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The uniqueness of the best L_1-approximant of continuous Banach-valued functions under interpolatory constraints

Celebrating the 75th birthdays of Professors V. F. Babenko and V. O. Kofanov

Abstract. The best L_1-approximation with interpolatory constraints for continuous mapping of a metric compact set Q into a Banach space X is considered. The inicity set's criterion is obtained. The result is generalized the result for real functions that was proved by A. Pinkus and H. Strauss.
Key words: L_1-approximation, continuous Banach-valued functions, criterion of the best L_1-approximant

Анотація. У статті розглядається найкраще L_1-наближення з інтерполяційними обмеженнями для неперервного відображення метричного компакта Q у банаховий простір X. Отримано критерій множини єдиності елемента найкращого L_1-наближення. Узагальнено результат для дійсних функцій, доведений А. Пінкусом та Г. Штраусом.
Ключові слова: L_1-наближення, неперервні банаховозначні функції, критерій елемента найкращого L_1-наближення

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1. Introduction

A. Pinkus and H. Strauss studied problems of best L_1-approximation to continuous functions from finite-dimensional subspaces U under variety constraints included approximation with interpolation. They investigated the unicity problem for L_1-approximation of real continuous functions f on compact set K from the set $M(f) = \{u \in U : u(t_i) = f(t_i), t_i \in K, i = 1, \dots, m; t_i \neq t_j, i \neq j\}$. Obviously $M(f)$ depends on f. A. Pinkus and H. Strauss reduced the unicity problem of best L_1-approximant for continuous function from $M(f)$ to the unicity problem of the best L_1-approximant for these functions from the finite-dimensional subspace $V = \{v \in U : v(t_i) =$

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$0, i = 1, \ldots, m\}$ (the points $t_1, \ldots, t_m$ are the same as in $M(f)$). Now the approximating set doesn't depends on f.

In this article we extend the result of A. Pinkus and H. Strauss. We study the uniqueness of the best L_1-approximant for continuous functions defined on metric compact with values in a strictly convex Banach space. The theorem 3 is proved by the methods from [4] and [1].

We are very grateful to our research advisor professor V. Babenko for setting the the problem and continuous attention.

2. Preliminaries and notations

Let (Q, ρ) be a metric compact set with a metric ρ , Σ – a σ-field of Borell subsets of Q, and μ is a nonnegative finite nonatomic measure on Σ. Moreover, the measure μ is positive for every nonempty open subset of the σ-field Σ.

Let X be a Banach space with the norm $||\cdot||_X$.

By $C_1(Q, X)$ denote the space of continuous functions $f : Q \to X$ with the L_1-norm $||f||_1 = \int\limits_Q ||f(x)||_X d\mu(x)$.

For $f \in C_1(Q, X)$, $U \subset C_1(Q, X)$ the value

$$E(f, U)_1 = \inf_{g \in U} \|f - g\|_1 \tag{2.1}$$

is called the best L_1-approximation of a function f by a set U in the metric L_1. The function $g^* \in U$ is the best L_1-approximant of a function f by elements of a set H if g^* realizes the infimum in (2.1). Let $P_U(f)$ be the set of the best L_1-approximants of a function f by a set U. By Z_f denote the set of zeros for a function f, and $N_f = Q \setminus Z_f$.

For $f, g \in C_1(Q, X)$ and $x \in Q$ put

$$\tau_-(f(x), g(x))_X = \lim_{t \to 0^-} \frac{||(f + tg)(x)||_X - ||f(x)||_X}{t}.$$

This functional was considered in [2, 3, 4].

Let U be a subspace of $C_1(Q, X)$ and $t_1, ..., t_m$ be distinct points in Q. For $f \in C_1(Q, X)$ set

$$M(f) = \{g \in U : g(t_i) = f(t_i), i = 1, ..., m\}.$$

Consider the subspace of U:

$$V = \{g \in U : g(t_i) = 0, i = 1, ..., m\}.$$

In this article by A. Pinkus and H. Strauss [4] we will consider the problem of the uniqueness of the best L_1-approximant for the functions $f \in C_1(Q, X)$ from $M(f)$. And by A. Pinkus and H. Strauss we note that if $\overline{g} \in M(f)$, then

$$M(f) = \{\overline{g} + v : v \in V\}.$$

3. Criterion of the best L_1-approximant from $M(f)$ in $C_1(Q, X)$

The following theorem (see [1, 2, 5]) is needed for further statements.

Theorem 1. *Let X be a Banach space, U be a subspace of space $C_1(Q, X)$. The element $g^* \in U$ is the best L_1-approximant of $f \in C_1(Q, X)$ from U iff for any $g \in U$*

$$\int\limits_{N_{f-g^*}} \tau_-((f-g^*)(x), g(x))_X d\mu(x) \leq \int\limits_{Z_{f-g^*}} ||g(x)||_X d\mu(x).$$

Using this representation, as in [4] we obtain a generalization of the Theorem 4.1 from [4] for the approximation of functions $f \in C_1(Q, X)$ by the set $M(f)$.

Theorem 2. *Let X be a Banach space, U be a subspace of space $C_1(Q, X)$. The element $g^* \in M(f)$ is the best L_1-approximant of $f \in C_1(Q, X)$ from $M(f)$ iff for any $v \in V$*

$$\int\limits_{N_{f-g^*}} \tau_-((f-g^*)(x), v(x))_X d\mu(x) \leq \int\limits_{Z_{f-g^*}} ||v(x)||_X d\mu(x).$$

4. Uniqueness of the best L_1-approximant from $M(f)$ in $C_1(Q, X)$

The following theorem is the main result of this article. It generalizes Theorem 4.2 from [4] for L_1-approximation of functions $f \in C_1(Q, X)$ from $M(f)$.

Theorem 3. *Let X be a strictly convex Banach space. Each function $f \in C_1(Q, X)$ has at most one best L_1-approximant from $M(f)$ iff each function $f \in C_1(Q, X)$ has at most one best L_1-approximant from V.*

Proof. Sufficiency. Suppose each function $f \in C_1(Q, X)$ has at most one best L_1-approximant by elements from V. Let $f \in C_1(Q, X)$ and $\overline{g} \in M(f)$; then $f - \overline{g} \in C_1(Q, X)$ and $f - \overline{g}$ has at most one best L_1-approximant from V.

If $f - \overline{g}$ has a unique best L_1-approximant $v^* \in V$, then the function $g^* = \overline{g} + v^* \in M(f)$ is the unique best L_1-approximant to f from $M(f)$.

Suppose that the function $f - \overline{g}$ does not have a best L_1-approximant from V. If $g^* \in M(f)$ is a best L_1-approximant to f from $M(f)$, then $v^* = g^* - \overline{g} \in V$ is the best L_1-approximant to $f - \overline{g}$ from V. It follows that f does not have a best L_1-approximant from $M(f)$.

Necessity. Suppose each function $f \in C_1(Q, X)$ has at most one best L_1-approximant by elements from $M(f)$, but there exists a function $f \in C_1(Q, X)$

that has more than one best L_1-approximant from V. Let $v_1, v_2 \in P_V(f)$; then $\frac{v_1+v_2}{2} \in P_V(f)$. This means that

$$||f - v_1||_1 = ||f - v_2||_1 = ||f - \frac{v_1 + v_2}{2}||_1,$$

in other words

$$\int_Q ||(f - \frac{v_1 + v_2}{2})(x)||_X d\mu(x) =$$

$$= \frac{1}{2}\int_Q ||(f - v_1)(x)||_X d\mu(x) + \frac{1}{2}\int_Q ||(f - v_2)(x)||_X d\mu(x).$$

Since the integrands are continuous and nonnegative on Q, and the measure μ is positive on every nonempty open subset of the σ-field Σ, we have $\forall x \in Q$

$$||(f - \frac{v_1 + v_2}{2})(x)||_X = \frac{1}{2}||(f - v_1)(x)||_X + \frac{1}{2}|(f - v_2)(x)||_X. \tag{4.1}$$

Then, since X is a strictly convex Banach space, it follows that for $x \in Q$ either $(f - v_1)(x) = 0$, or $(f - v_2)(x) = 0$, or $(f - v_1)(x) = c(x) \cdot (f - v_2)(x)$, where $c(x)$ is a real positive function. Note that $c(x) \neq 1, \forall x \in N_{v_2-v_1}$.

Let $f_0(x) = f(x) - \frac{(v_1+v_2)(x)}{2}$. We obviously have $f_0 \in C_1(Q, X)$ and $0 \in P_V(f_0)$.

It is clear that

$$f_0(x) = \pm\frac{(v_2 - v_1)(x)}{2}$$

for $x \in Q$ such that either $(f - v_1)(x) = 0$, or $(f - v_2)(x) = 0$;

$$f_0(x) = \frac{1 + c(x)}{2(c(x) - 1)} \cdot (v_2 - v_1)(x)$$

for $x \in N_{v_2-v_1}$ such that $(f - v_1)(x) = c(x) \cdot (f - v_2)(x)$.

Hence there exists a real function $\gamma(x)$ such that $\forall x \in N_{v_2-v_1}$

$$f_0(x) = \gamma(x) \cdot (v_2 - v_1)(x).$$

Using (4.1), we get $Z_{f_0} \subset Z_{v_2-v_1}$.

Moreover, $Z_\gamma \subset Z_{v_2-v_1}$ and $|\gamma(x)| > \frac{1}{2}, \forall x \in N_{v_2-v_1}$.

Define

$$v_0(x) = \begin{cases} sgn\gamma(x) \cdot (v_2 - v_1)(x), & x \in N_{v_2-v_1}, \\ 0, & x \in Z_{v_2-v_1}. \end{cases}$$

We obviously have $v_0 \in C_1(Q, X)$ and $v_0(t_i) = 0, i = 1, ..., m$.

We now prove that $v_\varepsilon = \varepsilon \cdot (v_2 - v_1) \in P_{M(v_0)}(v_0), \forall \varepsilon \in (0; 1)$.

We will need such well-known properties of the functional $\tau_-(f(x), g(x))_X$:

(i) $\tau_-(\alpha(x)f(x), g(x))_X = \tau_-(f(x), g(x))_X, \quad \forall \alpha(x) > 0, \quad \forall x \in Q,$

where $\alpha : Q \to \mathbb{R}$;

(ii) $\tau_-(f(x),g(x))_X \geq -||g(x)||_X, \quad \forall x \in Q.$

Since

$$(v_0 - v_\varepsilon)(x) = (sgn\gamma(x) - \varepsilon)(v_2 - v_1)(x) =$$
$$= |sgn\gamma(x) - \varepsilon| \cdot sgn\gamma(x) \cdot (v_2 - v_1)(x), \quad \forall x \in N_{v_2 - v_1},$$

we have

$$\tau_-((v_0 - v_\varepsilon)(x), v(x))_X = \tau_-(sgn\gamma(x) \cdot (v_2 - v_1)(x), v(x))_X =$$
$$= \tau_-(\frac{\gamma(x)}{|\gamma(x)|} \cdot (v_2 - v_1)(x), v(x))_X = \tau_-(\gamma(x) \cdot (v_2 - v_1)(x), v(x))_X =$$
$$= \tau_-(f_0(x), v(x))_X, \quad \forall x \in N_{v_2 - v_1}.$$

In addition $Z_{v_0 - v_\varepsilon} = Z_{v_2 - v_1}$.

Using $0 \in P_V(f_0)$ and (ii), we get for any $v \in V$

$$\int\limits_{N_{v_0 - v_\varepsilon}} \tau_-((v_0 - v_\varepsilon)(x), v(x))_X d\mu(x) = \int\limits_{N_{v_2 - v_1}} \tau_-(f_0(x), v(x))_X d\mu(x) =$$
$$= \int\limits_{N_{f_0}} \tau_-(f_0(x), v(x))_X d\mu(x) - \int\limits_{Z_{v_2 - v_1} \setminus Z_{f_0}} \tau_-(f_0(x), v(x))_X d\mu(x) \leq$$
$$\leq \int\limits_{Z_{f_0}} ||v(x)||_X d\mu(x) + \int\limits_{Z_{v_2 - v_1} \setminus Z_{f_0}} ||v(x)||_X d\mu(x) = \int\limits_{Z_{v_2 - v_1}} ||v(x)||_X d\mu(x).$$

Now, by Theorem 2, $v_\varepsilon \in P_{M(v_0)}(v_0), \forall \varepsilon \in (0;1)$.

This completes the proof of Theorem 3.

Remark 1. Following A. Pinkus and H. Strauss we will note that the approximating subspace V is the unicity set for continuous functions on Q with values in strictly convex Banach space X iff V is the unicity set for functions from the subspace $\{f \in C_1(Q,X) : f(t_i) = 0, i = 1, ..., m\}$.

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