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Torsion Groups with the Norm of pd-Subgroup of Finite Index

Abstract. The authors study the relations between the properties of torsion groups and their norms of *pd*-subgroups. The norm N_G^{pdI} of *pd*-subgroups of a group G is the intersection of the normalizers of all its *pd*-subgroups or a group itself, if the set of such subgroups is empty in a group. The structure of the norm of *pd*-subgroups in torsion groups is described and the conditions of Dedekindness of this norm is proved (Dedekind group is a group in which all subgroups are normal). It is proved that a torsion group is a finite extension of its norm of *pd*-subgroups if and only if it is a finite extension of its center. By this fact and the structure of the norm of *pd*-subgroups, we get that any torsion group that is a finite extension of this norm is locally finite.
Key words: torsion group, normalizer, norm of a group, norm of *pd*-subgroups, non-Dedekindness, subgroup index in a group, center of a group

Анотація. Автори досліджують зв'язки між властивостями періодичних груп та їх норм *pd*-підгруп. Нормою N_G^{pdI} *pd*-підгруп групи G називається перетин нормалізаторів усіх її *pd*-підгруп або сама група, якщо система таких підгруп в групі порожня. У статті встановлено будову норми *pd*-підгруп у періодичних групах та досліджено умови, за яких вказана норма є дедекіндовою (тобто, є групою, в якій всі підгрупи нормальні). Доведено, що періодична група є скінченним розширенням своєї норми *pd*-підгруп тоді і тільки тоді, коли вона є скінченним розширенням свого центру. З цього та з будови норми *pd*-підгруп випливає, що будь-яка періодична група, що є скінченним розширенням вказаної норми, локально скінченна.
Ключові слова: періодична група, нормалізатор, норма групи, норма *pd*-підгруп, недедекіндовість, індекс підгрупи в групі, центр групи

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1. Introduction

In group theory, an important place is occupied by results related to the study of groups in which all subgroups of some non-empty system Σ are normal.

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It is clear that each group with this property can be defined as a group that coincides with the intersection of the normalizers of all subgroups from the system Σ. If the system Σ contains all subgroups of a group G that have a given theoretical group property, then this intersection is called the Σ-norm of a group G. When studying Σ-norms and the impact of their properties on the properties of a group, many questions arise related to the study of the properties of groups, depending on the choice of the system Σ of subgroups and the restrictions that the defined Σ-norm satisfies.

Similar problems were considered by many algebraists, but in the majority of findings, Σ-norm coincides with a group. In other words, groups with the condition of normality for all subgroups of the system Σ were studied. Groups with different systems Σ of normal subgroups were actively studied in the second half of the 20th century. Therefore, for many systems Σ of subgroups, the structure of groups coinciding with their Σ-norm is known, and it naturally raises questions about the properties of the groups in which the Σ-norm satisfies some restrictions and is a proper subgroup of a group.

For the first time, such a problem was considered by R. Baer in the 1930s [1] for the system Σ, which consists of all subgroups of this group. R. Baer called the corresponding Σ-norm the norm $N(G)$ of a group G and defined it as the intersection of the normalizers of all subgroups of a group G. It is clear that the norm $N(G)$ is contained in all other Σ-norms, which, in turn, can be considered as generalizations.

Further, Baer's idea of the group norm was transferred to other systems Σ of subgroups (see, for example, [3] – [12], [14] – [17]) and found application in ring theory in the study of the Baer kernel, which is an analogue of the group norm for rings [2].

The goal of this article is to study the properties of other Σ-norm of a group – the norm N_G^{pdI} of pd-subgroups in the class of torsion groups and to establish the conditions under which this norm has a finite index in a group. As will be shown below, all infinite torsion groups that are finite extensions of the norm N_G^{pdI} are central-by-finite. This result agrees with the results of F. M. Lyman [13] about the properties of torsion pd-groups with the condition $G = N_G^{pdI}$ (that is, non-Abelian groups in which all pd-subgroups are normal).

2. Preliminary results

A group G is called a pd-group if it contains elements of prime order p. A non-Abelian group containing non-identity p-elements, in which every pd-subgroup is normal, is called a pdI-group. It is known (Lemma 1 and 2 [13]) that any torsion non-Hamiltonian pdI-group contains a unique subgroup of prime order p and does not contain cyclic subgroups of order p^2.

Proposition 1 presents a complete description of non-Hamiltonian pdI-groups.

Proposition 1 ([13, Theorem 1]). Any torsion non-Hamiltonian group G is a pdI-group if and only if it can be represented as a semidirect product $G = \langle a \rangle \leftthreetimes D$, where $\langle a \rangle$ is a Sylow p-subgroup of order $p \neq 2$ of a group G and D is a torsion Dedekind group.

Recall that a Dedekind group is a group in which every subgroup is normal (that is, such a group is either Abelian or Hamiltonian).

Definition 1. The intersection of the normalizers of all pd-subgroups of a group G (provided that the set of such subgroups in a group is non-empty) is called the norm of pd-subgroups or the pd-norm of this group and denoted by N_G^{pdI}.

In other words, the pd-norm of a group G is the maximal subgroup of this group that normalizes every pd-subgroup in G. For groups with an empty set of pd-subgroups, let's assume that $G = N_G^{pdI}$.

Note that any pdI-group is a meta-Hamiltonian, i.e., a non-Abelian group in which each non-Abelian subgroup is normal. So, the pd-norm is related to the norm of non-Abelian subgroups or the metanorm of a group ([3], [4]).

By the definition of pd-norm, in the case $G = N_G^{pdI}$ the set of pd-subgroups is empty or all such subgroups are normal in a group G (i.e., G is Abelian or pdI-group). Let us consider the properties of the pd-norm N_G^{pdI} and a group G, in which this norm is a proper subgroup.

Further, only groups with a non-empty set of pd-subgroups will be considered, and by the term "group"we will mean a pd-group.

By the definition of the pd-norm N_G^{pdI}, we get the following result.

Lemma 1. *Let N_G^{pdI} be the norm of pd-subgroups of a group G. Then the following propositions take place:*

1. *N_G^{pdI} is a characteristic subgroup of a group G;*
2. *$N_G^{pdI} \geq N(G) \geq Z(G)$, where $N(G)$ is the Baer's norm and $Z(G)$ is the center of a group G;*
3. *any pd-subgroup of the norm N_G^{pdI} is its normal subgroup;*
4. *$N_G^{pdI} = N_{N_G^{pdI}}^{pdI}$;*
5. *if H is a pd-subgroup of a group G, then $H \cap N_G^{pdI} \subseteq N_H^{pdI}$;*
6. *if $H \supseteq N_G^{pdI}$, then $N_H^{pdI} \supseteq N_G^{pdI}$;*
7. *if $H \lhd G$, then $HN_G^{pdI}/H \subseteq N_{G/H}^{pdI}$;*
8. *if H is a normal p-subgroup of a group G and $H \subseteq N_G^{pdI}$, then $N_G^{pdI}/H \subseteq N(G/H)$, where $N(G/H)$ is the Baer's norm of G/H;*

9. *if H is an Abelian pd-subgroup and $H \subseteq C_G(N_G^{pdI})$, then $G_1 = N_{G_1}^{pdI}$, where $G_1 = H \cdot N_G^{pdI}$*

Proof. The proof of (1)-(7) is obvious and follows directly from the definition of the *pd*-norm N_G^{pdI} of a group G. Let us prove properties (8) and (9).

Let H be a normal p-subgroup of a group G and $H \subseteq N_G^{pdI}$. In the quotient-group $\overline{G} = G/H$, the full preimage NH of an arbitrary subgroup $\overline{N} \leq \overline{G}$ is a *pd*-subgroup, therefore, NH is a N_G^{pdI}-admissible subgroup and $N_G^{pdI} \leq N_G(NH)$. Turning to the quotient-group, we get $\overline{N}_G^{pdI} \leq N_{\overline{G}}(\overline{N})$. Thus, $\overline{N}_G^{pdI} = N_G^{pdI}/H$ is contained in the normalizer of each subgroup of the group $\overline{G}$, that is $\overline{N}_G^{pdI} \subseteq N(\overline{G})$, which proves proposition (8) of Lemma 1.

To prove the proposition (9), let us consider the group $G_1 = H \cdot N_G^{pdI}$, where H is an Abelian *pd*-subgroup and $H \subseteq C_G(N_G^{pdI})$. According to (6) of Lemma 1, the *pd*-norm $N_{G_1}^{pdI}$ of the group G_1 contains the *pd*-norm N_G^{pdI} of a group G. It is also obvious that $H \subseteq Z(G_1)$, so $H \subseteq N_{G_1}^{pdI}$. Thus, $G_1 = H \cdot N_G^{pdI} \subseteq N_{G_1}^{pdI}$ and we get the required equality. The lemma is proved.

Lemma 2. *If the norm N_G^{pdI} of a pd-group G does not contain elements of order p, then it is Dedekind.*

Proof. Let the norm N_G^{pdI} satisfy the conditions of the lemma and let a be an arbitrary element of order p of a group G. Then $[\langle a\rangle, N_G^{pdI}] \subseteq \langle a\rangle \cap N_G^{pdI} = E$ and for an arbitrary element $x \in N_G^{pdI}$ the subgroup $\langle x, a\rangle = \langle x\rangle \times \langle a\rangle$ is N_G^{pdI}-admissible. So, $\langle x, a\rangle \lhd G_1 = \langle x, a\rangle \cdot N_G^{pdI}$ and

$$\langle x\rangle = \langle x, a\rangle \cap N_G^{pdI} \lhd N_G^{pdI},$$

that is, the norm N_G^{pdI} is Dedekind. The lemma is proved.

The following example confirms the existence of non-Abelian *pd*-groups with the norm N_G^{pdI} which does not contain non-identity p-elements.

Example 1. Let us consider the group $G = \langle a\rangle \leftthreetimes \langle b\rangle \times C$, in which $|a| = 3$, $|b| = 2$, $b^{-1}ab = a^{-1}$ and C is an infinite torsion Abelian group without elements of orders 2 and 3.

In this group, the $2d$-norm coincides with the center of the group $N_G^{2dI} = Z(G) = C$ and $2 \nmid |N_G^{2dI}|$. On the other hand, since any $3d$-subgroup contains the commutator $G' = \langle a\rangle$, G is a $3dI$-group and coincides with the $3d$-norm: $N_G^{3dI} = G$.

Lemma 3. *The norm N_G^{pdI} of a torsion pd-group G is Dedekind or a non-Hamiltonian pdI-group of the type of Proposition 1.*

Proof. If the norm N_G^{pdI} is Dedekind, then the lemma is proved. Let N_G^{pdI} be a non-Dedekind group. Then, by Lemma 2, N_G^{pdI} contains non-identity p-elements and therefore *pd*-subgroups. Since N_G^{pdI} normalizes all *pd*-subgroups,

any pd-subgroup of N_G^{pdI} is normal in it. Thus, N_G^{pdI} is a non-Hamiltonian pdI-group. The lemma is proved.

3. Main results

By Lemma 3 and Proposition 1 the norm N_G^{pdI} of a torsion locally nilpotent pd-group is Dedekind. Other conditions of Dedekindness of the norm N_G^{pdI} in a torsion group G are given in the following theorem.

Theorem 1. *The norm N_G^{pdI} of a torsion pd-group G is Dedekind if at least one of the following conditions takes place:*

1. *a group G contains a pd-subgroup identity-intersected with N_G^{pdI};*
2. *a group G contains a cyclic p-subgroup of order greater than p;*
3. *a group G contains an elementary Abelian subgroup of order p^2;*
4. *in the case of $p = 2$;*
5. *the center $Z(G)$ of a group G contains non-identity p-elements;*
6. *the norm N_G^{pdI} is locally nilpotent.*

Proof. 1) Let N_G^{pdI} be the norm of pd-subgroups of a torsion group G and G contain a pd-subgroup H such that $H \cap N_G^{pdI} = E$. Then, for an arbitrary element $x \in N_G^{pdI}$ the subgroup $\langle x, H\rangle = \langle x\rangle \times H$ contains p-elements and is normal in the group $G_1 = H \cdot N_G^{pdI}$. Since $\langle x\rangle = \langle x, H\rangle \cap N_G^{pdI} \lhd N_G^{pdI}$, the norm N_G^{pdI} is Dedekind.

2) Let's assume that the norm N_G^{pdI} is non-Dedekind. According to Lemma 3, N_G^{pdI} is a non-Hamiltonian pdI-group, which does not contain cyclic subgroups of order p^2 by Proposition 1. Thus, if G contains a cyclic subgroup $\langle x\rangle$ of order p^2, then $\langle x\rangle \not\subset N_G^{pdI}$ and $\langle x\rangle \cap N_G^{pdI} \neq E$ as proved above. Since N_G^{pdI} has a normal Sylow subgroup $\langle a\rangle$ of order p, $\langle x\rangle \cap N_G^{pdI} = \langle a\rangle$. Then for an arbitrary element $y \in N_G^{pdI}$ we have

$$[y, x] \in \langle x\rangle \cap N_G^{pdI} = \langle a\rangle.$$

So, $y^{-1}x^p y = (xa^k)^p = x^p a^{kp} = x^p$ and $[y, a] = 1$, which contradicts the defining relations of the norm N_G^{pdI}. Thus, N_G^{pdI} is Dedekind.

3) Let the norm N_G^{pdI} be non-Dedekind. By Lemma 3, N_G^{pdI} is a non-Hamiltonian pdI-group, which contains a unique subgroup $\langle a\rangle$ of order p by Proposition 1. Since G contains an elementary Abelian subgroup of order p^2, there exists the subgroup $\langle x\rangle$ of order p such that $\langle x\rangle \not\subset N_G^{pdI}$. But then $\langle x\rangle \cap N_G^{pdI} = E$ and N_G^{pdI} is Dedekind by the proved above. This contradicts the assumption.

4) Let $p = 2$. If the norm N_G^{pdI} does not contain elements of order 2, then it is Dedekind by Lemma 2. Let N_G^{pdI} contain 2-elements. Then all $2d$-subgroups are normal in it and N_G^{pdI} is either Abelian or a $2dI$-group. In the latter case, N_G^{pdI} is a Hamiltonian $2dI$-group by Proposition 1.

5) The proof follows from Proposition 1 and Lemma 3.

6) Let the norm N_G^{pdI} be locally nilpotent. If N_G^{pdI} does not contain p-elements, then it is Dedekind. Let N_G^{pdI} contain p-elements. Then it is a pdI-group and N_G^{pdI} cannot be non-Hamiltonian by Proposition 1. Therefore, the norm N_G^{pdI} is Dedekind in this case. The theorem is proved.

Corollary 1. *If the norm N_G^{pdI} of a torsion pd-group G is non-Dedekind, then G does not contain subgroups of order p^2.*

Lemma 4. *If the norm N_G^{pdI} of a torsion pd-group G is non-Dedekind, then $p \neq 2$ and the Sylow p-subgroup of G is of order p, coincides with the Sylow p-subgroup of the norm N_G^{pdI} and is a normal subgroup of G.*

Proof. Let the norm N_G^{pdI} of a torsion group G be non-Dedekind. Then, by Lemma 3, it is a non-Hamiltonian pdI-group of the type $N_G^{pdI} = \langle a \rangle \leftthreetimes D$, where $\langle a \rangle$ is a Sylow p-subgroup of order $p \neq 2$ of a group G, D is a torsion Dedekind group. By Theorem 1, a group G does not contain p-subgroups, which is different from $\langle a \rangle$. Therefore, $\langle a \rangle$ is the maximal p-subgroup of a group G, i.e., its Sylow p-subgroup. The lemma is proved.

Theorem 2. *Any infinite torsion pd-group G is a finite extension of the norm N_G^{pdI} if and only if G is central-by-finite.*

Proof. The sufficiency of the conditions of the theorem is obvious, so we will prove only their necessity. Let $[G : N_G^{pdI}] < \infty$. By Lemma 3, the norm N_G^{pdI} is Dedekind or is a group of the type $N_G^{pdI} = \langle a \rangle \leftthreetimes D$, where $\langle a \rangle$ is a Sylow p-subgroup of order $p \neq 2$ of a group G and D is a torsion Dedekind group of infinite order. If $G = N_G^{pdI}$, then $[G : Z(G)] < \infty$ in both cases.

Further, let's assume that $G \neq N_G^{pdI}$, that is, $1 < [G : N_G^{pdI}] < \infty$. Since the norm N_G^{pdI} is locally finite, a group G, which is its finite extension, is also locally finite. Then

$$G = \langle g_1, g_2, ..., g_n \rangle N_G^{pdI}.$$

Let the norm N_G^{pdI} be non-Dedekind and $Z = Z(N_G^{pdI})$ be the center of the norm N_G^{pdI}, $M = \langle g_1, g_2, ..., g_n \rangle$. Since $Z \lhd G$ and $[N_G^{pdI} : Z] < \infty$, $N_G^{pdI} = FZ$, where $|F| < \infty$. So,

$$G = \langle M, F \rangle Z$$

and $[G : Z] < \infty$. Considering that $N = \langle M, F \rangle$ is a finite pd-subgroup, we conclude that $N \lhd G$. Further, by the condition $|N| < \infty$ we get $[G : C_G(N)] < \infty$. But $C_G(N) \cap Z \subseteq Z(G)$, so, in this case $[G : Z(G)] < \infty$, which is desired conclusion.

Let the norm N_G^{pdI} be a Dedekind group. If it does not contain non-identity p-elements, then the finite group $M = \langle g_1, g_2, ..., g_n \rangle$ contains such elements. Since M is an N_G^{pdI}-admissible subgroup, $M \lhd G$ and $[G : C_G(M)] < \infty$. Taking into account that $[N_G^{pdI} : Z(N_G^{pdI})] \leq 4$ and

$$C_G(M) \cap Z(N_G^{pdI}) \subseteq Z(G),$$

we conclude that $[G : Z(G)] < \infty$ in this case.

Let N_G^{pdI} be a Dedekind group and $x \in N_G^{pdI}$ be an element of order p. Then $\langle M, x \rangle = \langle x, g_1, g_2, ..., g_n \rangle$ is finite and, therefore, an N_G^{pdI}-admissible pd-subgroup. Thus, $\langle M, x \rangle \lhd G$ and $[G : Z(G)] < \infty$ can be proved in the same way. The theorem is proved.

Corollary 2. *Any infinite torsion pd-group G with pd-norm N_G^{pdI} of a finite index is locally finite.*

Corollary 3. *Any infinite torsion pd-group G with pd-norm N_G^{pdI} of a finite index is Abelian-by-finite.*

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