Res. Math. **32**(1), 2024, p. 33–44 ISSN (Print) 2664-4991
doi:10.15421/242403 ISSN (Online) 2664-5009

UDK 517.98, 517.5

M. I. Dmytryshyn[*], **L. I. Dmytryshyn**[**]

[*] Vasyl Stefanyk Precarpathian National University,
Ivano-Frankivsk 76018. *E-mail: marian.dmytryshyn@pnu.edu.ua*
[**] Vasyl Stefanyk Precarpathian National University,
Ivano-Frankivsk 76018. *E-mail: lesia.dmytryshyn@pnu.edu.ua*

Projective tensor products of approximation spaces associated with positive operators

Abstract. In this paper the projective tensor products of approximation spaces associated with positive operators in Banach spaces are characterized. We show that the tensor products of approximation spaces can be considered as the interpolation spaces generated by K-method of real interpolation. The inequalities that provide a sharp estimates of best approximations by analytic vectors of positive operators on projective tensor products are established. Application to spectral approximations of the regular elliptic operators on projective tensor products of Lebesgue spaces is shown.

Key words: tensor products, positive operators, approximation spaces, Bernstein-Jackson inequalities

Анотація. У статті описано проєктивні тензорні добутки апроксимаційних просторів, асоційованих з позитивними операторами в банахових просторах. Показано, що тензорні добутки апроксимаційних просторів можна розглядати як інтерполяційні простори, породжені K-методом дійсної інтерполяції. Встановлено нерівності, що дають точні оцінки найкращих наближень аналітичними векторами позитивних операторів на проєктивних тензорних добутках. Показано застосування до спектральних апроксимацій регулярних еліптичних операторів на проєктивних тензорних добутках просторів Лебега.

Ключові слова: тензорні добутки, позитивні оператори, апроксимаційні простори, нерівності Бернштейна і Джексона

MSC2020: Pri 47A58, Sec 46M05, 41A17

1. Introduction

One of the basic notion of approximation theory is the concept of best approximations by E-functional, through which approximation spaces are determined [3, 16, 18]. A close connection between approximation and interpolation spaces allows to obtain the explicit formulas for estimating the

© *M. I. DMYTRYSHYN, L. I. DMYTRYSHYN, 2024*

approximations. The results obtained in this direction are formulated as inverse and direct theorems or Bernstein-Jackson inequalities [1, 2, 10, 12].

Applications of analytic vectors associated with the unbounded linear operators to approximation problems can be found in [8, 11, 17]. A relation between spectral subspaces and invariant subspaces of analytic vectors allows considering spectral approximations [9].

A special scale of approximation spaces associated with positive operators are investigated in [4]. Estimates of best approximations by exponential type analytic functions in Gaussian random variables with respect to the Malliavin derivative are established in [13].

Our goal is to extend inverse and direct theorems in the form of Bernstein-Jackson inequalities on a case of best approximations on the projective tensor products of approximation spaces associated with positive operators in Banach spaces, and furthermore to calculate the explicit formulas for constants in these inequalities. We also consider the spectral approximations of the regular elliptic operators on projective tensor products of Lebesgue spaces.

2. Positive operators on projective tensor products of Banach spaces

Let $\left\{\mathfrak{X}_j, \|.\|_{\mathfrak{X}_j}\right\}_{j=1}^{J}$ be a finite collection of Banach spaces on $\mathbb{C}$ and $\otimes_j \mathfrak{X}_j := \mathfrak{X}_1 \otimes \ldots \otimes \mathfrak{X}_J$ be a tensor product with projective norm

$$\|w\|_{\otimes_j \mathfrak{X}_j} = \inf_{w=\sum_n \otimes_j x_n^j} \sum_{n=1}^{N} \|x_n^1\|_{\mathfrak{X}_1} \cdot \ldots \cdot \|x_n^J\|_{\mathfrak{X}_J},$$

where inf takes on all representation of $w \in \otimes_j \mathfrak{X}_j$ as $w = \sum_{n=1}^{N} \otimes_j x_n^j$ with finite N, $x_n^j \in \mathfrak{X}_j$ and $\otimes_j x_n^j := x_n^1 \otimes \ldots \otimes x_n^J \in \otimes_j \mathfrak{X}_j$. Completeness of $\otimes_j \mathfrak{X}_j$ on the projective norm we denote by $\tilde{\otimes}_j \mathfrak{X}_j := \mathfrak{X}_1 \tilde{\otimes} \ldots \tilde{\otimes} \mathfrak{X}_J$.

Let A_j $(j = 1, \ldots, J)$ be a positive operator with dense domain $\mathcal{C}^1(A_j)$ in $\mathfrak{X}_j$ (see [19, Definition 1.14.1]. Then there exists a number $c > 0$ such that $\|(A_j - tI)^{-1}\| \leq c/(1 + |t|)$, $t \in (-\infty, 0]$. An example of a positive operator is any positive-definite self-adjoint operator in a Hilbert space. If A_j is the infinitesimal operator of strongly continuous semi-group $U(t)$, and there exist $M \geq 0, \beta < 0$, such that $\|U(t)\| \leq Me^{\beta t}$, $0 \leq t < \infty$, then by Hille-Iosida theorem, $-A_j$ is a positive operator. However, there exist positive operators which are not infinitesimal operators of suitable semi-groups.

Let $\mathcal{C}(A_j^k)$ $(k \in \mathbb{Z}_+)$ be a domain of A_j^k and $\mathcal{C}(A_j^\infty) := \bigcap_{k \in \mathbb{Z}_+} \mathcal{C}(A_j^k)$. We also consider a fractional power A_j^α for $\alpha \in \mathbb{C}$ [19, Section 1.15.1]. The domain $\mathcal{C}(A_j^\alpha)$ of A_j^α we consider as a Banach space with the norm $\|x\|_{\mathcal{C}(A_j^\alpha)} = \|A_j^\alpha x\|_{\mathfrak{X}_j}$, $x \in \mathcal{C}(A_j^\alpha)$. A_j^α is a continuous operator for $\operatorname{Re}\alpha < 0$ and A_j^α is an isomorphic mapping from $\mathcal{C}(A_j^\alpha)$ onto $\mathfrak{X}_j$ for $\operatorname{Re}\alpha > 0$ [19, Theorem 1.15.2].

Following [4], for any $\nu > 0$ we define the spaces of analytic vectors of A_j

$$\mathcal{E}^\nu(A_j^\alpha) = \left\{x \in \mathcal{C}(A_j^\infty) \colon \|x\|_{\mathcal{E}^\nu(A_j^\alpha)} = \sum_{k \in \mathbb{Z}_+} \|(A_j/\nu)^k x\|_{\mathcal{C}(A_j^\alpha)} < \infty\right\}.$$

Let $\mathcal{E}(A_j^\alpha)=\bigcup_{\nu>0}\mathcal{E}^\nu(A_j^\alpha)$ be a subspace with the quasi-norm

$$\|x\|_{\mathcal{E}(A_j^\alpha)}=\|x\|_{\mathfrak{X}_j}+\inf\left\{\nu>0\colon x\in\mathcal{E}^\nu(A_j^\alpha)\right\},$$

so that $\|x+y\|_{\mathcal{E}(A_j^\alpha)}\le\|x\|_{\mathcal{E}(A_j^\alpha)}+\|y\|_{\mathcal{E}(A_j^\alpha)}$ for all $x,y\in\mathcal{E}(A_j^\alpha)$. Denote by $\otimes_j\mathcal{E}(A_j^\alpha):=\mathcal{E}(A_1^\alpha)\otimes\ldots\otimes\mathcal{E}(A_J^\alpha)$ the tensor product with the projective quasi-norm

$$\|w\|_{\otimes_j\mathcal{E}(A_j^\alpha)}=\inf_{w=\sum_n\otimes_j x_n^j}\sum_{n=1}^N\|x_n^1\|_{\mathcal{E}(A_1^\alpha)}\cdot\ldots\cdot\|x_n^J\|_{\mathcal{E}(A_J^\alpha)},$$

and $\tilde\otimes_j\mathcal{E}(A_j^\alpha):=\mathcal{E}(A_1^\alpha)\tilde\otimes\ldots\tilde\otimes\mathcal{E}(A_J^\alpha)$ its completeness on this quasi-norm.

Note that any element w of completeness $\tilde\otimes_j\mathcal{E}(A_j^\alpha)$ is the sum of an absolutely convergent series $w=\sum_n\otimes_j x_n^j$, where $\otimes_j x_n^j\in\otimes_j\mathcal{E}(A_j^\alpha)$.

Indeed, $\forall\,\varepsilon>0$, $\exists\,w_n\in\otimes_j\mathcal{E}(A_j^\alpha)$ such that $\|w-w_n\|_{\tilde\otimes_j\mathcal{E}(A_j^\alpha)}<\varepsilon/2^{n+2}$. Since $\|w_{n+1}-w_n\|_{\otimes_j\mathcal{E}(A_j^\alpha)}\le\|w_{n+1}-w\|_{\tilde\otimes_j\mathcal{E}(A_j^\alpha)}+\|w-w_n\|_{\tilde\otimes_j\mathcal{E}(A_j^\alpha)}<\varepsilon/2^{n+1}$ for all $n\in\mathbb{N}$, then every $w_{n+1}-w_n$ in $\otimes_j\mathcal{E}(A_j^\alpha)$ has representation $w_{n+1}-w_n=\sum_{l=l_n+1}^{l_{(n+1)}}\otimes_j x_l^j$, where $\sum_{l=l_n+1}^{l_{(n+1)}}\|x_l^1\|_{\mathcal{E}(A_1^\alpha)}\ldots\|x_l^J\|_{\mathcal{E}(A_J^\alpha)}\le\|w_{n+1}-w_n\|_{\otimes_j\mathcal{E}(A_j^\alpha)}+\varepsilon/2^{n+1}\le\varepsilon/2^n$. Taking into account that $\|w_1\|_{\otimes_j\mathcal{E}(A_j^\alpha)}\le\|w\|_{\tilde\otimes_j\mathcal{E}(A_j^\alpha)}+\|w-w_1\|_{\tilde\otimes_j\mathcal{E}(A_j^\alpha)}<\|w\|_{\tilde\otimes_j\mathcal{E}(A_j^\alpha)}+\varepsilon/2$, an element w_1 can be representation in the form $w_1=\sum_{l=1}^{l_1}\otimes_j x_l^j$, such that $\sum_{l=1}^{l_1}\|x_l^1\|_{\mathcal{E}(A_1^\alpha)}\ldots\|x_l^J\|_{\mathcal{E}(A_J^\alpha)}\le\|w_1\|_{\otimes_j\mathcal{E}(A_j^\alpha)}+\varepsilon/2\le\|w\|_{\tilde\otimes_j\mathcal{E}(A_j^\alpha)}+\varepsilon$. So, we have $\forall\varepsilon>0$, $w=w_1+\sum_n(w_{n+1}-w_n)=\sum_l\otimes_j x_l^j$, while $\sum_l\|x_l^1\|_{\mathcal{E}(A_1^\alpha)}\ldots\|x_l^J\|_{\mathcal{E}(A_J^\alpha)}\le\|w\|_{\tilde\otimes_j\mathcal{E}(A_j^\alpha)}+\varepsilon+\sum_{n=1}^\infty\varepsilon/2^n=\|w\|_{\tilde\otimes_j\mathcal{E}(A_j^\alpha)}+2\varepsilon$.

Let us define the approximation E-functional $E(t,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j)=\inf\left\{\|w-w_0\|_{\tilde\otimes_j\mathfrak{X}_j}\colon w_0\in\tilde\otimes_j\mathcal{E}(A_j^\alpha),\|w_0\|_{\tilde\otimes_j\mathcal{E}(A_j^\alpha)}\le t\right\}$ for all $w\in\tilde\otimes_j\mathfrak{X}_j$.

For a pair indexes $\{0<s<\infty,0<\tau\le\infty\}$ the approximation spaces is defined as

$$\mathcal{B}^{s,\alpha}_{\tau,\otimes_j}:=\mathcal{B}^s_\tau(\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j)=\{w\in\tilde\otimes_j\mathfrak{X}_j:\|w\|_{\mathcal{B}^{s,\alpha}_{\tau,\otimes_j}}<\infty\},$$

with the quasi-norm

$$\|w\|_{\mathcal{B}^{s,\alpha}_{\tau,\otimes_j}}=\begin{cases}\left(\int_0^\infty\left[t^sE(t,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j)\right]^\tau dt/t\right)^{1/\tau}, & 0<\tau<\infty,\\ \sup\limits_{t>0}t^sE(t,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j), & \tau=\infty.\end{cases}$$

For $0<\theta<1$ and $1\le q\le\infty$, we define the interpolation space

$$\left(\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j\right)_{\theta,q}=\left\{w\in\tilde\otimes_j\mathfrak{X}_j\colon\|w\|_{\left(\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j\right)_{\theta,q}}<\infty\right\},$$

with the quasi-norm

$$\|w\|_{\left(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j\right)_{\theta,q}}=\begin{cases}\Big(\displaystyle\int_0^\infty\left[t^{-\theta}K(t,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)\right]^q\frac{dt}{t}\Big)^{\frac{1}{q}}, & q<\infty,\\ \sup\limits_{t>0}t^{-\theta}K(t,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j), & q=\infty,\end{cases}$$

where $K(t,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)=\inf\limits_{w=w_0+w_1}\left(\|w_0\|^2_{\tilde{\otimes}_j\mathcal{E}(A_j^\alpha)}+t^2\|w_1\|^2_{\tilde{\otimes}_j\mathfrak{X}_j}\right)^{1/2}$, $w_0\in\tilde{\otimes}_j\mathcal{E}(A_j^\alpha)$, $w_1\in\tilde{\otimes}_j\mathfrak{X}_j$, for $t>0$.

For details of K-method real interpolation of quasi-normed spaces we refer the reader to [3,16].

Theorem 1. *For $0<\theta<1$, $\tau=\theta q$, $s=1/\theta-1$ the following isomorphism holds,*

$$\mathcal{B}^s_\tau(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)=\left(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j\right)^{1/\theta}_{\theta,q}, \tag{2.1}$$

where $\left(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j\right)^{1/\theta}_{\theta,q}$ is defined to be $\left(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j\right)_{\theta,q}$ endowed with the quasi-norm $\|w\|^{1/\theta}_{(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)_{\theta,q}}$.

Proof. Let $(x_n)_{n=1}^\infty\subset\mathcal{E}(A_j^\alpha)$ be a fundamental sequence, i.e., $\forall\varepsilon>0$, $\exists n_\varepsilon$: $\|x_n-x_m\|_{\mathcal{E}(A_j^\alpha)}<\varepsilon$, $(\forall n,m\geq n_\varepsilon)$. Then

$$\inf\{\nu\colon x_n-x_m\in\mathcal{E}^\nu(A_j^\alpha)\}<\varepsilon\quad\text{and}\quad\|x_n-x_m\|_{\mathfrak{X}_j}<\varepsilon,$$

in particular, $x_n-x_m\in\mathcal{E}^{\varepsilon_0}(A_j^\alpha)$, $(\forall n,m\geq n_\varepsilon)$ for $\varepsilon_0<\varepsilon$. By Laplace's transform L, for each $x\in\mathcal{E}^\varepsilon(A_j^\alpha)$ there exists the entire function $x(\lambda)$ in variable $\lambda\in\mathbb{C}$ of an exponential type $h^x<\varepsilon$,

$$x(\lambda)=\sum_{k\in\mathbb{Z}_+}\|A_j^kx\|_{\mathcal{C}(A_j^\alpha)}\lambda^k/k!$$

such that

$$\|\varepsilon x\|_{\mathcal{E}^\varepsilon(A_j^\alpha)}=L[x(t)]=\int_0^\infty x(t)\exp(-\varepsilon t)\,dt,$$

where $g(z)=\|zx\|_{\mathcal{E}^{|z|}(A_j^\alpha)}$ is the function in variable $\{z\in\mathbb{C}\colon|z|>h^x\}$. Using an inverse transformation L^{-1}, we obtain

$$x(\lambda)=L^{-1}[g(z)]=\frac{1}{2\pi i}\oint\exp(\lambda z)\|zx\|_{\mathcal{E}^{|z|}(A_j^\alpha)}dz,\quad|z|=\varepsilon_0,\quad\lambda\in\mathbb{C},$$

where $h^x<\varepsilon_0<\varepsilon$. Since $x_n-x_m\in\mathcal{E}^{\varepsilon_0}(A_j^\alpha)$, $(\forall n,m\geq n_\varepsilon)$, that a set of functions

$$\Phi=\left\{\exp(-\varepsilon_0t)\|x_n-x_m\|_{\mathcal{E}^{|z|}(A_j^\alpha)}\colon n,m\geq n_\varepsilon,\ |z|=\varepsilon_0,\ t\in[0,\infty)\right\}$$

is uniformly bounded in $\{z \in \mathbb{C}\colon |z| = \varepsilon_0\} \times [0,\infty)$. Thus, a set of functions

$$\{[0,\infty) \ni t \mapsto (x_n - x_m)(t)\exp(-\varepsilon_0 t)\colon n,m \geq n_\varepsilon\}$$

is uniformly bounded by a constant K_ε for all $t \geq 0$ and $n,m \geq n_\varepsilon$.

By Bernstein's compactness theorem [15, Theorem 3.3.6] there exists a convergent subsequence $(x_{n_i} - x_{m_i})(t)\exp(-\varepsilon_0 t)$, $(i \in \mathbb{N})$ with respect to the topology of uniform convergence in variable $t \in [0,r]$ for all $r > 0$. Thus, $\forall n_i, m_i \geq n_\varepsilon$:

$$\begin{aligned}
\|x_{n_i} - x_{m_i}\|_{\mathcal{E}^\varepsilon(A_j^\alpha)} &= \frac{1}{\varepsilon}L[(x_{n_i} - x_{m_i})(t)] = \frac{1}{\varepsilon}\int_0^\infty (x_{n_i} - x_{m_i})(t)\exp(-\varepsilon t)dt \\
&= \frac{1}{\varepsilon}\int_0^\infty [(x_{n_i} - x_{m_i})(t)\exp(-\varepsilon_0 t)]\exp[-(\varepsilon-\varepsilon_0)t]\,dt \\
&= \frac{1}{\varepsilon}\lim_{r\to\infty}\int_0^r [(x_{n_i} - x_{m_i})(t)\exp(-\varepsilon_0 t)]\exp[-(\varepsilon-\varepsilon_0)t]\,dt \\
&\leq \frac{1}{\varepsilon}K_\varepsilon \lim_{r\to\infty}\int_0^r \exp[-(\varepsilon-\varepsilon_0)t]\,dt = M_\varepsilon.
\end{aligned}$$

Since $\|x_{n_i} - x_{m_i}\|_{\mathcal{E}^\varepsilon(A_j^\alpha)} = \sum_k \|(A_j/\varepsilon)^k(x_{n_i} - x_{m_i})\|_{\mathcal{C}(A_j^\alpha)}$, that

$$\|A_j^k(x_{n_i} - x_{m_i})\|_{\mathcal{C}(A_j^\alpha)} \leq \varepsilon^k \|x_{n_i} - x_{m_i}\|_{\mathcal{E}^\varepsilon(A_j^\alpha)} \leq \varepsilon^k M_\varepsilon, \quad (\forall n_i, m_i \geq n_\varepsilon,\ \forall k \geq 1).$$

So, $\left(A_j^k x_{n_i}\right)_{i=1}^\infty$, $(\forall k \geq 1)$ are fundamental in $\mathcal{C}(A_j^\alpha)$.

If $x_n \to 0$ in $\mathcal{C}(A_j^\alpha)$, then $\|A^k x_{n_i}\|_{\mathcal{C}(A_j^\alpha)} \to 0$, $(\forall k \geq 1)$ and $x_{n_i} \to 0$ as $i \to \infty$ in $\mathcal{E}^\varepsilon(A_j^\alpha)$, $(\forall \varepsilon > 0)$. Thus,

$$\forall \varepsilon > 0,\ \exists n_\varepsilon\colon \|x_{n_i}\|_{\mathcal{E}(A_j^\alpha)} = \|x_{n_i}\|_{\mathfrak{X}_j} + \inf\{\nu\colon x_{n_i} \in \mathcal{E}^\nu(A_j^\alpha)\} < 2\varepsilon, \quad (\forall n_i \geq n_\varepsilon).$$

It means that $x_{n_i} \to 0$ as $i \to \infty$ and, as a consequence, $x_n \to 0$ in $\mathcal{E}(A_j^\alpha)$.

Considering a fundamental sequence $(w_n)_{n=1}^\infty$ in $\tilde{\otimes}_j \mathcal{E}(A_j^\alpha)$ such that $w_n \to 0$ in $\tilde{\otimes}_j \mathfrak{X}_j$, we get $w_n \to 0$ in $\tilde{\otimes}_j \mathcal{E}(A_j^\alpha)$. Taking into account the continuous embedding $\tilde{\otimes}_j \mathcal{E}(A_j^\alpha) \hookrightarrow \tilde{\otimes}_j \mathfrak{X}_j$, the required equality (2.1) directly follows from [3, Theorem 7.1.7].

3. Estimates of best approximations on projective tensor products

In this section, we prove the inverse and direct approximation theorems on the projective tensor products of Banach spaces in the form of Bernstein-Jackson inequalities with exact constants. Such inequalities are expressed by the quasi-norms of approximation spaces $\mathcal{B}_\tau^s(\tilde{\otimes}_j \mathcal{E}(A_j^\alpha), \tilde{\otimes}_j \mathfrak{X}_j)$ associated with

the positive operators A_j $(j = 1, \dots, J)$. The approximation constants determined by the normalization factor

$$N_{\theta,q} := \begin{cases} \Big(\int_0^\infty t^{q(1-\theta)-1}(1+t^2)^{-q/2} dt \Big)^{-1/q}, & 1 \le q < \infty, \\ \theta^{-\theta/2}(1-\theta)^{-(1-\theta)/2}, & q = \infty. \end{cases}$$

We note that $N_{\theta,q} = N_{1-\theta,q}$ and

$$\min(1,s)/\sqrt{2} \le s/\sqrt{1+s^2} \le \min(1,s),$$

so that $N'_{\theta,q} \le N_{\theta,q} \le \sqrt{2} N'_{\theta,q}$, where

$$N'_{\theta,q} := \begin{cases} [q\theta(1-\theta)]^{1/q}, & 1 \le q < \infty, \\ 1, & q = \infty. \end{cases}$$

The normalization factors $N_{\theta,q}$ and $N'_{\theta,q}$ are used in [5, 6] and [9] correspondingly.

Theorem 2. *Let $0 < s < \infty, 0 < \tau \le \infty$ and $\kappa_{s,\tau} = (\tau^2(1+s))^{1/\tau}$. The following inequalities hold:*

$$\|w\|_{\mathcal{B}^{s,\alpha}_{\tau,\tilde{\otimes}_j}} \le c_{s,\tau} \|w\|^s_{\tilde{\otimes}_j \mathcal{E}(A_j^\alpha)} \|w\|_{\tilde{\otimes}_j \mathfrak{X}_j}, w \in \tilde{\otimes}_j \mathcal{E}(A_j^\alpha), \tag{3.1}$$

$$E(t, w; \tilde{\otimes}_j \mathcal{E}(A_j^\alpha), \tilde{\otimes}_j \mathfrak{X}_j) \le C_{s,\tau} t^{-s} \|w\|_{\mathcal{B}^{s,\alpha}_{\tau,\tilde{\otimes}_j}}, w \in \mathcal{B}^s_\tau(\tilde{\otimes}_j \mathcal{E}(A_j^\alpha), \tilde{\otimes}_j \mathfrak{X}_j), \tag{3.2}$$

with $c_{s,\tau} = \kappa_{s,\tau} N^{-(1+s)}_{1/(1+s),\tau(1+s)}$, $C_{s,\tau} = 2^{(1+s)/2} \kappa^{-1}_{s,\tau} N^{(1+s)}_{1/(1+s),\tau(1+s)}$ if $\tau < \infty$ and $c_{s,\infty} = N^{-(1+s)}_{1/(1+s),\infty}$, $C_{s,\infty} = 1$.

Proof. Let $1 \le q < \infty$ and $w \ne 0$. We have

$$\begin{aligned} \|w\|^q_{(\tilde{\otimes}_j \mathcal{E}(A_j^\alpha), \tilde{\otimes}_j \mathfrak{X}_j)_{\theta,q}} &\le \|w\|^q_{\tilde{\otimes}_j \mathfrak{X}_j} \int_0^r t^{-1+q(1-\theta)} dt + \|w\|^q_{\tilde{\otimes}_j \mathcal{E}(A_j^\alpha)} \int_r^\infty t^{-1-\theta q} dt \\ &= \frac{1}{q(1-\theta)} r^{q(1-\theta)} \|w\|^q_{\tilde{\otimes}_j \mathfrak{X}_j} + \frac{1}{\theta q} r^{-\theta q} \|w\|^q_{\tilde{\otimes}_j \mathcal{E}(A_j^\alpha)} \\ &= \frac{1}{q\theta(1-\theta)} \left(\|w\|^{1-\theta}_{\tilde{\otimes}_j \mathcal{E}(A_j^\alpha)} \|w\|^\theta_{\tilde{\otimes}_j \mathfrak{X}_j} \right)^q \end{aligned}$$

with $r = \|w\|_{\tilde{\otimes}_j \mathcal{E}(A_j^\alpha)} / \|w\|_{\tilde{\otimes}_j \mathfrak{X}_j}$.

If $q = \infty$, then

$$\|w\|_{(\tilde{\otimes}_j \mathcal{E}(A_j^\alpha), \tilde{\otimes}_j \mathfrak{X}_j)_{\theta,\infty}} \le \sup_{t>0} \frac{t^{1-\theta}}{\sqrt{1+t^2}} \|w\|^{1-\theta}_{\tilde{\otimes}_j \mathcal{E}(A_j^\alpha)} \|w\|^\theta_{\tilde{\otimes}_j \mathfrak{X}_j}.$$

Thus, for $1 \le q \le \infty$, we get

$$\|w\|_{(\tilde{\otimes}_j \mathcal{E}(A_j^\alpha), \tilde{\otimes}_j \mathfrak{X}_j)_{\theta,q}} \le N^{-1}_{\theta,q} \|w\|^{1-\theta}_{\tilde{\otimes}_j \mathcal{E}(A_j^\alpha)} \|w\|^\theta_{\tilde{\otimes}_j \mathfrak{X}_j}. \tag{3.3}$$

Consider the functional

$$K_\infty(t,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j)=\inf_{w=w_0+w_1}\max\left(\|w_0\|_{\tilde\otimes_j\mathcal{E}(A_j^\alpha)},t\|w_1\|_{\tilde\otimes_j\mathfrak{X}_j}\right).$$

As follows from [16, Remark 3.1],

$$\begin{aligned}K_\infty(t,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j) &\le K(t,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j)\\ &\le \sqrt{2}K_\infty(t,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j). \qquad (3.4)\end{aligned}$$

Integrating by parts with the change $v=t/E(t,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j)$, we get

$$\begin{aligned}&\int_0^\infty (v^{-\theta}K_\infty(v,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j))^q dv/v\\ &=-\frac{1}{\theta q}\int_0^\infty K_\infty(v,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j)^q dv^{-\theta q}\\ &=\frac{1}{\theta q}\int_0^\infty v^{-\theta q}dK_\infty(v,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j)^q\\ &=\frac{1}{\theta q}\int_0^\infty (t/E(t,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j))^{-\theta q}dt^q\\ &=\frac{1}{\theta q^2}\int_0^\infty (t^sE(t,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j))^{\theta q}dt/t \quad \text{with} \quad s=1/\theta-1.\end{aligned}$$

Using (3.4), we have

$$\begin{aligned}\frac{1}{\theta q^2}\|w\|^{\theta q}_{\mathcal{B}^{s,\alpha}_{\tau,\otimes_j}} &= \frac{1}{\theta q^2}\int_0^\infty (t^sE(t,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j))^{\theta q}dt/t\\ &=\int_0^\infty (v^{-\theta}K_\infty(v,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j))^q dv/v\\ &\le\int_0^\infty (v^{-\theta}K(v,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j))^q dv/v\\ &=\|w\|^q_{\left(\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j\right)_{\theta,q}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|w\|^q_{\left(\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j\right)_{\theta,q}} &= \int_0^\infty (v^{-\theta}K(v,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j))^q dv/v\\ &\le 2^{q/2}\int_0^\infty (v^{-\theta}K_\infty(v,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j))^q dv/v\\ &=2^{q/2}\frac{1}{\theta q^2}\int_0^\infty (t^sE(t,w;\tilde\otimes_j\mathcal{E}(A_j^\alpha),\tilde\otimes_j\mathfrak{X}_j))^{\theta q}dt/t\\ &=2^{q/2}\frac{1}{\theta q^2}\|w\|^{\theta q}_{\mathcal{B}^{s,\alpha}_{\tau,\otimes_j}}.\end{aligned}$$

As a result, we get

$$\|w\|^q_{\left(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j\right)_{\theta,q}} \le 2^{q/2}(\theta q^2)^{-1}\|w\|^{\theta q}_{\mathcal{B}^{s,\alpha}_{\tau,\otimes_j}} \le 2^{q/2}\|w\|^q_{\left(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j\right)_{\theta,q}} \tag{3.5}$$

with $\tau = \theta q$.

Using [3, Lemma 7.1.2], we get

$$\begin{aligned} t^sE(t,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j))^\theta &\le v^{-\theta}K_\infty(v,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j) \\ &\le \left(t^sE(t-0,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)\right)^\theta, \end{aligned} \tag{3.6}$$

which gives $\|w\|^\theta_{\mathcal{B}^{s,\alpha}_{\infty,\otimes_j}} \le \|w\|_{\left(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j\right)_{\theta,\infty}}$.

On the other hand,

$$\begin{aligned} v^{-\theta}K(v,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j) &\le \sqrt{2}v^{-\theta}K_\infty(v,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j) \\ &\le \sqrt{2}(t^sE(t-0,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j))^\theta \\ &\le \sqrt{2}(\sup_{t>0} t^sE(t,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j))^\theta, \end{aligned}$$

which gives $\|w\|_{\left(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j\right)_{\theta,\infty}} \le \sqrt{2}\|w\|^\theta_{\mathcal{B}^{s,\alpha}_{\infty,\otimes_j}}$. Now applying (3.3), we get

$$\|w\|^\theta_{\mathcal{B}^{s,\alpha}_{\tau,\otimes_j}} \le \begin{cases} (\theta q^2)^{1/q}N_{\theta,q}^{-1}\|w\|^{1-\theta}_{\tilde{\otimes}_j\mathcal{E}(A_j^\alpha)}\|w\|^\theta_{\tilde{\otimes}_j\mathfrak{X}_j}, & q<\infty \\ N_{\theta,\infty}^{-1}\|w\|^{1-\theta}_{\tilde{\otimes}_j\mathcal{E}(A_j^\alpha)}\|w\|^\theta_{\tilde{\otimes}_j\mathfrak{X}_j}, & q=\infty. \end{cases} \tag{3.7}$$

Setting $s = 1/\theta - 1$ and $\tau = \theta q$ in (3.7), we get the required inequalities (3.1).

By integration of both sides of

$$\min(1,v/t)K(t,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j) \le K(v,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)$$

we get

$$\left(\int_0^\infty \left(v^{-\theta}\min(1,v/t)\right)^q\frac{dv}{v}\right)^{1/q} K(t,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j) \le$$

$$\le \left(\int_0^\infty \left(v^{-\theta}K(v,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)\right)^q\frac{dv}{v}\right)^{1/q} = \|w\|_{\left(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j\right)_{\theta,q}},$$

$$\left(\int_0^\infty \left(v^{-\theta}\min(1,v/t)\right)^q\frac{dv}{v}\right)^{1/q} = \left(\int_0^t v^{(1-\theta)q-1}t^{-q}dv + \int_t^\infty v^{-\theta q-1}dv\right)^{1/q}$$

$$= \frac{1}{[q\theta(1-\theta)]^{1/q}t^\theta}.$$

It follows that

$$K(t,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j) \le [q\theta(1-\theta)]^{1/q}t^\theta\|w\|_{\left(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j\right)_{\theta,q}}.$$

Taking into account (3.4) and (3.6), we have

$$\begin{aligned} v^{1-\theta}E(v,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)^\theta &\leq t^{-\theta}K_\infty(t,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j) \\ &\leq [q\theta(1-\theta)]^{1/q}\|w\|_{\left(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j\right)_{\theta,q}}. \end{aligned}$$

Applying (3.5), we get

$$v^{1-\theta}E(v,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)^\theta \leq \sqrt{2}(\theta q^2)^{-1/q}N_{\theta,q}\|w\|^\theta_{\mathcal{B}^{s,\alpha}_{\tau,\otimes_j}}.$$

So, if $s=(1-\theta)/\theta$ and $\tau=\theta q$, one obtains that (3.2) for $1\leq q<\infty$. If $q=\infty$ then

$$t^sE(t,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)\leq\sup_{t>0}t^sE(t,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)=\|w\|_{\mathcal{B}^{s,\alpha}_{\infty,\otimes_j}}$$

for all $w\in\mathcal{B}^{s,\alpha}_{\infty,\otimes_j}$. Thus, the inequality (3.2) is valid and proof is complete.

Remark 1. If $\tau=2/(1+s)$ then for $w\in\mathcal{B}^s_{2/(1+s)}(\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)$, we have

$$E(t,w;\tilde{\otimes}_j\mathcal{E}(A_j^\alpha),\tilde{\otimes}_j\mathfrak{X}_j)\leq C_st^{-s}\|w\|_{\mathcal{B}^{s,\alpha}_{2/(1+s),\otimes_j}}, \tag{3.8}$$

with $C_s=[((1+s)/\pi)\sin(\pi/(1+s))]^{(1+s)/2}$.

Indeed, by [14, Exercise B.5], we have $N_{\theta,2}=((2/\pi)\sin(\pi\theta))^{1/2}$. So, if $\tau=2/(1+s)$ the inequality (3.2) yields (3.8).

4. Application

In this section, we consider the projective tensor products of approximation spaces associated with the regular elliptic differential operators.

In the space $L_{p_j}(\Omega)$ $(1<p_j<\infty, j=1,\dots,J)$ over an open bounded set $\Omega\subset\mathbb{R}^n$ with the infinitely smooth boundary $\partial\Omega$, we consider the closed linear operator A_j with the domain $W^{2m}_{p_j,A_j}(\Omega)=\left\{u\in W^{2m}_{p_j}(\Omega)\colon b_{jl}u\mid_{\partial\Omega}=0,\ l=1,\dots,m\right\}$ via the regular elliptic system [19, Definition 5.2.1/4]

$$\begin{aligned} (A_ju)(\xi)&=\sum_{|\beta|\leq 2m}a_{j\beta}(\xi)D^\beta u(\xi),\quad a_{j\beta}\in C^\infty(\bar\Omega),\quad \bar\Omega=\Omega\cup\partial\Omega,\\ (b_{jl}u)(\xi)&=\sum_{|\beta|\leq m_l}b_{jl,\beta}(\xi)D^\beta u(\xi),\quad b_{jl,\beta}\in C^\infty(\partial\Omega),\ l=1,\dots,m. \end{aligned}$$

We assume that $0\in\rho(A_j)$. Then for sufficiently large positive numbers $\rho\geq\rho_0$ the operator $A_j+\rho$ is a positive [19, Theorem 4.9.1]. Since the spaces $\mathcal{E}(A_j^\alpha)$ and $\mathcal{E}((A_j+\rho)^\alpha)$ coincide, we can assume that A_j is a positive operator in $L_{p_j}(\Omega)$. It follows that the spectrum $\sigma(A_j)$ is discrete and is independent on p_j [19, Section 5.4.4].

Let $\mathcal{R}_{\lambda_n}(A_j) = \left\{u \in L_{p_j}(\Omega)\colon (\lambda_n - A_j)^{r_n} u = 0\right\}$ be the spectral subspace, corresponding to the eigenvalue λ_n of multiplicity r_n. In [7, Theorem 2.2] it is proved that the following equality holds,

$$\mathcal{E}^{\nu}(A_j^k) = \operatorname{span}\left\{\mathcal{R}_{\lambda_n}(A_j)\colon |\lambda_n|^{k+1} < \nu\right\} \tag{4.1}$$

for any $k \in \mathbb{N}$. Let $0 < \theta < 1$, $k_0, k_1 \in \mathbb{N}$ and $\alpha = (1-\theta)k_0 + \theta k_1$. Applying [19, Theorem 1.15.3], as well as, taking into account (4.1), we obtain

$$\mathcal{E}^{\nu}(A_j^{\alpha}) = \operatorname{span}\left\{\mathcal{R}_{\lambda_n}(A_j)\colon |\lambda_n| < \min\left(\nu^{1/(k_0+1)}, \nu^{1/(k_1+1)}\right)\right\}. \tag{4.2}$$

Denote by $\mathcal{R}^{\nu}(A_j^{\alpha})$ the complex linear span in $L_{p_j}(\Omega)$ of all spectral subspaces $\mathcal{R}_{\lambda_n}(A_j)$ such that $|\lambda_n|^{\alpha+1} < \nu$. Let us define on $\mathcal{R}(A_j^{\alpha}) = \bigcup_{\nu>0} \mathcal{R}^{\nu}(A_j^{\alpha})$ the quasi-norm

$$\|u\|_{\mathcal{R}(A_j^{\alpha})} = \|u\|_{L_{p_j}(\Omega)} + \inf\left\{\nu > 0\colon u \in \mathcal{R}^{\nu}(A_j^{\alpha})\right\}.$$

Thus, we have

$$\mathcal{E}(A_j^{\alpha}) = \mathcal{R}(A_j^{\alpha}), \quad \|u\|_{\mathcal{E}(A_j^{\alpha})} = \|u\|_{\mathcal{R}(A_j^{\alpha})}, \quad u \in \mathcal{E}(A_j^{\alpha}). \tag{4.3}$$

Theorem 3. *The following inequalities hold:*

$$\|w\|_{\mathcal{B}^{s,\alpha}_{\tau,\tilde{\otimes}_j}} \le c_{s,\tau} \|w\|^{s}_{\tilde{\otimes}_j \mathcal{R}(A_j^{\alpha})} \|w\|_{\tilde{\otimes}_j L_{p_j}(\Omega)}, \ w \in \tilde{\otimes}_j \mathcal{R}(A_j^{\alpha}), \tag{4.4}$$

$$E(t, w; \tilde{\otimes}_j \mathcal{R}(A_j^{\alpha}), \tilde{\otimes}_j L_{p_j}(\Omega)) \le C_{s,\tau} t^{-s} \|w\|_{\mathcal{B}^{s,\alpha}_{\tau,\tilde{\otimes}_j}}, \tag{4.5}$$

for $w \in \mathcal{B}^{s}_{\tau}(\tilde{\otimes}_j \mathcal{R}(A_j^{\alpha}), \tilde{\otimes}_j L_{p_j}(\Omega))$, *with the constants* $c_{s,\tau}$ *and* $C_{s,\tau}$ *from Theorem 2, where is denoted* $E(t, w; \tilde{\otimes}_j \mathcal{R}(A_j^{\alpha}), \tilde{\otimes}_j L_{p_j}(\Omega)) = \inf\left\{\|w - w_0\|_{\tilde{\otimes}_j L_{p_j}(\Omega)}\colon w_0 \in \tilde{\otimes}_j \mathcal{R}(A_j^{\alpha}),\ \|w_0\|_{\tilde{\otimes}_j \mathcal{R}(A_j^{\alpha})} \le t\right\}$ *for* $w \in \tilde{\otimes}_j L_{p_j}(\Omega)$.

In addition, for $\tau = 2/(1+s)$ *the following inequality holds,*

$$E(t, w; \tilde{\otimes}_j \mathcal{R}(A_j^{\alpha}), \tilde{\otimes}_j L_{p_j}(\Omega)) \le C_s t^{-s} \|w\|_{\mathcal{B}^{s,\alpha}_{2/(1+s),\tilde{\otimes}_j}}, \tag{4.6}$$

$w \in \mathcal{B}^{s}_{2/(1+s)}(\tilde{\otimes}_j \mathcal{R}(A_j^{\alpha}), \tilde{\otimes}_j L_{p_j}(\Omega))$, *with* $C_s = [((1+s)/\pi)\sin(\pi/(1+s))]^{(1+s)/2}$.

Proof. Using (4.3) and Theorem 1, one obtains that

$$\begin{aligned} \mathcal{B}^{s}_{\tau}(\tilde{\otimes}_j \mathcal{E}(A_j^{\alpha}), \tilde{\otimes}_j L_{p_j}(\Omega)) &= \mathcal{B}^{s}_{\tau}(\tilde{\otimes}_j \mathcal{R}(A_j^{\alpha}), \tilde{\otimes}_j L_{p_j}(\Omega)) \\ &= \left(\tilde{\otimes}_j \mathcal{R}(A_j^{\alpha}), \tilde{\otimes}_j L_{p_j}(\Omega)\right)^{1/\theta}_{\theta,q}, \end{aligned}$$

with $0 < \theta < 1$, $\tau = \theta q$ and $s = 1/\theta - 1$.

Thus, the inequalities (4.4) and (4.5) follow directly from (3.1) and (3.2), while (4.6) follows from (3.8).

5. Conclusions

The approach to approximation problems considered by us is constructive because it combines approximation with the interpolation theory that allows to obtain the explicit formulas for estimating the approximations. The obtained inequalities that provide a sharp estimates of best approximations by analytic vectors of positive operators on projective tensor products are only initial results in this direction. The problem of investigating tensor products of approximation spaces in norms other than the projective one, remains open. Also, the problem of establishing estimates of best approximations by analytic vectors of different classes of operators on tensor products remains open.

References

1. *Abdullayev F. G., Savchuk V. V., Shidlich A. L., Ozkartepe P.*: Exact constants in direct and inverse approximation theorems of functions of several variables in the spaces S^p. Filomat 2019; 33(5): pp. 1471–1484. doi:10.2298/FIL1905471A
2. *Babenko V.F., Konareva S.V.*: Jackson-Stechkin-Type Inequalities for the Approximation of Elements of Hilbert Spaces. Ukr. Math. J. 2019; 70(9): pp. 1331–1344. doi:10.1007/s11253-019-01573-3
3. *Bergh J., Löfström J.*: Interpolation Spaces. Springer-Verlag, 1976.
4. *Dmytryshyn M.*: Approximation by analytic vectors of positive operators. Carpathian Math. Publ. 2020; 12(2): pp. 412–418. doi:10.15330/cmp.12.2.412-418
5. *Dmytryshyn M.*: Approximation by interpolation spectral subspaces of operators with discrete spectrum. Mat. Stud. 2021; 55(2): pp. 162–170. doi:10.30970/ms.55.2.162-170
6. *Dmytryshyn M., Dmytryshyn L.*: Bernstein-Jackson-type inequalities with exact constants in Orlicz spaces. Carpathian Math. Publ. 2022; 14(2): pp. 364–370. doi:10.15330/cmp.14.2.364-370
7. *Dmytryshyn M., Lopushansky O.*: Operator calculus on the exponential type vectors of operators with point spectrum; In book Topology in Banach Spaces; Nova Publ., 2001: pp. 137–145.
8. *Dmytryshyn M., Lopushansky O.*: Bernstein-Jackson-type inequalities and Besov spaces associated with unbounded operators. J. Inequal. Appl. 2014; 2014(105). doi:10.1186/1029-242X-2014-105
9. *Dmytryshyn M., Lopushansky O.*: On Spectral Approximations of Unbounded Operators. Complex Anal. Oper. Theory. 2019; 13(8): pp. 3659–3673. doi:10.1007/s11785-019-00923-0
10. *Giulini S.*: Bernstein and Jackson theorems for the Heisenberg group. J. Austral. Math. Soc. 1985; 38(2): pp. 241–254. doi:10.1017/S1446788700023107
11. *Gorbachuk M., Gorbachuk V.*: Approximation of smooth vectors of a closed operator by entire vectors of exponential type. Ukrainian Math. J. 1995; 47(5): pp. 713–726. doi:10.1007/BF01059045
12. *LinSen X., JiaCheng L., SenHua L., DunYan Y.*: Jackson-type and Bernstein-type inequalities for multipliers on Herz-type Hardy spaces. Sci. China Math. 2009; 52(3): pp. 481–492. doi:10.1007/s11425-008-0128-6
13. *Lopushansky O.*: Bernstein-Jackson Inequalities on Gaussian Hilbert Spaces. J. Fourier Anal. Appl. 2023; 29:58. doi:10.1007/s00041-023-10035-1

14. *McLean W.*: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge Univ. Press, 2000.
15. *Nikolskii S.*: Approximation of functions of several variables and imbedding theorems. Springer-Verlag, 1975.
16. *Peetre J., Sparr G.*: Interpolation of normed abelian groups. Ann. Mat. Pura ed Appl. 1972; 92(1): pp. 217–262. doi:10.1007/BF02417949
17. *Radzievskii G.V.*: On the best approximations and rate of convergence of decompositions in the root vectors of an operator. Ukrainian Math. J. 1997; 49(6): pp. 844–864. doi:10.1007/BF02513425
18. *Pietsch A.*: Approximation spaces. J. Approx. Theory. 1981; 32(2): pp. 115–134. doi:10.1016/0021-9045(81)90109-X
19. *Triebel H.*: Interpolation theory. Function spaces. Differential operators. North-Holland Publishing Company, 1978.

Received: 11.02.2024. *Accepted:* 20.06.2024