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Automorphism groups of some non-nilpotent Leibniz algebras [1]

Abstract. Let L be an algebra over a field F with the binary operations $+$ and $[,]$. Then L is called a left Leibniz algebra if it satisfies the left Leibniz identity: $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$ for all $a,b,c \in L$. A linear transformation f of L is called an endomorphism of L, if $f([a,b]) = [f(a),f(b)]$ for all elements $a,b \in L$. A bijective endomorphism of L is called an automorphism of L. It is easy to show that the set of all automorphisms of Leibniz algebra is a group with respect to the operation of multiplication of automorphisms. The description of the structure of the automorphism groups of Leibniz algebras is one of the natural and important problems of the general Leibniz algebra theory. The main goal of this article is to describe the structure of the automorphism group of a certain type of non-nilpotent three-dimensional Leibniz algebras.
Key words: Leibniz algebra, Lie algebra, automorphism group

Анотація. Нехай L — алгебра над полем F з бінарними операціями $+$ та $[,]$. Тоді L називатимемо лівою алгеброю Лейбніца, якщо вона задовольняє лівій тотожності Лейбніца: $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$ для всіх елементів $a,b,c \in L$. Лінійне перетворення f алгебри Лейбніца L називають ендоморфізмом алгебри L, якщо $f([a,b]) = [f(a),f(b)]$ для всіх елементів $a,b \in L$. Бієктивний ендоморфізм алгебри Лейбніца L називають автоморфізмом алгебри L. Легко показати, що множина всіх автоморфізмів алгебри Лейбніца є групою відносно операції множення автоморфізмів. Опис будови груп автоморфізмів алгебр Лейбніца є однією з природних та важливих задач загальної теорії алгебр Лейбніца. Головною метою цієї статті є опис будови групи автоморфізмів деякого типу ненільпотентних тривимірних алгебр Лейбніца.
Ключові слова: алгебра Лейбніца, алгебра Лі, група автоморфізмів

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1. Introduction.

Let L be an algebra over a field F with the binary operations $+$ and $[,]$. Then L is called a (*left*) *Leibniz algebra* if it satisfies the (*left*) *Leibniz identity*:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

for all $a, b, c \in L$.

Leibniz algebras appeared first in the paper [2], but the term "Leibniz algebra" appears in the book [12] and the article [13]. The theory of Leibniz algebras is developing very intensively in various directions of research. A significant part of the classical results of this theory can be found in the monograph [1]. It is worth noting that Leibniz algebras are a fairly broad generalization of Lie algebras. Furthermore, if L is a Leibniz algebra, in which

$$[a,a] = 0$$

for every element $a \in L$, then it is a Lie algebra. In other words, Lie algebras can be characterized as anticommutative Leibniz algebras. However, there is a very significant difference between these types of non-associative algebras (see, for example, survey papers [3, 4, 10, 14]).

A linear transformation f of a Leibniz algebra L is called an *endomorphism* of L, if

$$f([a,b]) = [f(a), f(b)]$$

for all $a, b \in L$. A bijective endomorphism of L is called an *automorphism* of L. It is not hard to prove that the set $Aut_{[,]}(L)$ of all automorphisms of L is a group by a multiplication (see, for example, [8]).

It is logical to start researching automorphism groups of Leibniz algebras, the structure of which has already been studied very well. The first natural step was to consider one-generated Leibniz algebras. A description of the structure of automorphism groups of infinite-dimensional one-generated Leibniz algebras was obtained in [11], and of finite-dimensional one-generated Leibniz algebras in [8]. The question arises about automorphism groups of low-dimensional Leibniz algebras. The case of two-dimensional Leibniz algebras is quite simple. The automorphism groups of such algebras were described in [9]. The situation with Leibniz algebras of dimension 3 is much more complicated. The most detailed description of three-dimensional Leibniz algebras can be found in [7]. The first step was taken in the article [9], in which the descriptions of the automorphism groups of nilpotent Leibniz algebras with nilpotency class 3 and of nilpotent Leibniz algebras with nilpotency class 2 and a two-dimensional center were obtained. In [5, 6], a description of the structure of automorphism groups of nilpotent Leibniz algebras with nilpotency class 2 and one-dimensional center was obtained.

In this article, we begin to study the structure of automorphism groups of some non-nilpotent Leibniz algebras of dimension 3.

2. Preliminaries and lemmas.

All necessary definitions can be found in [4].

Let L be a non-nilpotent Leibniz algebra, having dimension 3. As usual, we will suppose that L is not a Lie algebra, so that $Leib(L)$ is non-zero. Thus, we obtain the following two situations:

$$dim_F(Leib(L)) = 2 \text{ or } dim_F(Leib(L)) = 1.$$

Consider a first situation. Let $K = Leib(L)$. Since L is not a Lie algebra, there is an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. Using the information about the structure of Leibniz algebras of dimension 2 (see, for example, [4]), we obtain that either $[a_1, a_3] = 0$, or we can choose an element a_1 such that $[a_1, a_3] = a_3$. Suppose that $[a_1, a_3] = 0$. Since a subalgebra K is abelian of dimension 2, $K = Fa_3 \oplus Fb$ for some element b. Thus, $[a_3, x] = [x, a_3] = 0$ for every element $x \in K$. The fact that $L = \langle K, a_1\rangle$ implies that $a_3 \in \zeta(L)$. Since L is not nilpotent, the Lie factor-algebra L/Fa_3 is not abelian. Then L/Fa_3 has a coset $c+Fa_3$ such that $\langle c, Fa_3\rangle = Leib(L)$, $[a_1+Fa_3, c+Fa_3] = c+Fa_3$. Put $c = a_2$, then $[a_1, a_2] = a_2 + \lambda a_3$ for some scalar $\lambda \in F$. Thus, we come to the following type of Leibniz algebra:

$$Lei_5(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_3, [a_1, a_2] = a_2 + \lambda a_3, \lambda \in F,$$
$$[a_1, a_3] = [a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

In other words, $Lei_5(3, F) = L$ is a sum of the Leibniz kernel

$$Leib(L) = Fa_2 \oplus Fa_3 = A_2$$

and nilpotent one-generator Leibniz algebra

$$A_1 = Fa_1 \oplus Fa_3$$

of dimension 2, $[A_1, A_2] = A_2$, $[A_1, A_2] = \langle 0\rangle$, $Leib(L) = [L, L] = \zeta^{\text{left}}(L)$, $\zeta^{\text{right}}(L) = \zeta(L) = Fa_3$.

Here are some general useful properties of automorphism groups of Leibniz algebras (see Lemmas 1, 2 in [9]).

Lemma 1. *Let L be a Leibniz algebra over a field F and f be an automorphism of L. Then $f(\zeta^{\text{left}}(L)) = \zeta^{\text{left}}(L)$, $f(\zeta^{\text{right}}(L)) = \zeta^{\text{right}}(L)$, $f(\zeta(L)) = \zeta(L)$, $f([L, L]) = [L, L]$.*

Let L be a Leibniz algebra over a field F, A be a subalgebra of L and $G = Aut_{[,]}(L)$. Put

$$C_G(A) = \{\alpha \in G |\ \alpha(x) = x \text{ for every element } x \in A\}.$$

If A is an ideal of L, then put

$$C_G(L/A) = \{\alpha \in G |\ \alpha(x + A) = x + A \text{ for every element } x \in L\}.$$

Lemma 2. *Let L be a Leibniz algebra over a field F and $G = Aut_{[,]}(L)$. If A is a G-invariant subalgebra, then $C_G(A)$ and $C_G(L/A)$ are normal subgroups of G.*

Lemma 3. *Let L be a Lie algebra over a field F, having dimension 2, $L = Fa_1 \oplus Fa_2$, $[a_1, a_2] = a_2$, $G = Aut_{[,]}(L)$. Then G is isomorphic to a subgroup of $GL_2(F)$, consisting of the matrices of the form*

$$\begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix},$$

$\alpha, \beta \in F$. Furthermore, G is a semidirect product of a normal subgroup, which is isomorphic to additive group of a field F, and subgroup, which is isomorphic to multiplicative group of a field F.

Proof. Let $f \in Aut_{[,]}(L)$. By Lemma 1, $f(Fa_2) = Fa_2$, $f(a_1) = \alpha_1 a_1 + \alpha_2 a_2$, $f(a_2) = \beta a_2$. Then

$$\begin{aligned} f(a_2) &= f([a_1, a_2]) = [f(a_1), f(a_2)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2, \beta a_2] = \alpha_1 \beta [a_1, a_2] = \alpha_1 \beta a_2, \end{aligned}$$

and we obtain $\alpha_1 \beta a_2 = \beta a_2$. If we suppose that $\alpha_1 = 0$, then $f(a_1) \in [L, L]$. It follows that $f(L) \leq [L, L]$. But $dim_F(L) = 2$, $dim_F([L, L]) = 1$, and we obtain a contradiction with the fact that f is an automorphism. This contradiction shows that $\alpha_1 \neq 0$. In a similar way, if we suppose that $\beta = 0$, then $f(Fa_2) = \langle 0 \rangle$, and again we obtain a contradiction. This contradiction shows that $\beta \neq 0$. Then $\alpha_1 = 1$.

Denote by Ξ the canonical monomorphism of $Aut_{[,]}(L)$ in $M_2(F)$. Then $\Xi(f)$ is a following matrix

$$\begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix},$$

$\alpha, \beta \in F$. Denote by B the subset of $\Xi(G)$, consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix},$$

$\beta \in F$, and by A the subset of $\Xi(G)$, consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix},$$

$\alpha \in F$. It is not hard to prove that A, B are subgroups of $\Xi(G)$, moreover A is isomorphic to the additive group of a field F, B is isomorphic to the multiplicative group of a field F. The equality

$$\begin{pmatrix} 1 & 0 \\ -\alpha\beta^{-1} & \beta^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta^{-1}\alpha & 1 \end{pmatrix}$$

shows that a subgroup A is normal. The equality

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$$

shows that $\Xi(G)$ is a product of subgroups A and B. It is not hard to see that this product is semidirect.

3. Automorphism group of $Lei_5(3, F)$.

Theorem. *Let G be an automorphism group of Leibniz algebra $Lei_5(3, F)$. If $\lambda = 0$, then G is isomorphic to a subgroup of $GL_3(F)$, consisting of the matrices of the form*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ \alpha & 0 & 1 \end{pmatrix},$$

$\alpha, \beta \in F$. Furthermore, G is abelian, $G = A \times B$ where A is isomorphic to additive group of a field F and B is isomorphic to multiplicative group of a field F.

If $\lambda \neq 0$, then G is isomorphic to a subgroup of $GL_3(F)$, consisting of the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ \alpha_3 & \lambda(\beta_2 - 1) & 1 \end{pmatrix},$$

$\alpha_3, \beta_2 \in F$. Furthermore, G has a normal subgroup $C = C_G(L/\zeta(L))$, which is isomorphic to additive group of a field F, a factor-group G/C is a semidirect product of a normal subgroup, which is isomorphic to additive group of a field F, and subgroup, which is isomorphic to multiplicative group of a field F.

Proof. Let $L = Lei_5(3, F)$ and let $f \in Aut_{[,]}(L)$. By Lemma 1, $f(Fa_3) = Fa_3$, $f([L, L]) = [L, L]$, so that $f(a_1) = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$, $f(a_2) = \beta_2 a_2 + \beta_3 a_3$, $f(a_3) = \gamma a_3$. Then

$$\begin{aligned} f(a_3) &= f([a_1, a_1]) = [f(a_1), f(a_1)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \alpha_1^2 [a_1, a_1] + \alpha_1 \alpha_2 [a_1, a_2] = \alpha_1^2 a_3 + \alpha_1 \alpha_2 (a_2 + \lambda a_3) \\ &= \alpha_1 \alpha_2 a_2 + (\alpha_1^2 + \lambda \alpha_1 \alpha_2) a_3; \\ f([a_1, a_2]) &= f(a_2 + \lambda a_3) = f(a_2) + \lambda f(a_3) = \beta_2 a_2 + \beta_3 a_3 + \lambda \gamma a_3 \\ &= \beta_2 a_2 + (\beta_3 + \lambda \gamma) a_3; \\ f([a_1, a_2]) &= [f(a_1), f(a_2)] = [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \beta_2 a_2 + \beta_3 a_3] \\ &= \alpha_1 \beta_2 [a_1, a_2] = \alpha_1 \beta_2 (a_2 + \lambda a_3) = \alpha_1 \beta_2 a_2 + \lambda \alpha_1 \beta_2 a_3. \end{aligned}$$

Thus, $\alpha_1 \alpha_2 a_2 = 0$, $\gamma = \alpha_1^2 + \lambda \alpha_1 \alpha_2$, $\beta_2 a_2 + (\beta_3 + \lambda \gamma) a_3 = \alpha_1 \beta_2 a_2 + \lambda \alpha_1 \beta_2 a_3$, so that $\alpha_1 \alpha_2 = 0$, $\beta_2 = \alpha_1 \beta_2$, $\beta_3 + \lambda \gamma = \lambda \alpha_1 \beta_2$.

If we suppose that $\alpha_1 = 0$, then $f(a_1) \in [L, L]$. It follows that $f(L) \leq [L, L]$. But $dim_F(L) = 3$, $dim_F([L, L]) = 2$, and we obtain a contradiction with the fact that f is an automorphism. This contradiction shows that $\alpha_1 \neq 0$. Then $\alpha_2 = 0$. In a similar way, if we suppose that $\beta_2 = 0$, then $f(a_2) \in \zeta(L)$. It follows that $f([L, L]) \leq \zeta(L)$. But $dim_F([L, L]) = 2$, $dim_F(\zeta(L)) = 1$, and we obtain a contradiction with the fact that f is an automorphism. This contradiction shows that $\beta_2 \neq 0$. Then $\alpha_1 = 1$ and $\gamma = 1$.

Let $\lambda = 0$. Then we obtain that $\beta_3 = 0$. Denote by Ξ the canonical monomorphism of $Aut_{[,]}(L)$ in $M_3(F)$. Then $\Xi(f)$ is a following matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ \alpha_3 & 0 & 1 \end{pmatrix},$$

$\alpha_3, \beta_2 \in F$.

Conversely, let f be a linear transformation of L, having in a basis $\{a_1, a_2, a_3\}$ the above matrix. Let x, y be arbitrary elements of L,

$$x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3,$$
$$y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3,$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in F$. Then

$$\begin{aligned}
[x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&= \xi_1 \eta_1 [a_1, a_1] + \xi_1 \eta_2 [a_1, a_2] \\
&= \xi_1 \eta_2 a_2 + \xi_1 \eta_1 a_3; \\
f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) = \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) \\
&= \xi_1 (a_1 + \alpha_3 a_3) + \xi_2 \beta_2 a_2 + \xi_3 a_3 \\
&= \xi_1 a_1 + \xi_1 \alpha_3 a_3 + \xi_2 \beta_2 a_2 + \xi_3 a_3 \\
&= \xi_1 a_1 + \xi_2 \beta_2 a_2 + (\xi_1 \alpha_3 + \xi_3) a_3; \\
f(y) &= \eta_1 a_1 + \eta_2 \beta_2 a_2 + (\eta_1 \alpha_3 + \eta_3) a_3; \\
f([x, y]) &= f(\xi_1 \eta_2 a_2 + \xi_1 \eta_1 a_3) = \xi_1 \eta_2 f(a_2) + \xi_1 \eta_1 f(a_3) \\
&= \xi_1 \eta_2 \beta_2 a_2 + \xi_1 \eta_1 a_3; \\
[f(x), f(y)] &= [\xi_1 a_1 + \xi_2 \beta_2 a_2 + (\xi_1 \alpha_3 + \xi_3) a_3, \eta_1 a_1 + \eta_2 \beta_2 a_2 + (\eta_1 \alpha_3 + \eta_3) a_3] \\
&= \xi_1 \eta_1 [a_1, a_1] + \xi_1 \eta_2 \beta_2 [a_1, a_2] \\
&= \xi_1 \eta_2 \beta_2 a_2 + \xi_1 \eta_1 a_3.
\end{aligned}$$

Hence,

$$f([x, y]) = [f(x), f(y)]$$

for all elements $x, y \in L$, so that every linear transformation of L, whose matrices has the above form, is an automorphism of Leibniz algebra L. The matrices equalities

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ \alpha & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ \alpha & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha+\beta & 0 & 1 \end{pmatrix}$$

shows that G is abelian, $G = A \times B$ where $\Xi(A)$ is a subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix},$$

$\alpha \in F$, and $\Xi(B)$ is a subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\beta \in F$. It is clear that A is isomorphic to additive group of a field F and B is isomorphic to multiplicative group of a field F.

Consider now the case when $\lambda \neq 0$. As above we obtain that $\alpha_2 = 0$, $\alpha_1 = 1$. By $\gamma = \alpha_1^2 + \lambda\alpha_1\alpha_2$ we obtain that $\gamma = 1$. Furthermore, by $\beta_3 + \lambda\gamma = \lambda\alpha_1\beta_2$ we obtain $\beta_3 + \lambda = \lambda\beta_2$, so that

$$\beta_3 = \lambda(\beta_2 - 1).$$

Then $\Xi(f)$ is a following matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ \alpha_3 & \lambda(\beta_2 - 1) & 1 \end{pmatrix},$$

$\alpha_3, \beta_2 \in F$.

Conversely, let f be a linear transformation of L, having in a basis $\{a_1, a_2, a_3\}$ the above matrix. Let x, y be arbitrary elements of L,

$$x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3,$$
$$y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3,$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in F$. Then

$$\begin{aligned}
[x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\
&= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] \\
&= \xi_1\eta_2(a_2 + \lambda a_3) + \xi_1\eta_1 a_3 \\
&= \xi_1\eta_2 a_2 + (\xi_1\eta_2\lambda + \xi_1\eta_1)a_3; \\
f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) \\
&= \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) \\
&= \xi_1(a_1 + \alpha_3 a_3) + \xi_2(\beta_2 a_2 + \lambda(\beta_2 - 1)a_3) + \xi_3 a_3 \\
&= \xi_1 a_1 + \xi_2\beta_2 a_2 + (\xi_1\alpha_3 + \xi_2\lambda(\beta_2 - 1) + \xi_3)a_3; \\
f(y) &= \eta_1 a_1 + \eta_2\beta_2 a_2 + (\eta_1\alpha_3 + \eta_2\lambda(\beta_2 - 1) + \eta_3)a_3; \\
f([x, y]) &= f(\xi_1\eta_2 a_2 + (\xi_1\eta_2\lambda + \xi_1\eta_1)a_3) \\
&= \xi_1\eta_2 f(a_2) + (\xi_1\eta_2\lambda + \xi_1\eta_1)f(a_3) \\
&= \xi_1\eta_2(\beta_2 a_2 + \lambda(\beta_2 - 1)a_3) + (\xi_1\eta_2\lambda + \xi_1\eta_1)a_3 \\
&= \xi_1\eta_2\beta_2 a_2 + (\xi_1\eta_2\lambda(\beta_2 - 1) + \xi_1\eta_2\lambda + \xi_1\eta_1)a_3 \\
&= \xi_1\eta_2\beta_2 a_2 + (\xi_1\eta_2\lambda\beta_2 - \xi_1\eta_2\lambda + \xi_1\eta_2\lambda + \xi_1\eta_1)a_3 \\
&= \xi_1\eta_2\beta_2 a_2 + (\xi_1\eta_2\lambda\beta_2 + \xi_1\eta_1)a_3; \\
[f(x), f(y)] &= [\xi_1 a_1 + \xi_2\beta_2 a_2 + (\xi_1\alpha_3 + \xi_2\lambda(\beta_2 - 1) + \xi_3)a_3, \\
&\quad \eta_1 a_1 + \eta_2\beta_2 a_2 + (\eta_1\alpha_3 + \eta_2\lambda(\beta_2 - 1) + \eta_3)a_3] \\
&= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2\beta_2[a_1, a_2] \\
&= \xi_1\eta_1 a_3 + \xi_1\eta_2\beta_2(a_2 + \lambda a_3) \\
&= \xi_1\eta_2\beta_2 a_2 + (\xi_1\eta_1 + \lambda\xi_1\eta_2\beta_2)a_3.
\end{aligned}$$

Hence,

$$f([x, y]) = [f(x), f(y)]$$

for all elements $x, y \in L$, so that every linear transformation of L, whose matrices has the above form, is an automorphism of Leibniz algebra L.

By Lemma 2,

$$C_G(L/\zeta(L)) = C$$

is a normal subgroup of G. It is clear that $C_G(L/\zeta(L))$ consists of the automorphisms f such that $\beta_2 = 1$, so that $\Xi(C_G(L/\zeta(L)))$ consists of the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_3 & 0 & 1 \end{pmatrix},$$

$\alpha_3 \in F$. It follows that C is isomorphic to the additive group of a field F. The factor-group G/C is isomorphic to the automorphism group of the factor $L/\zeta(L)$, and we can use Lemma 3.

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