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Description of the automorphism groups of some Leibniz algebras [1]

Abstract. Let L be an algebra over a field F with the binary operations $+$ and $[,]$. Then L is called a left Leibniz algebra if it satisfies the left Leibniz identity: $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ for all elements $a,b,c \in L$. A linear transformation f of L is called an endomorphism of L, if $f([a,b]) = [f(a), f(b)]$ for all elements $a, b \in L$. A bijective endomorphism of L is called an automorphism of L. It is easy to show that the set of all automorphisms of the Leibniz algebra is a group with respect to the operation of multiplication of automorphisms. The description of the structure of the automorphism groups of Leibniz algebras is one of the natural and important problems of the general Leibniz algebra theory. The main goal of this article is to describe the structure of the automorphism group of a certain type of nilpotent three-dimensional Leibniz algebras.
Key words: Leibniz algebra, automorphism group

Анотація. Нехай L — алгебра над полем F з бінарними операціями $+$ та $[,]$. Тоді L називатимемо лівою алгеброю Лейбніца, якщо вона задовольняє лівій тотожності Лейбніца: $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ для всіх елементів $a,b,c \in L$. Лінійне перетворення f алгебри Лейбніца L називають ендоморфізмом алгебри L, якщо $f([a,b]) = [f(a), f(b)]$ для всіх елементів $a, b \in L$. Бієктивний ендоморфізм алгебри Лейбніца L називають автоморфізмом алгебри L. Легко показати, що множина всіх автоморфізмів алгебри Лейбніца є групою відносно операції множення автоморфізмів. Опис будови груп автоморфізмів алгебр Лейбніца є однією з природних та важливих задач загальної теорії алгебр Лейбніца. Головною метою цієї статті є опис будови групи автоморфізмів деякого типу нільпотентних тривимірних алгебр Лейбніца.
Ключові слова: алгебра Лейбніца, група автоморфізмів

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1. Introduction.

Let L be an algebra over a field F with the binary operations $+$ and $[,]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity:

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]]$$

for all elements $a, b, c \in L$.

Leibniz algebras appeared first in the paper of A. Blokh [2], but the term "Leibniz algebra" appears in the book of J.-L. Loday [11], and the article of J.-L. Loday [12]. In [13], J.-L. Loday and T. Pirashvili began the real study of the properties of Leibniz algebras. The theory of Leibniz algebras has developed very intensively in many different directions. Some of the results of this theory were presented in the book [1]. Note that Lie algebras are a partial case of Leibniz algebras. Conversely, if L is a Leibniz algebra, in which $[a,a] = 0$ for every element $a \in L$, then it is a Lie algebra. Thus, Lie algebras can be characterized as anticommutative Leibniz algebras. At the same time, there is a very significant difference between Lie algebras and Leibniz algebras (see, for example, survey papers [3, 4, 9, 14]).

Let L be a Leibniz algebra. As usual, a linear transformation f of L is called an *endomorphism* of L, if $f([a,b]) = [f(a), f(b)]$ for all elements $a, b \in L$. Clearly, a product of two endomorphisms of L is also endomorphism, so that the set of all endomorphisms of L is a semigroup by a multiplication. We note that the sum of two endomorphisms of L is not necessary to be an endomorphism of L, so that we cannot speak about the endomorphism ring of L.

As usual, a bijective endomorphism of L is called an *automorphism* of L. It is not hard to show that the set $Aut_{[,]}(L)$ of all automorphisms of L is a group by a multiplication (see, for example, [7]).

As for other algebraic structures, the search for the structure of automorphism groups of Leibniz algebras is one of the important problems of this theory. It should be noted that automorphisms groups of Leibniz algebras have hardly been studied. It is natural to start studying automorphism groups of Leibniz algebras, the structure of which has been studied quite fully. A description of the structure of automorphism groups of infinite-dimensional cyclic Leibniz algebras was obtained in [10], and of finite-dimensional cyclic Leibniz algebras was obtained in [7]. The question naturally arises about automorphism groups of Leibniz algebras of low dimension. Unlike Lie algebras, the situation with Leibniz algebras of dimension 3 is very diverse. Leibniz algebras of dimension 3 are mostly described. Their most detailed description can be found in [6]. In [8], the description of automorphism groups of Leibniz algebras with dimension 3 was started. This description is quite large. The automorphism groups of only two types of nilpotent Leibniz algebras of dimension 3 are described in [8].

This article is devoted to the description of another type of nilpotent Leibniz algebras.

2. Some preliminary results and remarks.

Let L be a Leibniz algebra over a field F. Then L is called *abelian* if $[a, b] = 0$ for every elements $a, b \in L$. In particular, an abelian Leibniz algebra is a Lie algebra.

If A, B are subspaces of L, then $[A, B]$ will denote a subspace generated by all elements $[a, b]$ where $a \in A$, $b \in B$. A subspace A of L is called a *subalgebra* of L, if $[a, b] \in A$ for every $a, b \in A$. A subalgebra A of L is called a *left* (respectively *right*) *ideal* of L, if $[b, a] \in A$ (respectively $[a, b] \in A$) for every $a \in A$, $b \in L$. A subalgebra A of L is called an *ideal* of L (more precisely, *two-sided ideal*) if it is both a left ideal and a right ideal.

Every Leibniz algebra L possesses the following specific ideal. Denote by $Leib(L)$ the subspace generated by the elements $[a, a]$, $a \in L$. It is not hard to prove that $Leib(L)$ is an ideal of L. The ideal $Leib(L)$ is called the *Leibniz kernel* of L. We note the following important property of the Leibniz kernel: $[[a, a], x] = [a, [a, x]] - [a, [a, x]] = 0$.

The *left* (respectively *right*) *center* $\zeta^{\mathrm{left}}(L)$ (respectively $\zeta^{\mathrm{right}}(L)$) of a Leibniz algebra L is defined by the rule:

$$\zeta^{\mathrm{left}}(L) = \{x \in L |\ [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively,

$$\zeta^{\mathrm{right}}(L) = \{x \in L |\ [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but that is not true for the right center. Moreover, $Leib(L) \leqslant \zeta^{\mathrm{left}}(L)$ so that $L/\zeta^{\mathrm{left}}(L)$ is a Lie algebra. The right center is a subalgebra of L and, in general, the left and right centers are different (see, for example, [5]).

The *center* $\zeta(L)$ of L is defined by the rule:

$$\zeta(L) = \{x \in L |\ [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L.

Now we define the *upper central series*

$$\langle 0 \rangle = \zeta_0(L) \leqslant \zeta_1(L) \leqslant \ldots \zeta_\alpha(L) \leqslant \zeta_{\alpha+1}(L) \leqslant \ldots \zeta_\eta(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) = \bigcup_{\mu<\lambda} \zeta_\mu(L)$ for the limit ordinals λ. By definition, each term of this series is an ideal of L.

Define the *lower central series* of L

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \ldots \gamma_\alpha(L) \geqslant \gamma_{\alpha+1} \geqslant \ldots \gamma_\tau(L) = \gamma_\infty(L)$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ.

We say that a Leibniz algebra L is *nilpotent*, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class* c if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$. We denote the nilpotency class of L by $ncl(L)$.

Let L be a Leibniz algebra over a field F, M be non-empty subset of L and H be a subalgebra of L. Put

$$Ann_H^{\text{left}}(M) = \{a \in H | \ [a, M] = \langle 0 \rangle\},$$
$$Ann_H^{\text{right}}(M) = \{a \in H | \ [M, a] = \langle 0 \rangle\}.$$

The subset $Ann_H^{\text{left}}(M)$ is called the *left annihilator* of M in subalgebra H. The subset $Ann_H^{\text{right}}(M)$ is called the *right annihilator* of M in subalgebra H. The intersection

$$Ann_H(M) = Ann_H^{\text{left}}(M) \cap Ann_H^{\text{right}}(M) = \{a \in H | \ [a, M] = \langle 0 \rangle = [M, a]\}$$

is called the *annihilator* of M in subalgebra H. It is not hard to see that all of these subsets are subalgebras of L. Moreover, if M is an ideal of L, then $Ann_L(M)$ is an ideal of L (see, for example, [4]).

The first type of nilpotent Leibniz algebras of dimension 3 are nilpotent Leibniz algebras of nilpotency class 3. There is only one type of such algebras:

$$L_1 = Lei_1(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3, \text{ where } [a_1, a_1] = a_2, [a_1, a_2] = a_3,$$
$$[a_1, a_3] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

This is a cyclic Leibniz algebra, $Leib(L_1) = \zeta^{\text{left}}(L_1) = [L_1, L_1] = Fa_2 \oplus Fa_3$, $\zeta^{right}(L_1) = \zeta(L_1) = \gamma_3(L_1) = Fa_3$.

Let now L be a nilpotent Leibniz algebra, whose nilpotency class is 2. Of course, we assume that L is not a Lie algebra. Then the center $\zeta(L)$ has dimension 2 or 1. Suppose that $dim_F(\zeta(L)) = 2$. Since L is not a Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. Since a Leibniz algebra of dimension 1 is abelian, $a_3 \in \zeta(L)$. It follows that $[a_1, a_3] = [a_3, a_1] = [a_3, a_3] = 0$. Being an abelian algebra of dimension 2, $\zeta(L)$ has a direct decomposition $\zeta(L) = Fa_2 \oplus Fa_3$ for some element a_2. Thus we come to the following type of nilpotent Leibniz algebra:

$$L_2 = Lei_2(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3, \text{ where } [a_1, a_1] = a_3, [a_1, a_2] =$$
$$[a_1, a_3] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

In other words, L_2 is a direct sum of two ideals $A = Fa_1 \oplus Fa_3$ and $B = Fa_2$. Moreover, A is a nilpotent cyclic Leibniz algebra of dimension 2, $Leib(L_2) = [L_2, L_2] = Fa_3$, $\zeta^{\text{left}}(L_2) = \zeta^{\text{right}}(L_2) = \zeta(L_2) = Fa_2 \oplus Fa_3$.

We note that the structure of the automorphism groups of Leibniz algebras $Lei_1(3, F)$ and $Lei_2(3, F)$ was described in [8].

Suppose now that L is a nilpotent Leibniz algebra, $ncl(L) = 2$ and $dim_F(\zeta(L)) = 1$. Since L is not a Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. Since the factor-algebra $L/\zeta(L)$ is abelian, $a_3 \in \zeta(L)$. It follows that $[a_1, a_3] = [a_3, a_1] = [a_3, a_3] = 0$. Then $\zeta(L) = Fa_3$. For every element $x \in L$ we have $[a_1, x], [x, a_1] \in \zeta(L) \leqslant \langle a_1 \rangle = Fa_1 \oplus Fa_3$. It follows that subalgebra $\langle a_1 \rangle$ is an ideal of L. Since $dim_F(\langle a_1 \rangle) = 2$, $\langle a_1 \rangle \neq L$. Choose an element b such that $b \notin \langle a_1 \rangle$. We have $[b, a_1] = \gamma a_3$ for some $\gamma \in F$. If $\gamma \neq 0$, then put $b_1 = \gamma^{-1} b - a_1$. Then $[b_1, a_1] = 0$. The choice of b_1 shows that $b_1 \notin \langle a_1 \rangle$. If follows that subalgebra $Ann_L^{\text{left}}(a_1)$ has dimension 2. Suppose first that $Ann_L^{\text{left}}(a_1)$ is an abelian subalgebra. Then it has a direct decomposition $Ann_L^{\text{left}}(a_1) = Fa_2 \oplus Fb_2$ for some element b_2, where $[b_2, b_2] = 0$. Since $dim_F(\zeta(L)) = 1$, $b_2 \notin \zeta(L)$. Then $[a_1, b_2] = \lambda a_3$ where $0 \neq \lambda \in F$. Put now $a_2 = \lambda^{-1} b_2$. Thus, we come to the following type of nilpotent Leibniz algebra:

$$L_3 = Lei_3(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3, \text{ where } [a_1, a_1] = [a_1, a_2] = a_3,$$
$$[a_1, a_3] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

In other words, L_3 is a direct sum of ideal $A = Fa_1 \oplus Fa_3$ and subalgebra $B = Fa_2$. Moreover, A is a nilpotent cyclic Leibniz algebra of dimension 2, $Leib(L_3) = [L_3, L_3] = \zeta^{\text{right}}(L_3) = \zeta(L_3) = Fa_3$, $\zeta^{\text{left}}(L_3) = Fa_2 \oplus Fa_3$.

This article is devoted to the description of this type of nilpotent Leibniz algebras.

Here are some general useful properties of automorphism groups of Leibniz algebras. Their proofs are given in the article [8].

Lemma 1. *Let L be a Leibniz algebra over a field F and f be an automorphism of L. Then $f(\zeta^{\text{left}}(L)) = \zeta^{\text{left}}(L)$, $f(\zeta^{\text{right}}(L)) = \zeta^{\text{right}}(L)$, $f(\zeta(L)) = \zeta(L)$, $f([L, L]) = [L, L]$.*

Lemma 2. *Let L be a Leibniz algebra over a field F and f be an automorphism of L. Then $f(\zeta_\alpha(L)) = \zeta_\alpha(L)$, $f(\gamma_\alpha(L)) = \gamma_\alpha(L)$ for all ordinals α. In particular, $f(\zeta_\infty(L)) = \zeta_\infty(L)$ and $f(\gamma_\infty(L)) = \gamma_\infty(L)$.*

Lemma 3. *Let L be a Leibniz algebra over a field F and f be an endomorphism of L. Then $f(\gamma_\alpha(L)) \leqslant \gamma_\alpha(L)$ for all ordinals α. In particular, $f(\gamma_\infty(L)) \leqslant \gamma_\infty(L)$.*

Let L be a Leibniz algebra over a field F, A be a subalgebra of L, $G = Aut_{[,]}(L)$. Put

$$C_G(A) = \{\alpha \in G |\ \alpha(x) = x \text{ for every element } x \in A\}.$$

If A is an ideal of L, then put

$$C_G(L/A) = \{\alpha \in G |\ \alpha(x + A) = x + A \text{ for every element } x \in L\}.$$

Lemma 4. *Let L be a Leibniz algebra over a field F and $G = Aut_{[,]}(L)$. If A is a G-invariant subalgebra, then $C_G(A)$ and $C_G(L/A)$ are normal subgroups of G.*

3. Main result.

Theorem. *Let G be an automorphism group of Leibniz algebra $Lei_3(3, F)$. Then G is isomorphic to a subgroup of $GL_3(F)$, consisting of the matrices, having the following form:*

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_1 + \alpha_2 & 0 \\ \alpha_3 & \beta_3 & \alpha_1^2 + \alpha_1\alpha_2 \end{pmatrix}$$

where $\alpha_1 \neq 0$, $\alpha_1 + \alpha_2 \neq 0$. This subgroup is a semidirect product of normal subgroup $S_3(L, F)$, which is isomorphic to a subgroup of $GL_3(F)$, consisting of the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1 + \alpha_2 & 0 \\ \alpha_3 & \beta_3 & 1 + \alpha_2 \end{pmatrix}$$

and a subgroup $D_3(L, F)$, consisting of the matrices of the form

$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix}.$$

In particular, $D_3(L, F)$ is isomorphic to multiplicative group of a field F. Furthermore, $S_3(L, F)$ is a semidirect product of subgroup $T_3(L, F)$, which is normal in G and isomorphic to a subgroup of $GL_3(F)$, consisting of the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_3 & \beta_3 & 1 \end{pmatrix},$$

and a subgroup $J_3(L, F)$, which is isomorphic to a subgroup of $GL_3(F)$, consisting of the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 + \lambda & 0 \\ 0 & 0 & 1 + \lambda \end{pmatrix}.$$

A subgroup $T_3(L, F)$ is isomorphic to direct product of two copy of additive group of a field F, and a subgroup $J_3(L, F)$ is isomorphic to multiplicative group of a field F.

Proof. Let $L = Lei_3(3, F)$, $f \in Aut_{[,]}(L)$. By Lemma 1, $f(Fa_3) = Fa_3$, $f(Fa_2 \oplus Fa_3) = Fa_2 \oplus Fa_3$, so that

$$f(a_1) = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3,$$
$$f(a_2) = \beta_2 a_2 + \beta_3 a_3.$$

Then

$$f(a_3) = f([a_1, a_1]) = [f(a_1), f(a_1)] =$$
$$[\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] =$$
$$\alpha_1^2[a_1, a_1] + \alpha_1\alpha_2[a_1, a_2] = \alpha_1^2 a_3 + \alpha_1\alpha_2 a_3 = (\alpha_1^2 + \alpha_1\alpha_2)a_3;$$
$$f(a_3) = f([a_1, a_2]) = [f(a_1), f(a_2)] = [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \beta_2 a_2 + \beta_3 a_3] =$$
$$\alpha_1\beta_2[a_1, a_2] = \alpha_1\beta_2 a_3.$$

Thus, we obtain an equality $\alpha_1(\alpha_1 + \alpha_2) = \alpha_1\beta_2$. Being an automorphism, f is a non-degenerate linear transformation, so that $\alpha_1 \neq 0$. It follows that $\alpha_1 + \alpha_2 = \beta_2$. Thus, an automorphism f has in basis $\{a_1, a_2, a_3\}$ the following matrix

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_1 + \alpha_2 & 0 \\ \alpha_3 & \beta_3 & \alpha_1^2 + \alpha_1\alpha_2 \end{pmatrix}$$

Denote by Ξ the canonical monomorphism of $End_{[,]}(L)$ in $M_3(F)$.

Put

$$S = \{f |\ f \in End(L), f(a_1) \in a_1 + \zeta^{\text{left}}(L)\} = C_{End(L)}(L/\zeta^{\text{left}}(L)).$$

If $f \in S \cap Aut_{[,]}(L)$, then $f(a_1) = a_1 + \alpha_2 a_2 + \alpha_3 a_3$, $f(a_2) = (1+\alpha_2)a_2 + \beta_3 a_3$, $f(a_3) = (1 + \alpha_2)a_3$. If x is an arbitrary element of L, $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, then

$$f(x) = \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) =$$
$$\xi_1 a_1 + \xi_1\alpha_2 a_2 + \xi_1\alpha_3 a_3 + \xi_2((1 + \alpha_2)a_2 + \beta_3 a_3) + \xi_3(1 + \alpha_2)a_3 =$$
$$\xi_1 a_1 + (\xi_1\alpha_2 + \xi_2 + \xi_2\alpha_2)a_2 + (\xi_1\alpha_3 + \xi_2\beta_3 + \xi_3(1 + \alpha_2))a_3.$$

Conversely, let λ, μ, ν be the elements of F, $v_{\lambda,\mu,\nu}$ be a linear transformation of L, defined by the rule: if $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, then

$$v_{\lambda,\mu,\nu}(x) = \xi_1 a_1 + (\xi_1\lambda + \xi_2 + \xi_2\lambda)a_2 + (\xi_1\mu + \xi_2\nu + \xi_3(1 + \lambda))a_3.$$

Let x, y be the arbitrary elements of L, $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, $y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$, where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in F$. Then

$$[x, y] = [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] =$$
$$\xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] = \xi_1(\eta_1 + \eta_2)a_3;$$
$$v_{\lambda,\mu,\nu}([x, y]) = v_{\lambda,\mu,\nu}(\xi_1(\eta_1 + \eta_2)a_3) = \xi_1(\eta_1 + \eta_2)v_{\lambda,\mu,\nu}(a_3) =$$
$$\xi_1(\eta_1 + \eta_2)(1 + \lambda)a_3;$$
$$[v_{\lambda,\mu,\nu}(x), v_{\lambda,\mu,\nu}(y)] =$$
$$[\xi_1 a_1 + (\xi_1\lambda + \xi_2 + \xi_2\lambda)a_2 + (\xi_1\mu + \xi_2\nu + \xi_3(1 + \lambda))a_3,$$
$$\eta_1 a_1 + (\eta_1\lambda + \eta_2 + \eta_2\lambda)a_2 + (\eta_1\mu + \eta_2\nu + \eta_3(1 + \lambda))a_3] =$$
$$\xi_1\eta_1[a_1, a_1] + \xi_1(\eta_1\lambda + \eta_2 + \eta_2\lambda)[a_1, a_2] = (\xi_1\eta_1 + \xi_1(\eta_1\lambda + \eta_2 + \eta_2\lambda))a_3 =$$
$$\xi_1(\eta_1 + \eta_1\lambda + \eta_2 + \eta_2\lambda)a_3 = \xi_1(\eta_1 + \eta_2)(1 + \lambda)a_3,$$

so that $v_{\lambda,\mu,\nu}([x,y]) = [v_{\lambda,\mu,\nu}(x), v_{\lambda,\mu,\nu}(y)]$. It shows that $S \leqslant Aut_{[,]}(L)$. Moreover, by Lemma 4, S is a normal subgroup of $Aut_{[,]}(L)$. Furthermore, put $S_3(L,F) = \Xi(S)$. Then $S_3(L,F)$ is a subgroup of a group $T_3(F)$, which consist of the matrices, having the following form:

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1+\alpha_2 & 0 \\ \alpha_3 & \beta_3 & 1+\alpha_2 \end{pmatrix}.$$

Let

$$T = \{f \mid f \in End(L), f(a_1) \in a_1 + [L,L], f(a_2) \in a_2 + [L,L]\} = C_{End(L)}(L/[L,L]).$$

If $f \in T \cap Aut_{[,]}(L)$, then $f(a_1) = a_1 + \alpha_3 a_3$, $f(a_2) = a_2 + \beta_3 a_3$, $f(a_3) = a_3$. If x is an arbitrary element of L, $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, then

$$f(x) = \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) = \\ \xi_1 a_1 + \xi_1 \alpha_3 a_3 + \xi_2 a_2 + \xi_2 \beta_3 a_3 + \xi_3 a_3 = \\ \xi_1 a_1 + \xi_2 a_2 + (\xi_1 \alpha_3 + \xi_2 \beta_3 + \xi_3) a_3.$$

Conversely, let λ, μ be the elements of F, $z_{\lambda,\mu}$ be a linear transformation of L, defined by the rule: if $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, then

$$z_{\lambda,\mu}(x) = \xi_1 a_1 + \xi_2 a_2 + (\xi_1 \lambda + \xi_2 \mu + \xi_3) a_3.$$

Let x, y be the arbitrary elements of L, $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, $y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$, where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in F$. Then

$$[x,y] = [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] = \xi_1(\eta_1 + \eta_2) a_3; \\ z_{\lambda,\mu}([x,y]) = z_{\lambda,\mu}(\xi_1(\eta_1+\eta_2)a_3) = \xi_1(\eta_1+\eta_2) z_{\lambda,\mu}(a_3) = \xi_1(\eta_1+\eta_2)a_3; \\ [z_{\lambda,\mu}(x), z_{\lambda,\mu}(y)] = \\ [\xi_1 a_1 + \xi_2 a_2 + (\xi_1\lambda + \xi_2\mu + \xi_3)a_3, \eta_1 a_1 + \eta_2 a_2 + (\eta_1\lambda + \eta_2\mu + \eta_3)a_3] = \\ \xi_1\eta_1[a_1,a_1] + \xi_1\eta_2[a_1,a_2] = \xi_1(\eta_1+\eta_2)a_3,$$

so that $z_{\lambda,\mu}([x,y]) = [z_{\lambda,\mu}(x), z_{\lambda,\mu}(y)]$. It shows that T is a subgroup of $Aut_{[,]}(L)$. Furthermore, put $T_3(L,F) = \Xi(T)$. Then $T_3(L,F)$ is a subgroup of a group $UT_3(F)$ of all unitriangular matrices over a field F, which consist of the matrices, having the following form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_3 & \beta_3 & 1 \end{pmatrix}.$$

It is not hard to see, that $T_3(L,F)$ is abelian and it is isomorphic to direct product of two copy of additive group of a field F. Clearly $\Xi(Aut_{[,]}(L)) \cap UT_3(F) = T_3(L,F)$, so it follows that a subgroup T is normal in $Aut_{[,]}(L)$.

Let

$$J = \{f \mid f \in S, f(a_1) = a_1 + \lambda a_2, f(a_2) = (1+\lambda)a_2, f(a_3) = (1+\lambda)a_3, \lambda \in F\}.$$

Put $J_3(L, F) = \Xi(J)$. Then $J_3(L, F)$ is a subset of $T_3(F)$, which consist of the matrices, having the following form:

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1+\lambda & 0 \\ 0 & 0 & 1+\lambda \end{pmatrix}.$$

An equality

$$\begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1+\lambda & 0 \\ 0 & 0 & 1+\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \mu & 1+\mu & 0 \\ 0 & 0 & 1+\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ (1+\lambda)(1+\mu)-1 & (1+\lambda)(1+\mu) & 0 \\ 0 & 0 & (1+\lambda)(1+\mu) \end{pmatrix}$$

shows that $J_3(L, F)$ is a subgroup of $S_3(L, F)$. Moreover, it is not hard to see that $J_3(L, F)$ is isomorphic to multiplicative group of a field F. Furthermore, it is not hard to see that the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1+\alpha_2 & 0 \\ \alpha_3 & \beta_3 & 1+\alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ z & 1+z & 0 \\ 0 & 0 & 1+z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ z & 1+z & 0 \\ x+yz & y+yz & 1+z \end{pmatrix}$$

has the solutions. It proves that $S_3(L, F) = T_3(L, F)J_3(L, F)$ and therefore $S = TJ$. Clearly the intersection $T \cap J$ is trivial.

Let

$$D = \{f \mid f \in Aut_{[,]}(L), f(a_1) = \sigma a_1, f(a_2) = \sigma a_2, \sigma \in F\}.$$

By proved above, $f(a_3) = \sigma^2 a_3$.

Put $D_3(L, F) = \Xi(D)$. Then $D_3(L, F)$ is a subset of $T_3(F)$, which consist of the matrices, having the following form:

$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix}.$$

An equality

$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} \nu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu^2 \end{pmatrix} = \begin{pmatrix} \sigma\nu & 0 & 0 \\ 0 & \sigma\nu & 0 \\ 0 & 0 & \sigma^2\nu^2 \end{pmatrix}$$

shows that $D_3(L, F)$ is a subgroup of $\Xi(Aut_{[,]}(L))$. Moreover, it is not hard to see that $D_3(L, F)$ is isomorphic to multiplicative group of a field F. Furthermore, an equality

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_1+\alpha_2 & 0 \\ \alpha_3 & \beta_3 & \alpha_1^2+\alpha_1\alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_2\alpha_1^{-1} & 1+\alpha_2\alpha_1^{-1} & 0 \\ \alpha_3\alpha_1^{-1} & \beta_3\alpha_1^{-1} & 1+\alpha_2\alpha_1^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1^2 \end{pmatrix}$$

proves that $\Xi(Aut_{[,]}(L)) = S_3(L, F)D_3(L, F)$ and therefore $Aut_{[,]}(L) = SD$. Clearly the intersection $S \cap D$ is trivial. Thus, we obtain that $Aut_{[,]}(L) = S \leftthreetimes D$, $D \cong F^{\times}$, $S = T \leftthreetimes J$, moreover T is normal in $Aut_{[,]}(L)$, $T \cong F_+ \times F_+$, $J \cong F^{\times}$.

References

1. *Ayupov Sh., Omirov B., Rakhimov I.*: Leibniz algebras: structure and classification. CRC Press, Taylor & Francis Group, 2020.
2. *Blokh A.*: On a generalization of the concept of Lie algebra. Dokl. Akad. Nauk SSSR 1965; 165(3): pp. 471–473.
3. *Chupordia V.A., Pypka A.A., Semko N.N., Yashchuk V.S.*: Leibniz algebras: a brief review of current results. Carpathian Math. Publ. 2019; 11(2): pp. 250–257. doi:10.15330/cmp.11.2.250-257
4. *Kirichenko V.V., Kurdachenko L.A., Pypka A.A., Subbotin I.Ya.*: Some aspects of Leibniz algebra theory. Algebra Discrete Math. 2017; 24(1): pp. 1–33.
5. *Kurdachenko L.A., Otal J., Pypka A.A.*: Relationships between the factors of the canonical central series of Leibniz algebras. Eur. J. Math. 2016; 2(2): pp. 565–577. doi:10.1007/s40879-016-0093-5
6. *Kurdachenko L.A., Pypka O.O., Subbotin I.Ya.*: On the structure of low-dimensional Leibniz algebras: some revision. Algebra Discrete Math. 2022; 34(1): pp. 68–104. doi:10.12958/adm2036
7. *Kurdachenko L.A., Pypka A.A., Subbotin I.Ya.*: On the automorphism groups of some Leibniz algebras. Int. J. Group Theory 2023; 12(1): pp. 1–20. doi:10.22108/IJGT.2021.130057.1735
8. *Kurdachenko L.A., Pypka O.O., Velychko T.V.*: On the automorphism groups for some Leibniz algebras of low dimensions. Ukrainian Math. J. 2023; 74(10): pp. 1526–1546. doi:10.1007/s11253-023-02153-2
9. *Kurdachenko L.A., Semko N.N., Subbotin I.Ya.*: Applying group theory philosophy to Leibniz algebras: some new developments. Adv. Group Theory Appl. 2020; 9: pp. 71–121. doi:10.32037/agta-2020-004
10. *Kurdachenko L.A., Subbotin I.Ya., Yashchuk V.S.*: On the endomorphisms and derivations of some Leibniz algebras. J. Algebra Appl. 2022; 2450002. doi:10.1142/S0219498824500026
11. *Loday J.-L.*: Cyclic homology. Grundlehren der Mathematischen Wissenschaften, 301, Springer Verlag, 1992. doi:10.1007/978-3-662-11389-9
12. *Loday J.-L.*: Une version non commutative des algèbres de Lie; les algèbras de Leibniz. Enseign. Math. 1993; 39: pp. 269–293.
13. *Loday J.-L., Pirashvili T.*: Universal enveloping algebras of Leibniz algebras and (co)homology. Math. Ann. 1993; 296(1): pp. 139–158. doi:10.1007/BF01445099
14. *Subbotin I.Ya.*: Methods of group theory in Leibniz algebras: some compelling results. Res. Math. 2021; 29(2): pp. 44–54. doi:10.15421/242108

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