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The homology groups of the Cartesian product $\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2)$

Celebrating the 75th birthdays of Professors V. F. Babenko and V. O. Kofanov

Abstract. The paper continues the investigation of the spaces of complex-valued perfect splines $\Omega_n(m)$. These spaces were introduced as generalization of the spaces Ω_n, the topology of which has been studied by V.I. Ruban, V.A. Koshcheev, A.M. Pasko. In our previous papers the homology groups of the spaces $\Omega_n(m)$ have been found and their simply connectedness was established. The topic of the paper is finding of the homology groups of the Cartesian product $\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2)$. In order to find the homology groups of this Cartesian product the Kunneth theorem has been used. Using the Kunneth theorem and the fact that $\mathrm{Tor}(A, B) = 0$ if at least one of the group A, B is free we presented the homology group of the Cartesian product $\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2)$ as the sum of the tensor products of the homology groups of this spaces. Calculating the tensor products we found the homology groups of $\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2)$.

Key words: generalized perfect spline, the Cartesian product, homology groups

Анотація. Стаття продовжує вивчення просторів узагальнених комплекснозначних досконалих сплайнів $\Omega_n(m)$. Ці простори було введено як узагальнення важливих у теорії наближень просторів Ω_n, топологія яких вивчалася В.І. Рубаном, В.А. Кощєєвим, А.М. Паськом. У попередніх роботах автора було знайдено групи гомологій просторів $\Omega_n(m)$, а також доведено їх однозв'язність. Предметом цього дослідження є гомологічні групи декартових добутків $\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2)$. Задля їх знаходження у статті використано відому теорему Кюнета. Застосовуючи теорему Кюнета і той факт, що $\mathrm{Tor}(A, B) = 0$, якщо бодай одна з груп A, B - вільна, ми подали групи гомологій декартового добутку $\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2)$ як пряму суму тензорних добутків гомологічних груп цих просторів. Обчислюючи ці тензорні добутки, ми знайшли гомологічні групи просторів $\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2)$.

Ключові слова: узагальнені досконалі сплайни, декартів добуток, групи гомологій

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Let $\omega(t), t \geq 0$, be the non-negative, continuous increasing function, $\omega(0) = 0$. Consider integer $q \geq 0$, integer $m \geq 2$ and the system of the knots

$$0 = \eta_0 < \eta_1 < \ldots < \eta_q < \eta_{q+1} = 1.$$

Let $F(\eta, s, t)$ be the function (ω-spline)

$$F(\eta, s, t) = s_k \cdot \min\{\omega(t - \eta_{k-1}), \omega(\eta_k - t)\}, \qquad for \quad t \in [\eta_{k-1}, \eta_k], \tag{1}$$

with

$$s_k \in G_m = \left\{ e^{i\frac{2\pi l}{m}} : l = 0, 1, \ldots, m-1 \right\}.$$

Denote by $\Omega_n(m)$, $n \geq 2$, the subspace of the space $L_1[0,1]$ that consists of the splines (1) for $q \leq n$. The spaces $\Omega_n(m)$ are the generalization of the space Ω_n the topology of which has been studied by V.I. Ruban [6, 7], particularly the space $\Omega_n(2)$ coincides with the space Ω_n. The spaces $\Omega_n(m)$ have been introduced in [4]. Further the topology of the spaces $\Omega_n(m)$ has been studied in [2, 3]. Particularly in [3] their homology groups were found:

$$H_k\left(\Omega_n(m)\right) = \begin{cases} \mathbb{Z}, & k = 0, \\ 0, & k \neq 0, n, \\ \mathbb{Z}^{N(n,m)}, & k = n, \end{cases} \tag{2}$$

where $m \geq 2$, $n \geq 2$ and

$$N(n,m) = \frac{m^{n+2} + (-1)^{n+1}}{m+1}.$$

Further we will need the definition and some properties of tensor product. Let A, B be abelian groups. Accordingly to [1, p. 218] the tensor product $A \otimes B$ is group with generators $a \otimes b$ for $a \in A, b \in B$ and relations $(a + a') \otimes b = a \otimes b + a' \otimes b$, $a \otimes (b + b') = a \otimes b + a \otimes b'$. It is known (see [1, p. 218]) that for any abelian groups A, B:
1) $(A \oplus A') \otimes B = (A \otimes B) \oplus (A' \otimes B)$;
2) $\mathbb{Z} \otimes A = A$.

The properties 1), 2) of tensor product imply that

$$\mathbb{Z}^n \otimes G = G^n \tag{3}$$

for any abelian group G.

Let A, A', B, B' be abelian groups, $\varphi : A \to A', \psi : B \to B'$ are homomorphisms. They induce the homomorphism $\varphi \otimes \psi : A \otimes B \to A' \otimes B'$ defined on the generators as

$$(\varphi \otimes \psi)(a \otimes b) = \varphi(a) \otimes \psi(b).$$

Let A be abelian group. It is known (see [5, p. 38]) that the exact sequence holds

$$0 \to R \xrightarrow{i} F \xrightarrow{p} A \to 0,$$

where R, F are some free abelian groups. It induces the exact sequence

$$R \otimes B \xrightarrow{i\otimes 1} F \otimes B \xrightarrow{p\otimes 1} A \otimes B \to 0$$

for any abelian group B. Denote by $\mathrm{Tor}(A, B)$ the kernel

$$\mathrm{Tor}(A, B) = \mathrm{Ker}(i \otimes 1).$$

It is known (see [5, pp. 40, 41]) that if at least one of the groups A, B is free

$$\mathrm{Tor}(A, B) = 0. \tag{4}$$

If $A_i, B_i, i = 0, 1, 2, ...$, are the sequences of abelian groups denote

$$(\mathrm{Tor}(A_*, B_*))_k = \bigoplus_{p+q=k} \mathrm{Tor}(A_p, B_q).$$

Let K, L be the finite CW-complexes. Accordingly to Kunneth theorem [5, p. 110] the exact sequence holds

$$\begin{aligned} 0 \longrightarrow (H_*(K) \otimes H_*(L))_k \longrightarrow H_k(K \times L) \longrightarrow \\ \longrightarrow (\mathrm{Tor}(H_*(K), H_*(L)))_{k-1} \longrightarrow 0, \end{aligned} \tag{5}$$

where

$$(H_*(K) \otimes H_*(L))_k = \bigoplus_{i=0}^{k} H_i(K) \otimes H_{k-i}(L).$$

In order to find the homology groups of the Cartesian product $\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2)$ consider (5) with $K = \Omega_{n_1}(m_1), L = \Omega_{n_2}(m_2)$.

The homology groups of these spaces are free, so by the virtue of (4)

$$(\mathrm{Tor}(H_*(\Omega_{n_1}(m_1)), H_*(\Omega_{n_2}(m_2))))_{k-1} = 0.$$

Therefore the exact sequence (5) turns to the shorter sequence

$$0 \longrightarrow (H_*(\Omega_{n_1}(m_1)) \otimes H_*(\Omega_{n_2}(m_2)))_k \longrightarrow H_k(\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2)) \longrightarrow 0.$$

It means that

$$H_k(\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2)) \cong \bigoplus_{i=0}^{k} H_i(\Omega_{n_1}(m_1)) \otimes H_{k-i}(\Omega_{n_2}(m_2)). \tag{6}$$

By the virtue of (2) the only summands in the right hand of (6) that differ from the zero group are those corresponding the values of the indexes:
1) $i = k - i = 0$;
2) $i = 0, k - i = n_2$;
3) $i = n_1, k - i = 0$;
4) $i = n_1, k - i = n_2$.

Then $H_k(\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2))$ differs from the zero group if and only if $k = 0, n_1, n_2, n_1 + n_2$. The equalities (2), (3) lead us to the theorem.

Theorem 1. *If $n_1 \neq n_2$ the homology groups of the Cartesian product $\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2)$ are equal*

$$H_k\left(\Omega_{n_1}(m_1) \times \Omega_{n_2}(m_2)\right) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}^{N(n_1,m_1)}, & k = n_1, \\ \mathbb{Z}^{N(n_2,m_2)}, & k = n_2, \\ \mathbb{Z}^{N(n_1,m_1)N(n_2,m_2)}, & k = n_1 + n_2, \\ 0, & k \neq 0, n_1, n_2, n_1 + n_2. \end{cases}$$

If $n_1 = n_2$ then denoting $n_1 = n_2 = n$ we have

$$H_k\left(\Omega_n(m_1) \times \Omega_n(m_2)\right) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}^{N(n,m_1)+N(n,m_2)}, & k = n, \\ \mathbb{Z}^{N(n,m_1)N(n,m_2)}, & k = 2n, \\ 0, & k \neq 0, n, 2n. \end{cases}$$

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