Res. Math. **31**(2), 2023, p. 36–48 ISSN (Print) 2664-4991
doi:10.15421/242313 ISSN (Online) 2664-5009

UDK 517.5

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# Norm attaining bilinear forms of $\mathcal{L}({}^2d_*(1,w)^2)$ at given vectors

**Abstract.** For given unit vectors $x_1, \cdots, x_n$ of a real Banach space $E$, we define

$$NA(\mathcal{L}({}^nE))(x_1, \cdots, x_n) = \{T \in \mathcal{L}({}^nE) : |T(x_1, \cdots, x_n)| = \|T\| = 1\},$$

where $\mathcal{L}({}^nE)$ denotes the Banach space of all continuous $n$-linear forms on $E$ endowed with the norm $\|T\| = \sup_{\|x_k\|=1, 1\leq k\leq n} |T(x_1, \ldots, x_n)|$.
In this paper, we classify $NA(\mathcal{L}({}^2d_*(1,w)^2))(Z_1, Z_2)$ for unit vectors $Z_1, Z_2 \in d_*(1,w)^2$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the norm of weight $0 < w < 1$ endowed with $\|(x,y)\|_{d_*(1,w)} = \max\left\{|x|, |y|, \frac{|x|+|y|}{1+w}\right\}$.

**Key words:** norm attaining bilinear forms, the plane with an octagonal norm

**Анотація.** Для заданих одиничних векторів $x_1, \cdots, x_n$ дійсного банахового простору $E$, ми визначаємо

$$NA(\mathcal{L}({}^nE))(x_1, \cdots, x_n) = \{T \in \mathcal{L}({}^nE) : |T(x_1, \cdots, x_n)| = \|T\| = 1\},$$

де $\mathcal{L}({}^nE)$ позначає банаховий простір усіх неперервних $n$-лінійних форм на $E$ з нормою $\|T\| = \sup_{\|x_k\|=1, 1\leq k\leq n} |T(x_1, \ldots, x_n)|$.
У цій статті ми класифікуємо $NA(\mathcal{L}({}^2d_*(1,w)^2))(Z_1, Z_2)$ для одиничних векторів $Z_1, Z_2 \in d_*(1,w)^2$, де $d_*(1,w)^2 = \mathbb{R}^2$ з нормою ваги $0 < w < 1$ визначеною як $\|(x,y)\|_{d_*(1,w)} = \max\left\{|x|, |y|, \frac{|x|+|y|}{1+w}\right\}$.

**Ключові слова:** нормодосягаючі білінійні форми, площина з восьмикутною нормою

**MSC2020:** 46A22

## 1. Introduction

Let us introduce a brief history of norm attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [2] showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result

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to more general settings, especially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jimenez-Sevilla and Paya [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}, n \geq 2$. We write $S_E$ for the unit sphere of a real Banach space $E$. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous $n$-linear forms on $E$ endowed with the norm $\|T\| = \sup_{\|x_k\|=1, 1\leq k\leq n} |T(x_1, \ldots, x_n)|$. The subspace of all continuous symmetric $n$-linear forms on $E$ is denoted by $\mathcal{L}_s(^nE)$. A mapping $P : E \to \mathbb{R}$ is a continuous $n$-homogeneous polynomial if there exists $T \in \mathcal{L}(^nE)$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous $n$-homogeneous polynomials from $E$ into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}(^nE)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$. In this case, $T$ is called a *norm attaining* $n$-linear form at $(x_1, \ldots, x_n)$. Similarly, an element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. In this case, $P$ is called a *norm attaining* $n$-homogeneous polynomial at $x$. Let $X = \mathcal{L}(^nE)$ or $\mathcal{L}_s(^nE)$. We define

$$NA(X) = \{T \in X : T \text{ is a norm attaining } n\text{-linear form}\}$$

and

$$NA(\mathcal{P}(^nE)) = \{P \in \mathcal{P}(^nE) : T \text{ is a norm attaining polynomial}\}.$$

For $x, x_1, \cdots, x_n \in S_E$, we define

$$NA(X)(x_1, \cdots, x_n) = \{T \in NA(X) : |T(x_1, \cdots, x_n)| = \|T\| = 1\}$$

and

$$NA(\mathcal{P}(^nE))(x) = \{P \in NA(\mathcal{P}(^nE)) : |P(x)| = \|P\| = 1\}.$$

The following remark provides a relation between $NA(X)(x_1, \cdots, x_n)$ and $NA(X)$.

**Remark.** *Let $E$ be a Banach space with $dim(E) \geq 2$. Then for each $x_1, \cdots, x_n \in S_E$,*

$$NA(X)(x_1, \cdots, x_n) \neq NA(X).$$

*Proof.* We will only prove $X = \mathcal{L}(^nE)$.

*Case 1.* $\dim(E) < \infty$

Let $f_j \in E^*$ be such that $f_1(x_1) = 0, \|f_1\| = 1$ and $f_j(x_j) = \|f_j\| = 1$ for $j = 2, \ldots, n$. Define $T \in \mathcal{L}(^nE)$ by

$$T(z_1, \cdots, z_n) = \prod_{1 \le j \le n} f_j(z_j).$$

Since $\dim(E) < \infty$, $NA(\mathcal{L}(^nE)) = \mathcal{L}(^nE)$. Hence $T \in NA(\mathcal{L}(^nE))$. Choose $x_1^{'} \in S_E$ such that $|f_1(x_1^{'})| > \frac{1}{2}$. It follows that

$$\|T\| \ge |T(x_1^{'}, x_2, \ldots, x_n)| = |f_1(x_1^{'})| > \frac{1}{2}.$$

Thus $T \notin NA(\mathcal{L}(^nE))(x_1, \cdots, x_n)$ because $T(x_1, \cdots, x_n) = 0$.

*Case 2.* $\dim(E) = \infty$

Let $f_j \in E^*$ be such that $f_j(x_j) = \|f_j\| = 1$ for $j = 2, \ldots, n$. Let $x_1^{''} \ne x_1 \in S_E$. Let $f_1^{'} \in E^*$ be such that $f_1^{'}(x_1^{''}) = 1 = \|f_1^{'}\| > |f_1^{'}(x_1)|$. Define $S \in \mathcal{L}(^nE)$ by

$$S(z_1, \cdots, z_n) = f_1^{'}(z_1) \cdot \prod_{2 \le j \le n} f_j(z_j).$$

Note that $\|S\| \le 1$. It follows that

$$1 \ge \|S\| \ge |S(x_1^{''}, x_2, \ldots, x_n)| = 1 > |S(x_1, x_2, \ldots, x_n)| = |f_1^{'}(x_1)|.$$

Thus $S \in NA(\mathcal{L}(^nE)) \backslash NA(\mathcal{L}(^nE))(x_1, \cdots, x_n)$. $\square$

Notice that

$$NA(\mathcal{L}(^nE))(x_1, \cdots, x_n) = NA(\mathcal{L}(^nE))(\pm x_1, \cdots, \pm x_n),$$

$$NA(\mathcal{L}_s(^nE))(x_1, \cdots, x_n) = NA(\mathcal{L}_s(^nE))(\pm x_{\sigma(1)}, \cdots, \pm x_{\sigma(n)})$$

and

$$NA(\mathcal{P}(^nE))(x) = NA(\mathcal{P})(^nE))(-x)$$

for all $x, x_1, \cdots, x_n \in S_E$ and for all permutation $\sigma$ on $\{1, \ldots, n\}$.

It seems to be natural and interesting to study about

$$NA(\mathcal{L}(^nE))(x_1, \cdots, x_n), NA(\mathcal{L}_s(^nE))(x_1, \cdots, x_n) \text{ and } NA(\mathcal{P}(^nE))(x)$$

for $x, x_1, \cdots, x_n \in S_E$. Kim [6] classified $NA(\mathcal{P}(^2\ell_p^2))((x_1, x_2))$ for $(x_1, x_2) \in S_{\ell_p^2}$ and $p = 1, 2, \infty$, where $\ell_p^2 = \mathbb{R}^2$ with the $\ell_p$-norm. Kim [8] classified $NA(\mathcal{L}(^2\ell_1^2))(Z_1, Z_2)$ for $Z_1, Z_2 \in S_{\ell_1^2}$. Recently, Kim [9] has classified $NA(\mathcal{L}(^n\ell_1^2))(Z_1, \ldots, Z_n)$ for $n \ge 2, Z_1, \ldots, Z_n \in S_{\ell_1^2}$.

Let $d_*(1, w)^2 = \mathbb{R}^2$ with the norm of weight $0 < w < 1$ endowed with $\|(x, y)\|_{d_*(1,w)} = \max\left\{|x|, |y|, \frac{|x|+|y|}{1+w}\right\}$. Notice that

$$\|(x, y)\|_{d_*(1,w)} = \|(\pm x, \pm y)\|_{d_*(1,w)} = \|(\pm y, \pm x)\|_{d_*(1,w)} \text{ for } (x, y) \in_{d_*(1,w)}.$$

Kim [7] classified the extreme points of the unit ball of $\mathcal{L}(^2d_*(1,w)^2)$. In this paper, we classify $NA(\mathcal{L}(^2d_*(1,w)^2))(Z_1,Z_2)$ for $Z_1,Z_2 \in S_{d_*(1,w)^2}$.

## 2. Results

Throughout this paper we let $0 < w < 1$. Let $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2d_*(1,w)^2)$ for some $a,b,c,d \in \mathbb{R}$. For simplicity, we denote $T = (a,b,c,d)$.

Kim [7] presented an explicit formulae for $\|T\|$ for every $T \in \mathcal{L}(^2d_*(1,w)^2)$.

**Theorem 1.** *([7]) Let $0 < w < 1$ and $T \in \mathcal{L}(^2d_*(1,w)^2)$ with $T((x_1,y_1),(x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1$ for some $a,b,c,d \in \mathbb{R}$. Then, there exists $T' = (a^*,b^*,c^*,d^*) \in \mathcal{L}(^2d_*(1,w)^2)$ such that $a^*,b^*,c^*,d^* \in \{\pm a,\pm b,\pm c,\pm d\}$ with $a^* \geq |b^*|$ and $a^* \geq c^* \geq d^* \geq 0$ and $\|T\| = \|T'\|$.*

**Theorem 2.** *([7]) Let $0 < w < 1$ and $T \in \mathcal{L}(^2d_*(1,w)^2)$ with $T((x_1,y_1),(x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1$ for some $a \geq |b|, a \geq c \geq d \geq 0$. Then*

$$\begin{aligned}\|T\| &= \max\{a+bw^2+(c+d)w,\ a-bw^2+(c-d)w,\ (a+b)w+c+dw^2,\\ &\quad (a-b)w+c-dw^2\}\\ &= \max\{a+cw+|bw+d|w,\ c+aw+|dw+b|w\}.\end{aligned}$$

**Theorem 3.** *Let $T = (a,b,c,d) \in \mathcal{L}(^2d_*(1,w)^2)$ for some $a,b,c,d \in \mathbb{R}$. The following are equivalent: let $(x_1,y_1),(x_2,y_2) \in S_{d_*(1,w)^2}$.*

*(a)* $T \in NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,y_1),(x_2,y_2))$;
*(b)* $T_1 := (-a,b,-c,d) \in NA(\mathcal{L}(^2d_*(1,w)^2))((-x_1,y_1),(x_2,y_2))$;
*(c)* $T_2 := (-a,b,c,-d) \in NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,y_1),(-x_2,y_2))$;
*(d)* $T_3 := (a,b,-c,-d) \in NA(\mathcal{L}(^2d_*(1,w)^2))((-x_1,y_1),(-x_2,y_2))$;
*(e)* $T_4 := (-a,-b,-c,-d) \in NA(\mathcal{L}(^2d_*(1,w)^2))((-x_1,-y_1),(x_2,y_2))$;
*(f)* $T_5 := (a,-b,-c,d) \in NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,y_1),(x_2,-y_2))$.

**Proof.** It is obvious.

**Theorem 4.** *Let $(x_1,y_1),(x_2,y_2) \in S_{d_*(1,w)^2}$. Then*

$$\begin{aligned}&NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,y_1),(x_2,y_2))\\ =\ &NA(\mathcal{L}(^2d_*(1,w)^2))((-x_1,-y_1),(x_2,y_2))\\ =\ &NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,y_1),(-x_2,-y_2))\\ =\ &NA(\mathcal{L}(^2d_*(1,w)^2))((-x_1,-y_1),(-x_2,-y_2)).\end{aligned}$$

**Proof.** It is obvious.

**Lemma 1.** *Let $x_j > 0$ and $\epsilon_j = \pm 1$ for $j = 1, \ldots, n$. Suppose that $\sum_{1 \le j \le n} x_j = 1$. Then $\left| \sum_{1 \le j \le n} \epsilon_j \ x_j \right| = 1$ if and only if ($\epsilon_j = 1$ for every $j = 1, \ldots, n$) or ($\epsilon_j = -1$ for every $j = 1, \ldots, n$).*

**Proof.** ($\Leftarrow$). It is obvious.

($\Rightarrow$). Use induction on $n$.

If $n = 1$, then it's ok. Suppose that ($\Rightarrow$) is true for $n = k$ for some $k \in \mathbb{N}$. Suppose that

$$\sum_{1 \le j \le k+1} x_j = 1 = \sum_{1 \le j \le k+1} \epsilon_j \ x_j.$$

Let $y := \sum_{1 \le j \le k} x_j > 0$.

**Claim.** $y = \left| \sum_{1 \le j \le k} \epsilon_j \ x_j \right|$

It follows that

$$1 = \left| \sum_{1 \le j \le k+1} \epsilon_j \ x_j \right| \le \left| \sum_{1 \le j \le k} \epsilon_j \ x_j \right| + x_{k+1} \le y + x_{k+1} = 1,$$

which shows that

$$1 = \left| \sum_{1 \le j \le k} \epsilon_j \ x_j \right| + x_{k+1} = y + x_{k+1}.$$

Hence, the claim holds. Notice that

$$1 = \left| \sum_{1 \le j \le k} \epsilon_j \left( \frac{x_j}{y} \right) \right| = \sum_{1 \le j \le k} \frac{x_j}{y}.$$

By the induction hypothesis, ($\epsilon_j = 1$ for every $j = 1, \ldots, k$) or ($\epsilon_j = -1$ for every $j = 1, \ldots, k$).

**Case 1.** $\epsilon_j = 1$ for every $j = 1, \ldots, k$

We claim that $\epsilon_{k+1} = 1$. Assume that $\epsilon_{k+1} = -1$. Then

$$y + x_{k+1} = |y - x_{k+1}| = \pm(y - x_{k+1}).$$

If $y + x_{k+1} = y - x_{k+1}$, then $x_{k+1} = 0$, which is a contradiction. If $y + x_{k+1} = -(y - x_{k+1})$, then $y = 0$, which is a contradiction. Hence, $\epsilon_{k+1} = 1$. Therefore, $\epsilon_j = 1$ for every $j = 1, \ldots, k+1$.

**Case 2.** $\epsilon_j = -1$ for every $j = 1, \ldots, k$

It follows that

$$y + x_{k+1} = |y - \epsilon_{k+1} \ x_{k+1}|.$$

By the proof of Case 1, $-\epsilon_{k+1} = 1$. Therefore, $\epsilon_j = -1$ for every $j = 1, \ldots, k+1$. We complete the proof.

Let

$$\begin{aligned}
\Omega &= \Big\{(0,1),(w,1),(1,w),(1,-w),(w,-1),(0,-1)\Big\},\\
B_1 &= \Big\{t(0,1)+(1-t)(w,1) : 0\le t\le 1\Big\},\\
B_2 &= \Big\{t(w,1)+(1-t)(1,w) : 0\le t\le 1\Big\},\\
B_3 &= \Big\{t(1,w)+(1-t)(1,-w) : 0\le t\le 1\Big\},\\
B_4 &= \Big\{t(1,-w)+(1-t)(w,-1) : 0\le t\le 1\Big\},\\
B_5 &= \Big\{t(w,-1)+(1-t)(0,-1) : 0\le t\le 1\Big\}.
\end{aligned}$$

Let $A_{ij} = B_i \times B_j$ for $i,j = 1,\dots,5$.

Notice that

$$S_{d_*(1,w)^2} = \bigcup_{1\le k\le 5} (\pm B_k).$$

We are in position to classify $NA(\mathcal{L}(^2d_*(1,w)^2))(X,Y)$ for every $X = (x_1,y_1), Y = (x_2,y_2) \in S_{d_*(1,w)^2}$. By Theorem 4, we may assume that $x_j \ge 0$ for every $j = 1,2$.

**Theorem 5.** *Let $X,Y \in A_{ij}$ for some $i,j = 1,\dots,5$. Write $X = (x_1,y_1), Y = (x_2,y_2)$, $x_k \ge 0$ for every $k = 1,2$. Then the following statements holds:*

**Case 1**. $(X,Y) \in (B_i\backslash\Omega)\times(B_j\backslash\Omega)$

$$\begin{aligned}
&NA(\mathcal{L}(^2d_*(1,w)^2))(X,Y) = \Big\{ \pm T \in \mathcal{L}(^2d_*(1,w)^2) : 1 = T(U,U')\\
&= T(U,V') = T(V,U') = T(V,V') \text{ if } \operatorname{ext} B_i = \{U,V\}, \operatorname{ext} B_j = \{U',V'\}\Big\},
\end{aligned}$$

*where* $\operatorname{ext} B_i$ *denotes the set of end points of* $B_i$.

*Moreover,* $NA(\mathcal{L}(^2d_*(1,w)^2))(X,Y) = \{\pm T_0\}$ *for some* $T_0 \in \mathcal{L}(^2d_*(1,w)^2)$.

**Case 2**. $(X,Y) \in (B_i\backslash\Omega)\times(B_j\cap\Omega)$

$$\begin{aligned}
&NA(\mathcal{L}(^2d_*(1,w)^2))(X,Y) = \Big\{ \pm T \in \mathcal{L}(^2d_*(1,w)^2) : 1 = T(U,Y)\\
&= T(V,Y) \text{ if } \operatorname{ext} B_i = \{U,V\}, |T(W_1,W_2)| \le 1 \text{ for all } W_k \in \operatorname{ext} B_{d_*(1,w)^2},\\
&\quad k = 1,2\Big\},
\end{aligned}$$

*where* $\operatorname{ext} B_{d_*(1,w)^2}$ *denotes the set of all extreme points of* $B_{d_*(1,w)^2}$.

**Case 3**. $(X,Y) \in (B_i\cap\Omega)\times(B_j\backslash\Omega)$

$$NA(\mathcal{L}(^2d_*(1,w)^2))(X,Y) = \Big\{ \pm T \in \mathcal{L}(^2d_*(1,w)^2) : 1 = T(X,U')$$

$= T(X, V^{'})$ *if* $\operatorname{ext} B_j = \{U^{'}, V^{'}\}, |T(W_1, W_2)| \leq 1$ *for all* $W_k \in \operatorname{ext} B_{d_*(1,w)^2}$, $k = 1, 2\Big\}$.

**Case 4**. $(X, Y) \in (B_i \cap \Omega) \times (B_j \cap \Omega)$

$$NA(\mathcal{L}({}^2d_*(1,w)^2))(X,Y) = \Big\{T \in \mathcal{L}({}^2d_*(1,w)^2) : 1 = |T(X,Y)|,$$
$$|T(W_1,W_2)| \leq 1 \textit{ for all } W_k \in \operatorname{ext} B_{d_*(1,w)^2}, k = 1,2\Big\}.$$

**Proof.** *Case 1.* Let $X = tU + sV$ and $Y = t^{'}U^{'} + s^{'}V^{'}$ for some $0 < t, t^{'}, s, s^{'} < 1$, $t + s = t^{'} + s^{'} = 1$. Let

$$\mathcal{F}_1 = \Big\{ \pm T \in \mathcal{L}({}^2d_*(1,w)^2) : 1 = T(U,U^{'}) = T(U,V^{'}) = T(V,U^{'})$$
$$= \quad T(V,V^{'}) \text{ if } \operatorname{ext} B_i = \{U,V\}, \operatorname{ext} B_j = \{U^{'},V^{'}\}\Big\}.$$

We claim that $NA(\mathcal{L}({}^2d_*(1,w)^2))(X,Y) = \mathcal{F}_1$.

$(\subseteq)$ : Suppose that $T \in NA(\mathcal{L}({}^2d_*(1,w)^2))(X,Y)$.

**Claim 1**. $\Big(T(U,U^{'}) = T(U,V^{'}) = T(V,U^{'}) = T(V,V^{'}) = 1\Big)$ or $\Big(T(U,U^{'}) = T(U,V^{'}) = T(V,U^{'}) = T(V,V^{'}) = -1\Big)$.

First, we will show that

$$|T(U,U^{'})| = |T(U,V^{'})| = |T(V,U^{'})| = |T(V,V^{'})| = 1.$$

Assume the contrary. Without loss of generality we may assume that $|T(U,U^{'})| < 1$. It follows that

$$\begin{aligned}
1 &= |T(X,Y)| = |T(tU + sV,\ t^{'}U^{'} + s^{'}V^{'})| \\
&= |tt^{'}T(U,U^{'}) + ts^{'}T(U,V^{'}) + t^{'}sT(V,U^{'}) + ss^{'}T(V,V^{'})| \\
&\leq tt^{'}|T(U,U^{'})| + ts^{'}|T(U,V^{'})| + t^{'}s|T(V,U^{'})| + ss^{'}|T(V,V^{'})| \\
&< tt^{'} + ts^{'}|T(U,V^{'})| + t^{'}s|T(V,U^{'})| + ss^{'}|T(V,V^{'})| \\
&\leq (t+s)(t^{'}+s^{'}) = 1,
\end{aligned}$$

which is a contradiction. Hence,

$$|T(U,U^{'})| = |T(U,V^{'})| = |T(V,U^{'})| = |T(V,V^{'})| = 1.$$

Since $tt^{'}, ts^{'}, t^{'}s, ss^{'} > 0$ and

$$|tt^{'}T(U,U^{'}) + ts^{'}T(U,V^{'}) + t^{'}sT(V,U^{'}) + ss^{'}T(V,V^{'})| = 1,$$

by Lemma 1, Claim 1 holds. Hence, $T \in \mathcal{F}_1$.

$(\supseteq)$ : Let $T \in \mathcal{F}_1$.

**Claim 2**. $\|T\| = 1$

Suppose that $(X,Y) \in A_{11}$.

Then $X = t(0,1) + s(w,1), Y = t^{'}(0,1) + s^{'}(w,1)$ and $T = \pm(0,1,0,0)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \{\pm(0,1,0,0)\}.$$

Suppose that $(X,Y) \in A_{12}$.

$X = t(0,1) + s(w,1), Y = t^{'}(w,1) + s^{'}(1,w)$ and $T = \pm\Big(0, \frac{1}{1+w}, 0, \frac{1}{1+w}\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \Big\{ \pm \Big(0, \frac{1}{1+w}, 0, \frac{1}{1+w}\Big)\Big\}.$$

Suppose that $(X,Y) \in A_{13}$.

Then $X = t(0,1) + s(w,1), Y = t^{'}(1,w) + s^{'}(1,-w)$ and $T = \pm(0,0,0,1)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \{\pm(0,0,0,1)\}.$$

Suppose that $(X,Y) \in A_{14}$.

Then $X = t(0,1) + s(w,1), Y = t^{'}(w,-1) + s^{'}(0,-1)$ and $T = \pm\Big(0, -\frac{1}{1+w}, 0, \frac{1}{1+w}\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \Big\{ \pm \Big(0, -\frac{1}{1+w}, 0, \frac{1}{1+w}\Big)\Big\}.$$

Suppose that $(X,Y) \in A_{15}$.

Then $X = t(0,1) + s(w,1), Y = t^{'}(1,-w) + s^{'}(w,-1)$ and $T = \pm(0,-1,0,0)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \{\pm(0,-1,0,0)\}.$$

Suppose that $(X,Y) \in A_{21}$.

Then $X = t(w,1) + s(1,w), Y = t^{'}(0,1) + s^{'}(w,1)$ and $T = \pm\Big(0, \frac{1}{1+w}, \frac{1}{1+w}, 0\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \Big\{ \pm \Big(0, \frac{1}{1+w}, \frac{1}{1+w}, 0\Big)\Big\}.$$

Suppose that $(X,Y) \in A_{22}$.

Then $(x_1,y_1) = t(w,1) + s(1,w), (x_2,y_2) = t^{'}(w,1) + s^{'}(1,w)$ and $T = \pm\Big(\frac{1}{(1+w)^2}, \frac{1}{(1+w)^2}, \frac{1}{(1+w)^2}, \frac{1}{(1+w)^2}\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$\begin{aligned} & NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) \\ = \; & \Big\{ \pm \Big(\frac{1}{(1+w)^2}, \frac{1}{(1+w)^2}, \frac{1}{(1+w)^2}, \frac{1}{(1+w)^2}\Big)\Big\}. \end{aligned}$$

Suppose that $(X, Y) \in A_{23}$.

Then $(x_1, y_1) = t(w, 1) + s(1, w), (x_2, y_2) = t^{'}(1, w) + s^{'}(1, -w)$ and $T = \pm\Big(\frac{1}{1+w}, 0, 0, \frac{1}{1+w}\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}({}^2d_*(1, w)^2))((x_1, x_2), (y_1, y_2)) = \Big\{ \pm \Big(\frac{1}{1+w}, 0, 0, \frac{1}{1+w}\Big)\Big\}.$$

Suppose that $(X, Y) \in A_{24}$.

Then $(x_1, y_1) = t(w, 1) + s(1, w), (x_2, y_2) = t^{'}(1, -w) + s^{'}(w, -1)$ and $T = \pm\Big(\frac{1}{1+w}, -\frac{1}{1+w}, \frac{w-1}{1+w}, 0\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}({}^2d_*(1, w)^2))((x_1, x_2), (y_1, y_2)) = \Big\{ \pm \Big(\frac{1}{1+w}, -\frac{1}{1+w}, \frac{w-1}{1+w}, 0\Big)\Big\}.$$

Suppose that $(X, Y) \in A_{25}$.

Then $(x_1, y_1) = t(w, 1) + s(1, w), (x_2, y_2) = t^{'}(w, -1) + s^{'}(0, -1)$ and $T = \pm\Big(0, -\frac{1}{1+w}, -\frac{1}{1+w}, 0\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}({}^2d_*(1, w)^2))((x_1, x_2), (y_1, y_2)) = \Big\{ \pm \Big(0, -\frac{1}{1+w}, -\frac{1}{1+w}, 0\Big)\Big\}.$$

Suppose that $(X, Y) \in A_{31}$.

Then $(x_1, y_1) = t(1, w) + s(1, -w), (x_2, y_2) = t^{'}(0, 1) + s^{'}(w, 1)$ and $T = \pm(0, 0, 1, 0)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}({}^2d_*(1, w)^2))((x_1, x_2), (y_1, y_2)) = \{\pm(0, 0, 1, 0)\}.$$

Suppose that $(X, Y) \in A_{32}$.

Then $(x_1, y_1) = t(1, w) + s(1, -w), (x_2, y_2) = t^{'}(w, 1) + s^{'}(1, w)$ and $T = \pm\Big(\frac{1}{1+w}, 0, \frac{1}{1+w}, 0\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}({}^2d_*(1, w)^2))((x_1, x_2), (y_1, y_2)) = \Big\{ \pm \Big(\frac{1}{1+w}, 0, \frac{1}{1+w}, 0\Big)\Big\}.$$

Suppose that $(X, Y) \in A_{33}$.

Then $(x_1, y_1) = t(1, w) + s(w, -1), (x_2, y_2) = t^{'}(1, w) + s^{'}(w, -1)$ and $T = \pm(1, 0, 0, 0)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}({}^2d_*(1, w)^2))((x_1, x_2), (y_1, y_2)) = \{\pm(1, 0, 0, 0)\}.$$

Suppose that $(X, Y) \in A_{34}$.

Then $(x_1, y_1) = t(1, w) + s(w, -1), (x_2, y_2) = t^{'}(1, -w) + s^{'}(w, -1)$ and $T = \pm\Big(\frac{1}{1+w}, 0, \frac{1}{1+w}, 0\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}({}^2d_*(1, w)^2))((x_1, x_2), (y_1, y_2)) = \Big\{ \pm \Big(\frac{1}{1+w}, 0, -\frac{1}{1+w}, 0\Big)\Big\}.$$

Suppose that $(X,Y) \in A_{35}$.
Then $(x_1,y_1) = t(1,w) + s(w,-1), (x_2,y_2) = t^{'}(w,-1) + s^{'}(0,-1)$ and $T = \pm(0,0,1,0)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \{\pm(0,0,1,0)\}.$$

Suppose that $(X,Y) \in A_{41}$.
Then $(x_1,y_1) = t(1,-w) + s(w,-1), (x_2,y_2) = t^{'}(0,1) + s^{'}(w,1)$ and $T = \pm\Big(0, -\frac{1}{1+w}, \frac{1}{1+w}, 0\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \Big\{ \pm \Big(0, -\frac{1}{1+w}, \frac{1}{1+w}, 0\Big)\Big\}.$$

Suppose that $(X,Y) \in A_{42}$.
Then $(x_1,y_1) = t(1,-w) + s(w,-1), (x_2,y_2) = t^{'}(w,1) + s^{'}(1,w)$ and $T = \pm\Big(\frac{1}{(1+w)^2}, -\frac{1}{(1+w)^2}, \frac{1}{(1+w)^2}, -\frac{1}{(1+w)^2},\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$\begin{aligned} & NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) \\ = \ & \Big\{ \pm \Big(\frac{1}{(1+w)^2}, -\frac{1}{(1+w)^2}, \frac{1}{(1+w)^2}, -\frac{1}{(1+w)^2},\Big)\Big\}. \end{aligned}$$

Suppose that $(X,Y) \in A_{43}$.
Then $(x_1,y_1) = t(1,-w) + s(w,-1), (x_2,y_2) = t^{'}(1,w) + s^{'}(1,-w)$ and $T = \pm\Big(\frac{1}{1+w}, 0, 0, -\frac{1}{1+w}\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \Big\{ \pm \Big(\frac{1}{1+w}, 0, 0, -\frac{1}{1+w}\Big)\Big\}.$$

Suppose that $(X,Y) \in A_{44}$.
Then $(x_1,y_1) = t(1,-w) + s(w,-1), (x_2,y_2) = t^{'}(1,-w) + s^{'}(w,-1)$ and $T = \pm\Big(\frac{1-w}{1+w}, 0, -\frac{1}{1+w}, -\frac{1}{1+w}\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \Big\{ \pm \Big(\frac{1-w}{1+w}, 0, -\frac{1}{1+w}, -\frac{1}{1+w}\Big)\Big\}.$$

Suppose that $(X,Y) \in A_{45}$.
Then $(x_1,y_1) = t(1,-w) + s(w,-1), (x_2,y_2) = t^{'}(w,-1) + s^{'}(0,-1)$ and $T = \pm\Big(0, \frac{1}{1+w}, -\frac{1}{1+w}, 0\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \Big\{ \pm \Big(0, \frac{1}{1+w}, -\frac{1}{1+w}, 0\Big)\Big\}.$$

Suppose that $(X,Y) \in A_{51}$.
Then $(x_1,y_1) = t(w,-1) + s(0,-1), (x_2,y_2) = t^{'}(0,1) + s^{'}(w,1)$ and

$T = \pm(0,-1,0,0)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \{\pm(0,-1,0,0)\}.$$

Suppose that $(X,Y) \in A_{52}$.
Then $(x_1,y_1) = t(w,-1) + s(0,-1), (x_2,y_2) = t'(w,1) + s'(1,w)$ and $T = \pm\Big(0, -\frac{1}{1+w}, 0, -\frac{1}{1+w}\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \Big\{ \pm \Big(0, \frac{1}{1+w}, -\frac{1}{1+w}, 0\Big)\Big\}.$$

Suppose that $(X,Y) \in A_{53}$.
Then $(x_1,y_1) = t(w,-1) + s(0,-1), (x_2,y_2) = t'(1,w) + s'(1,-w)$ and $T = \pm(0,0,0,-1)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \{\pm(0,0,0,-1)\}.$$

Suppose that $(X,Y) \in A_{54}$.
Then $(x_1,y_1) = t(w,-1) + s(0,-1), (x_2,y_2) = t'(1,-w) + s'(w,-1)$ and $T = \pm\Big(0, \frac{1}{1+w}, 0, -\frac{1}{1+w}\Big)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \Big\{ \pm \Big(0, \frac{1}{1+w}, 0, -\frac{1}{1+w}\Big)\Big\}.$$

Suppose that $(X,Y) \in A_{55}$.
Then $(x_1,y_1) = t(w,-1) + s(0,-1), (x_2,y_2) = t'(w,-1) + s'(0,-1)$ and $T = \pm(0,1,0,0)$. By Theorem 2, $\|T\| = 1$. Hence,

$$NA(\mathcal{L}(^2d_*(1,w)^2))((x_1,x_2),(y_1,y_2)) = \{\pm(0,1,0,0)\}.$$

Thus, Claim 2 holds.

It follows that

$$\begin{aligned} |T(X,Y)| &= |T(tU+sV,\ t'U' + s'V')| \\ &= |tt'T(U,U') + ts'T(U,V') + t'sT(V,U') + ss'T(V,V')| \\ &= |tt' + ts' + t's + ss'| = (t+s)(t'+s') = 1 = \|T\|, \end{aligned}$$

which shows that $T \in NA(\mathcal{L}(^2d_*(1,w)^2)(X,Y)$.

*Case 2.* Let $X = tU + sV$ and $Y = t'U' + s'V'$ for some $0 < t, s < 1,\ t+s = 1, t' = 0$ or 1. Then $Y = U'$ or $V'$. Let

$$\mathcal{F}_2 = \Big\{ \pm T \in \mathcal{L}(^2d_*(1,w)^2) : 1 = T(U,Y) = T(V,Y) \text{ if } \operatorname{ext} B_i = \{U,V\},$$
$$|T(W_1,W_2)| \le 1 \text{ for all } W_k \in \operatorname{ext} B_{d_*(1,w)^2}, k = 1,2\Big\}.$$

We claim that $NA(\mathcal{L}(^2d_*(1,w)^2))(X,Y) = \mathcal{F}_2$.

($\subseteq$) : Suppose that $T \in NA(\mathcal{L}(^2d_*(1,w)^2))(X,Y)$.

**Claim 3**. $\Big(T(U,Y) = T(V,Y) = 1\Big)$ or $\Big(T(U,Y) = T(V,Y) = -1\Big)$.

First, we will show that

$$|T(U,Y)| = |T(V,Y)| = 1.$$

Assume the contrary. Without loss of generality we may assume that $|T(V,Y)| < 1$. Let $t^{'} = 0$. Then $Y = V^{'}$. It follows that

$$\begin{aligned} 1 &= |T(X,Y)| = |T(tU + sV,\ V^{'})| = |tT(U,V^{'}) + sT(V,V^{'})| \\ &\leq t|T(U,Y)| + s|T(V,Y)| < t|T(U,Y)| + |T(V,Y)| \leq t + s = 1, \end{aligned}$$

which is a contradiction. Hence,

$$|T(U,Y)| = |T(V,Y)| = 1.$$

Since $t, s > 0$ and

$$|tT(U,Y) + sT(V,Y)| = 1,$$

by Lemma 1, Claim 3 holds.

Let $t^{'} = 1$. Then $Y = U^{'}$. It follows that

$$\begin{aligned} 1 &= |T(X,Y)| = |T(tU + sV,\ U^{'})| = |tT(U,U^{'}) + sT(V,U^{'})| \\ &\leq t|T(U,Y)| + s|T(V,Y)| < t|T(U,Y)| + |T(V,Y)| \leq t + s = 1, \end{aligned}$$

which is a contradiction. Hence,

$$|T(U,Y)| = |T(V,Y)| = 1.$$

Since $t, s > 0$ and

$$|tT(U,Y) + sT(V,Y)| = 1,$$

by Lemma 1, the claim 3 holds. Since

$$1 = \|T\| = \sup_{W_1,W_2 \in \operatorname{ext} B_{d_*(1,w)^2}} |T(W_1,W_2)|,$$

$T \in \mathcal{F}_2$. Hence, $NA(\mathcal{L}(^2d_*(1,w)^2))(X,Y) \subseteq \mathcal{F}_2$.

Notice that $\mathcal{F}_2 \subseteq NA(\mathcal{L}(^2d_*(1,w)^2))(X,Y)$ is obvious.

Therefore, $NA(\mathcal{L}(^2d_*(1,w)^2))(X,Y) = \mathcal{F}_2$.

*Case 3.* Let $X = tU + sV$ and $Y = t^{'}U^{'} + s^{'}V^{'}$ for some $0 < t^{'}, s^{'} < 1$, $t^{'} + s^{'} = 1, t = 0$ or $1$. Then $X = U$ or $V$. Let

$$\begin{aligned} \mathcal{F}_3 = \Big\{ &\pm T \in \mathcal{L}(^2d_*(1,w)^2) : 1 = T(X,U^{'}) = T(X,V^{'}) \\ &\text{if } \operatorname{ext} B_j = \{U^{'}, V^{'}\}, |T(W_1,W_2)| \leq 1 \text{ for all } W_k \in \operatorname{ext} B_{d_*(1,w)^2}, \\ &k = 1,2\Big\}. \end{aligned}$$

By analogous arguments as those of Case 2, $NA(\mathcal{L}({}^2d_*(1,w)^2))(X,Y) = \mathcal{F}_3$.

*Case 4.* Notice that

$$(X,Y) = (U,U'), (U,V'), (V,U') \text{ or } (V,V').$$

Let

$$\mathcal{F}_4 = \Big\{T \in \mathcal{L}({}^2d_*(1,w)^2) : 1 = |T(X,Y)|,\ |T(W_1,W_2)| \leq 1 \text{ for all } W_k \in \operatorname{ext} B_{d_*(1,w)^2}, k = 1,2\Big\}.$$

It is obvious that $NA(\mathcal{L}({}^2d_*(1,w)^2))(X,Y) = \mathcal{F}_4$. This completes the proof.

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*Received*: 23.08.2023. *Accepted:* 30.10.2023