A is an $\mathfrak{X}$-by-$\mathfrak{Y}$ brace if A contains an ideal D such that $D \in \mathfrak{X}$ and $A/D \in \mathfrak{Y}$.

Corollary 6. *Let A be a soluble-by-finite left brace. Then A is Noetherian if and only if $(A,+)$ is a finitely generated abelian group.*

Proof. Suppose first that A is Noetherian. Let D be a soluble ideal of A such that A/D is finite. By Proposition 12 D is Noetherian and Corollary 5 implies that $(D,+)$ is finitely generated. Since the additive group of D has finite index in the additive group of A we see that $(A,+)$ is also finitely generated.

Conversely, if $(A,+)$ is a finitely generated abelian group, then $(D,+)$ is also finitely generated and Corollary 5 implies that D is a Noetherian brace. Clearly every finite brace is Noetherian so that Proposition 11 implies that A is a Noetherian brace.

Next we need an easy technical lemma.

Lemma 2. *Let A be a left brace and let c_1, c_2 be elements of $\zeta(\star, A)$. Then $(a_1 + c_1) \star (a_2 + c_2) = a_1 \star a_2$ for all elements a_1, a_2 of A.*

Proof. By the definition of $\zeta(\star, A)$ we have

$$\begin{aligned}(a_1 + c_1) \star (a_2 + c_2) &= a_1c_1 \star a_2c_2 = a_1c_1a_2c_2 - a_1c_1 - a_2c_2\\ &= a_1a_2 + c_1 + c_2 - a_1 - a_2 - c_1 - c_2\\ &= a_1 \star a_2.\end{aligned}$$

We recall (see [2, Definition 2.11]) that if B is an ideal of the left brace A and $a \in A$, then $aB = a + B$ always. Thus every transversal to $(B, +)$ in $(A, +)$ is simultaneously a transversal to $(B, \cdot)$ in $(A, \cdot)$ and conversely. So we may unambiguously talk about a transversal to B in A without specifying the operation.

Corollary 7. *Let A be a left brace and let T be a transversal to $\zeta(\star, A)$ in A. If $A^{(2)} \leq \zeta(\star, A)$, then $A^{(2)}$ is generated as an additive abelian group by the elements $x \star y$, where $x, y \in T$.*

Proof. Let $a, b \in A$. Then there exist $x, y \in T$ and $u, v \in \zeta(\star, A)$ such that $a = x + u$ and $b = y + t$. It follows from Lemma 2 that $a \star b = x \star y$. Since $A^{(2)}$ is generated as an abelian group by the elements $a \star b$, the result follows.

Proposition 14. Let A be a left brace and suppose that the factor brace $A/\zeta(\star, A)$ is abelian and generated by k elements. Then $A^{(2)}$ is generated as an additive abelian group by at most k^2 elements. In particular $A^{(2)}$ is a finitely generated brace.

Proof. Let $a, b, x, y \in A$. First we note that $a = a \cdot 0 = a(y-y) = ay + a(-y) - a$ so that $a(-y) = 2a - ay$. Then Lemma 1 and an easy induction show that $a \star (nx) = n(a \star x)$ for all integers n.

Next we note that $a + b = ab + c_1, bx = b + x + c_2$ for certain elements $c_1, c_2 \in \zeta(\star, A)$. Using Lemma 2 we obtain

$$\begin{aligned}(a+b) \star x = (ab + c_1) \star x = ab \star x = abx - ab - x \\ = a(b + x + c_2) - ab - x = ab + ax + ac_2 - 2a - ab - x \\ = ax + a + c_2 - 2a - x = ax - a - x + c_2 \\ = ax - a - x + bx - b - x = a \star x + b \star x.\end{aligned}$$

In particular, we deduce that $(na) \star x = n(a \star x)$ for all integers n.

Now let $B = \zeta(\star, A)$ and let $x_1 + B, x_2 + B, \dots, x_k + B$ generate A/B as an abelian group. Let $a, b \in A$ and note that $a = n_1x_1 + n_2x_2 + \cdots + n_kx + k + u, b = m_1x_1 + \cdots + m_kx_k + v$ for certain integers n_i, m_i and elements $u, v \in B$.

Our work above and Lemma 2 show that

$$\begin{aligned}a \star b &= (n_1x_1 + \dots x_kn_k + u) \star b \\ &= n_1(x_1 \star b) + \dots n_k(x_k \star b)\end{aligned}$$

Moreover, again using our work above, we see that $x_i \star b = m_1(x_i \star x_1) + \dots m_k(x_i \star x_k)$ so that

$$a \star b = \sum_{i=1}^{k} \sum_{j=1}^{k} n_im_j(x_i \star x_j).$$

Thus as an abelian group $A^{(2)}$ is generated by the elements $x_i \star x_j$ for $i, j = 1, \dots k$ and in particular we see that $A^{(2)}$ is generated by k^2 elements.

Proposition 15. Let A be a left brace and let C, D be ideals of A such that $C \leq D \cap \zeta(\star, A)$ and $D/C \leq \zeta(\star, A/C)$. Let T be a transversal to D in A and let S be a transversal to C in D. Then $A \star D$ is generated as an additive abelian group by the elements $x \star y$ where $x \in T \cup S, y \in S$.

Proof. If $a, b \in A, x, y \in D$, then as in the proof of Proposition 14, $a \star (nx) = n(a \star x)$ for all integers n.

Since $A \star D \leq C$ by the definition of D it follows that $b \star x \in C$. Then $a \star (b \star x) = 0$ since $C \leq \zeta(\star, A)$. Lemma 1 implies that

$$(ab) \star x = a \star (b \star x) + b \star x + a \star x = a \star x + b \star x. \tag{3.1}$$

In particular we have $a^n \star x = n(a \star x)$ for all integers n.

Now, $a \in uD$ for some element $u \in T$, so $a = ud$ for some element $d \in D$. Also, we have $x \in w + C$ for some element $w \in S$ and hence $x = w + c$ for some element $c \in C$. Then, using (3.1) we have,

$$\begin{aligned} a \star x &= (ud) \star x = u \star x + d \star x = u \star (w + c) + d \star (w + c) \\ &= u \star w + d \star w, \end{aligned}$$

by Lemma 2. In turn, $d \in v + C$ for some element $v \in S$ so that $d = v + c_1$ for some element $c_1 \in C$ and Lemma 2 gives $d \star w = v \star w$, which proves the result.

Corollary 8. *Let A be a left brace and let C, D be ideals of A such that $C \leq D \cap \zeta(\star, A)$ and $D/C \leq \zeta(\star, A/C)$. Let M be a subset of A such that $(A/D, \cdot)$ is generated by the set $\{aD | a \in M\}$ and let R be the subset of D such that $(D/C, +)$ is generated by the set $\{u + C | u \in R\}$. Then $A \star D$ is generated as an abelian group by the elements $x \star y$, where $x \in M \cup R$ and $y \in R$.*

Proof. We can choose a transversal T to D in A such that $T \subseteq \langle M \rangle$ and a transversal S to C in D such that $S \subseteq \langle R \rangle$. Proposition 15 shows that $A \star D$ is generated as an additive abelian group by the elements $x \star y$ where $x \in T \cup S, y \in S$. Every element $x \in T$ has the form $x = g_1^{k_1} \cdot \ldots \cdot g_m^{k_m}$, where $g_1, \ldots, g_m \in M$ and $k_1, \ldots, k_m \in \mathbb{Z}$. Using (3.1) of Proposition 15 and induction we have

$$\begin{aligned} x \star y &= g_1^{k_1} \cdot \ldots \cdot g_m^{k_m} \star y = g_1^{k_1} \star y + \cdots + g_m^{k_m} \star y \\ &= k_1(g_1 \star y) + \cdots + k_m(g_m \star y). \end{aligned}$$

Every element $y \in S$ has the form $y = t_1 u_1 + \cdots + t_s u_s$ for certain elements $u_1, \ldots, u_s \in R, t_1, \ldots, t_s \in \mathbb{Z}$. Then, using Lemma 1, we obtain

$$\begin{aligned} g_j \star y &= g_j \star (t_1 u_1 + \cdots + t_s u_s) = g_j \star (t_1 u_1) + \cdots + g_j \star (t_s u_s) \\ &= t_1(g_j \star u_1) + \cdots + t_s(g_j \star u_s). \end{aligned}$$

Let z be a further element of S and suppose $z = r_1v_1 + \cdots + r_kv_k$ for certain $v_1, \ldots, v_k \in R, r_1, \ldots, r_k \in \mathbb{Z}$. Then, as in the proof of Proposition 14, we deduce

$$(r_1v_1 + \cdots + r_kv_k) \star (t_1u_1 + \cdots + t_su_s) = \sum_{1 \le n \le s, 1 \le j \le k} r_jt_n(v_j \star u_n)$$

which proves the result.

The next result is immediate.

Corollary 9. *Let A be a left brace and let C, D be ideals of A such that $C \le D \cap \zeta(\star, A)$ and $D/C \le \zeta(\star, A/C)$. If $(A/D, \cdot)$ is finitely generated and $(D/C, +)$ is finitely generated, then the additive group of $A \star D$ is finitely generated.*

We next prove results analogous to these for $D \star A$.

Proposition 16. Let A be a left brace and let C, D be ideals of A such that $C \le D \cap \zeta(\star, A)$ and $D/C \le \zeta(\star, A/C)$. Let T be a transversal to D in A and let S be a transversal to C in D. Then $D \star A$ is generated as an additive abelian group by the elements $y \star x$, where $y \in S, x \in T \cup S$.

Proof. Let $a \in A$ and $d \in D$ be arbitrary. We have $a \in x + D$ for some element $x \in T$ so that $a = x + u$ for some element $u \in D$. Lemma 1 shows that $d \star a = d \star x + d \star u$. For the element d we have $d \in yC$ for some element $y \in S$ and hence $d = yc$ for some element $c \in C$. Again by Lemma 1 we have, since $c \star x = 0$,

$$d \star x = yc \star x = y \star (c \star x) + y \star x + c \star x = y \star x.$$

Also $u \in vC = v + C$ for some element $v \in S$ we may write $u = v + c_1$ for some element $c_1 \in C$. Applying Lemma 1 and Lemma 2 again we finally obtain

$$d \star u = yc \star u = y \star (c \star u) + y \star u + c \star u = y \star u = y \star (v + c_1) = y \star v = y \star v.$$

Thus $d \star a = y \star x + y \star v$, which gives the result.

Corollary 10. *Let A be a left brace and let C, D be ideals of A such that $C \le D \cap \zeta(\star, A)$ and $D/C \le \zeta(\star, A/C)$. Let M be a subset of A such that $(A/D, +)$ is generated by the set $\{a + D | a \in M\}$ and let R be a subset of D such that $(D/C, \cdot)$ is generated by the set $\{uC | u \in R\}$. Then $D \star A$ is generated as an additive abelian group by the elements $y \star x$, where $y \in R, x \in M \cup R$.*

Proof. We can choose a transversal T to D in A such that $T \subseteq \langle M \rangle$ and a transversal S to C in D such that $S \subseteq \langle R \rangle$. Proposition 16 shows that $D \star A$ is generated as an additive abelian group by the elements $y \star x$, where

$y \in S, x \in T \cup S$. Every element $x \in T$ has the form $x = k_1 g_1 + \cdots + k_m g_m$ where $g_1, \ldots g_m \in M, k_1, \ldots k_m \in \mathbb{Z}$. By Lemma 1 we see that

$$y \star x = y \star (k_1 g_1 + \cdots + k_m g_m) = y \star (k_1 g_1) + \cdots + y \star (k_m g_m)$$
$$= k_1(y \star g_1) + \cdots + k_m(y \star g_m).$$

Every element $y \in S$ has the form $y = u_1^{t_1} \ldots u_s^{t_s}$, where $u_1, \ldots, u_s \in R, t_1, \ldots, t_s \in \mathbb{Z}$. Again using Lemma 1 we obtain

$$y \star g_j = (u_1^{t_1} \ldots u_s^{t_s}) \star g_j = u_1^{t_1} \star g_j + \cdots + u_s^{t_s} \star g_j = t_1(u_1 \star g_j) + \cdots + t_s(u_s \star g_j).$$

Let z be a further element of S. Then $z = v_1^{r_1} \ldots v_k^{r_k}$ where $v_1, \ldots, v_k \in R, r_1, \ldots, r_k \in \mathbb{Z}$. Since the additive and multiplicative groups of D/C coincide we have $z = r_1 v_1 + \ldots r_k v_k + c$ for some element $c \in C$. The arguments of the proof of Proposition 14 show that

$$\begin{aligned} y \star z &= u_1^{t_1} \ldots u_s^{t_s} \star z = t_1(u_1 \star z) + \cdots + t_s(u_s \star z) \\ &= t_1(u_1 \star r_1 v_1 + \cdots + r_k v_k + c) \cdots + t_s(u_s \star r_1 v_1 + \cdots + r_k v_k + c) \\ &= \sum_{1 \le j \le k, 1 \le n \le s} r_j t_n (v_j \star u_n) \end{aligned}$$

which proves the result.

Again, the following result is immediate.

Corollary 11. *Let A be a left brace and let C, D be ideals of A such that $C \le D \cap \zeta(\star, A)$ and $D/C \le \zeta(\star, A/C)$. If $(A/D, +)$ is finitely generated and $(D/C, \cdot)$ is finitely generated, then $D \star A$ is finitely generated as an additive abelian group.*

We now obtain our second main theorem.

Theorem B. Let A be a left brace and let A be Smoktunowicz-nilpotent. If A is finitely generated, then the additive and multiplicative groups of A are finitely generated.

Proof. Let

$$0 = Z_0 \le Z_1 \le Z_2 \le \cdots \le Z_{n-1} \le Z_n = A$$

be the upper $\star$-central series of A. We prove the result by induction $n = \mathrm{zl}(A)$.

If $n = 2$, then $B = \zeta(\star, A)$ contains $A^{(2)}$. Since A is a finitely generated brace A/B is generated as a brace by a finite set $\{a_1 + B \ldots, a_k + B\}$ of cosets. Proposition 14 implies that $A^{(2)}$ is generated as an additive abelian group by the elements $a_j \star a_m$ for $1 \le j, m \le k$. In particular $(A^{(2)}, +)$ is finitely generated. Since $(A/A^{(2)}, +)$ is also finitely generated, we deduce that $(A, +)$ is finitely generated. Since $(A/A^{(2)}, +)$ coincides with $(A/A^{(2)}, \cdot)$ and

$(A^{(2)},+)$ coincides with $(A^{(2)},\cdot)$ it follows that $(A^{(2)},\cdot)$ and $(A/A^{(2)},\cdot)$ are finitely generated. Consequently the multiplicative group of A is finitely generated, so our result holds for $n=2$.

Suppose now that $n>2$. By the natural induction hypothesis we may assume that $(A/Z_1,+)$ and $(A/Z_1,\cdot)$ are finitely generated. It follows that $(A/Z_2,+)$ and $(Z_2/Z_1,+)$ are likewise finitely generated. By definition of Z_2/Z_1, $(Z_2/Z_1,+)$ coincides with $(Z_2/Z_1,\cdot)$ so that $(Z_2/Z_1,\cdot)$ is finitely generated. Using Corollary 11 we see that $Z_2\star A$ is finitely generated as an additive abelian group. Proposition 5 implies that $Z_2\star A$ is an ideal of A so, passing to the factor brace $A/(Z_2\star A)$, we may assume without loss of generality that $Z_2\star A=0$.

This means that we may assume $Z_2\le \mathrm{Soc}(A)$ and Proposition 6 implies that $A\star Z_2$ is an ideal of A. Since $(A/Z_1,\cdot)$ is finitely generated so is $(A/Z_2,\cdot)$. As we noted above, $(Z_2/Z_1,+)$ is finitely generated. Using Corollary 10 we deduce that the additive group of $A\star Z_2$ is finitely generated.

Clearly, the $\star$-center of $A/(A\star Z_2)$ contains $Z_2/(A\star Z_2)$ and in particular we have $\mathrm{zl}(A/(A\star Z_2))<n$. By the induction hypothesis the additive group of $A/(A\star Z_2)$ is finitely generated. Hence $(A,+)$ is finitely generated.

Since every factor of a finitely generated abelian group is finitely generated, $(Z_j/Z_{j-1},+)$ is finitely generated for $1\le j\le n$. We note that $(Z_j/Z_{j-1},+)$ coincides with $(Z_j/Z_{j-1},\cdot)$ for $1\le j\le n$. Hence the multiplicative group of A has a finite series whose factors are finitely generated and it follows that $(A,\cdot)$ is also finitely generated.

Using Corollary 5 we obtain an analogue of the well-known result from group theory:

Corollary 12. *Let A be a left brace and suppose that A is Smoktunowicz-nilpotent. If A is finitely generated, then A is Noetherian.*

In particular we have, using Proposition 12, that

Corollary 13. *Let A be a left brace and suppose that A is Smoktunowicz-nilpotent. If A is finitely generated, then every subbrace of A is finitely generated.*

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