Researches in Mathematics `https://vestnmath.dnu.dp.ua`

Res. Math. **33**(3), 2025, p. 113–126 ISSN (Print) 2664-4991
doi:10.15421/242531 ISSN (Online) 2664-5009

UDK 517.5

A. A. Selivanov

Oles Honchar Dnipro National University, Dnipro 49045.
E-mail: seland2008@gmail.com

Some notes on polynomial splines on the real axis

Abstract. We prove a number of properties of polynomial splines from the L_p space on the real line, such as inclusion in other L_q spaces and behavior at infinity of splines and their derivatives. Separately, we find the general form of a set of functions orthogonal to the space of polynomial splines with arbitrary smoothness.

Key words: polynomial splines, Nikol'skii duality theorem, orthogonal sets

Анотація. Доведено низку властивостей поліноміальних сплайнів з простору L_p на дійсній осі, таких як включення в інші L_q простори та поведінка на нескінченності сплайнів та їх похідних. Також знайдено загальний вид множини функцій, ортогональних до простору поліноміальних сплайнів довільної гладкості.

Ключові слова: поліноміальні сплайни, теорема двоїстості Нікольського, ортогональні множини

MSC2020: Pri 41A15, 41A35, 41A44

1. Introduction

Polynomial splines as an approximation tool play a key role in many problems of approximation theory, in particular, in problems of functional class approximation. A significant number of results in this area are contained in the monograph [4], for further development of this area, see, for example, [8, 9] and references therein.

It should be noted that approaches to solving such problems often depend to a large extent on the domain, namely, approximation with polynomial splines is considered on a period, on a segment and on the real line. We also note that the above sources contain results mainly for the periodic case. Much less is known about results concerning non-periodic problems on the real axis (see, for example, [6, 10] and the references therein).

To justify the approximate properties of splines, it is often necessary to know the general behavior of the spline function itself. Properties of the polynomial splines in periodic case are well known and described, for example, in [4] and [1].

On the other hand, the properties of polynomial splines on the real line are not described in a similarly structured way and can only be found in specific cases as lemmas in related articles (for example, [6]). First part of the results presented in this article covers several general properties of the polynomial splines on the real line.

Separately, there exists a well-known approach used to find the upper bound for the value of the best approximation, which uses Nikol'skii duality theorem (originally described in [7]) to convert the problem of finding the exact value of the best approximation into a problem of finding the functional class orthogonal to the space of approximating functions. It was used, for example in [6] (real line case) and [5, 2, 3] (periodic case), where functional classes orthogonal to the space of polynomial splines with specific smoothness conditions were found. Second part of the results presented in this article covers finding the functional class orthogonal to the space of general polynomial splines on the real line with arbitrary smoothness.

The results obtained will allow solving extreme problems for functional classes on the real axis corresponding to their periodic analogues.

2. Notations

For a function f we will denote i-th derivative as $f^{(i)}$, additionally, by definition $f^{(0)} = f$.

$L_p(I)$ (or L_p for short when $I = \mathbb{R}$) is defined as the normed space of all summable functions $x : I \to \mathbb{R}$, with norm

$$\|x\|_{L_p(I)} = \left(\int\limits_I |x|^p\right)^{\frac{1}{p}} < \infty$$

We will denote $\|x\|_{L_p(I)}$ as $\|x\|_p$ when $I = \mathbb{R}$.

$C(I)$ is defined as the normed space of all continuous functions $x : I \to \mathbb{R}$, with norm

$$\|x\|_C = \sup_{t\in I} |x(t)| < \infty$$

$C^m(I)$ (or C^m for short when $I = \mathbb{R}$) is defined as the space of all m times continuously differentiable functions $x : I \to \mathbb{R}$. $C^0(I) = C(I)$ by definition.

$P_m(I)$ is defined as the space of all polynomials $p : I \to \mathbb{R}$ of a degree not greater than m.

$L^r_q(I)$ (or L^r_q for short when $I = \mathbb{R}$) is defined as the space of all functions $x : I \to \mathbb{R}$, such that $x^{(r-1)}$ is absolutely continuous (locally when I is not bounded) and $x^{(r)} \in L_q$.

Also, $L^r_{p,q}(I) = L_p(I) \cap L^r_q(I)$.

$W_q^r(I)$ (or W_q^r for short when $I = \mathbb{R}$) is defined as the space of all functions $x \in L_q^r$, such that $\left\|x^{(r)}\right\|_{L_q(I)} \leq 1$.

Also, $W_{p,q}^r(I) = L_p(I) \cap W_q^r(I)$.

3. Definitions

Definition 1. We say that set of points τ_j, such that $\tau_j < \tau_{j+1}, j \in \mathbb{Z}$ defines a partition Δ of $\mathbb{R}$ if

$$\mathbb{R} = \bigcup_{j \in \mathbb{Z}} \left[\tau_j; \tau_{j+1}\right]$$

Intervals $\left(\tau_j; \tau_{j+1}\right)$ are denoted as Δ_j.

Definition 2. If the partition Δ is such that all intervals Δ_j have the same length, we call it an **equidistant partition**.

Remark 1. For given $\theta \in \mathbb{R}, h \in \mathbb{R}_+$, points $\tau_j = \theta + jh$ define an equidistant partition $\Delta(\theta, h)$ of $\mathbb{R}$.

Definition 3. Given an equidistant partition $\Delta(\theta, h)$ of $\mathbb{R}$ and a non-negative integer m, we can define a function $s(t) : \mathbb{R} \to \mathbb{R}$ in the following way for all $j \in \mathbb{Z}$:

1. for $t \in \Delta_j$, $s(t) = s_j(t)$, where $s_j \in P_m(\Delta_j)$.
2. for $t = \tau_j$, $s(\tau_j) = \frac{s_{j-1}(\tau_j) + s_j(\tau_j)}{2}$

We call such function a **polynomial spline** defined on a partition $\Delta(\theta, h)$ with degree not greater than m.

A set of all splines with degree not greater than m on the partition $\Delta(\theta, h)$ will be denoted as $S_m(\Delta(\theta, h))$ or S_m for short when the partition is clear from the context.

Additionally, we will use the following notation: $S_m^k = S_m \bigcap C^{m-k}$ for $1 \leq k \leq m$. For convenience by definition S_m^{m+1} is equal to S_m.

Remark 2. For spline $s(t) \in S_m(\Delta(\theta, h))$, it is true that $s_j(t + jh) \in P_m(\Delta_0)$.

Remark 3. Clearly derivatives $s^{(k)}(t)$ for $k = 0, 1, 2, ...$ are defined and continuous for all $t \in \Delta_j, j \in \mathbb{Z}$. As for points $t = \tau_j$, we can define $s^{(k)}(\tau_j)$ as

$$s^{(k)}(\tau_j) = \frac{s_{j-1}^{(k)}(\tau_j) + s_j^{(k)}(\tau_j)}{2}$$

It is clear that if $s^{(k)}$ is naturally differentiable at point τ_j, then $s^{(k)}(\tau_j) = s_{j-1}^{(k)}(\tau_j) = s_j^{(k)}(\tau_j)$ and so the definition is also correct in this case.

Remark 4. If $s \in S_m(\Delta(\theta,h)) \cap C^r(\mathbb{R})$, then $s^{(k)} \in S_{m-k}(\Delta(\theta,h)) \cap C^{r-k}(\mathbb{R})$, $k = 0, 1, \dots, m$.

Remark 5. $S_m(\Delta(\theta,h))$ is a vector space (with $+$ operation defined naturally as point-wise summation), which may be obtained directly by checking vector space axioms.

Definition 4. Given a partition Δ of $\mathbb{R}$, we say that function $f(x) : \mathbb{R} \to \mathbb{R}$ **vanishes** at partition Δ if $f(\tau_j) = 0$, $j \in \mathbb{Z}$.

Definition 5. Let $x(t) \in L_p$. We say that function $f(t) \in L_{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$ is **orthogonal** to x (and denoted as $f \perp x$) if:

$$\int\limits_{\mathbb{R}} f(t)x(t)dt = 0$$

Definition 6. Let X be a subspace of L_p. We say that function $f \in L_{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$ is **orthogonal** to X (and denoted as $f \perp X$) if $f \perp x$ for all $x \in X$.

Definition 7. Given points $t_0 \le t_1 \le \dots \le t_{m+1}$ (called knots), B-splines $B_{i,k}$, $i = 0, \dots, m-k$, $k = 0, \dots, m$ are defined as follows:

$$B_{i,0}(x) = \begin{cases} 1 & t_i \le x < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$B_{i,k}(x) = (x - t_i)d_{i,k}B_{i,k-1}(x) + (t_{i+k+1} - x)d_{i+1,k}B_{i+1,k-1}(x),$$

where

$$d_{i,k} = \begin{cases} \frac{1}{t_{i+k} - t_i} & t_{i+k} < t_i \\ 0 & \text{otherwise} \end{cases}$$

Remark 6. $B_{i,k}(x)$ has the following well-known properties (see [11] for example):

- $B_{i,k}(x)$ has finite support $[t_i; t_{i+k+1}]$
- $B_{i,k}(x) \ge 0$ for all $x \in \mathbb{R}$
- $\frac{d}{dx}B_{i,k}(x) = k\left(d_{i,k}B_{i,k-1}(x) - d_{i+1,k}B_{i+1,k-1}(x)\right)$
- $B_{i,k}(x)$ is continuously differentiable on all intervals $(t_i; t_{i+1})$
- Consider knot t_j with multiplicity c (i.e. there are in total c knots equal to t_j). Then $B_{i,k}(x)$ $(k \ge c)$ is exactly $k-c$ times continuously differentiable. If $k < c$, then $B_{i,k}(x)$ is discontinuous at t_j.
- Let c_{max} be the largest knot multiplicity for given points $t_0, t_1, \dots, t_{m+1}$. Then $B_{i,k}(x) \in S_k^{c_{max}}(\Delta)$ for any partition Δ, which contains points $t_0, t_1, \dots, t_{m+1}$

4. Properties of $S_m \cap L_p$

The following properties of polynomial splines on the real line were obtained:

Theorem 1. *If* $s \in S_m(\Delta(\theta,h)) \cap L_p$, $1 \le p \le \infty$, *then also* $s \in S_m(\Delta(\theta,h)) \cap L_q$, *for all* q, *such that* $p \le q \le \infty$. *In other words,* $(S_m(\Delta(\theta,h)) \cap L_p) \subset L_q$.

Theorem 2. *If* $s \in S_m(\Delta(\theta,h)) \cap L_p$, $1 \le p \le \infty$, *then for every non-negative integers* r, k, *such that* $r \le k$ *and* $m - k + r \ge 0$ *and for arbitrary polynomial* $Q \in P_r(\mathbb{R})$, *it is true that* $s^{(k)}Q \in S_{m-k+r}(\Delta(\theta,h)) \cap L_p$.

Theorem 3. *If* $s \in S_m(\Delta(\theta,h)) \cap L_p$, $1 \le p < \infty$, *then for every* $k = 0, 1, \ldots, m$

$$\lim_{t \to \pm\infty} \left| s^{(k)}(t) \right| = 0$$

Corollary 1. *If* $s \in S_m(\Delta(\theta,h)) \cap L_p$, $1 \le p \le \infty$, *then* $s^{(k)} \in S_{m-k}(\Delta(\theta,h)) \cap L_p$ *for* $k = 0, 1, \ldots, m$.

Proof. Follows immediately from theorem 2 with $Q(t) = 1$.

Corollary 2. *If* $s \in S_m(\Delta(\theta,h)) \cap L_p$, $1 \le p \le \infty$, *then* $s^{(k)}$ *is bounded for* $k = 0, 1, \ldots, m$.

Proof. By corollary 1 we get $s^{(k)} \in S_{m-k}(\Delta(\theta,h)) \cap L_p$. By theorem 1 we then get that $s^{(k)} \in S_{m-k}(\Delta(\theta,h)) \cap L_\infty$. So $s^{(k)}$ is bounded.

To proof the provided properties of polynomial splines on the real line, the following auxiliary statements are needed:

Lemma 1. *If* $s \in S_m(\Delta(\theta,h)) \cap L_p$, $1 \le p < \infty$, *and* $\|\cdot\|_X$ *is an arbitrary norm of space* $P_m(\Delta_0)$, *then bi-infinite sequence*

$$\sum_{j \in \mathbb{Z}} \|s_j(\cdot + jh)\|_X^p$$

converges.

Proof. Firstly, taking into account remark 2, $s_j(t + jh) \in P_m(\Delta_0)$, so it is valid to calculate $\|s_j(\cdot + jh)\|_X$.

Since $s \in L_p$, $\|s\|_p = (\int\limits_{\mathbb{R}} |s(t)|^p dt)^{\frac{1}{p}} < \infty$. Let us denote $A = (\|s\|_p)^p$. Then

$$A = \int\limits_{\mathbb{R}} |s(t)|^p dt = \sum_{j \in \mathbb{Z}} \int\limits_{\Delta_j} |s_j(t)|^p \, dt = \sum_{j \in \mathbb{Z}} \int\limits_{\Delta_0} |s_j(t + jh)|^p \, dt = \sum_{j \in \mathbb{Z}} \|s_j(\cdot + jh)\|_{L_p(\Delta_0)}^p$$

Since $P_m(\Delta_0)$ is a finite-dimensional space, then every two norms in it are equivalent. So, there exist a number $C_X > 0$, depending on the norm X, such that for every $x \in P_m(\Delta_0)$:

$$\|x\|_{L_p(\Delta_0)} \geq C_X \|x\|_X$$

Therefore:

$$A \geq C_X^p \sum_{j\in\mathbb{Z}} \|s_j(\cdot + jh)\|_X^p$$

which implies that sequence on the right hand side converges.

Lemma 2. *If* $s \in S_m(\Delta(\theta, h)) \cap L_p$, $1 \leq p \leq \infty$, *and* $\|\cdot\|_X$ *is an arbitrary norm of space* $P_m(\Delta_0)$, *then*

$$\sup_{j\in\mathbb{Z}} \|s_j(\cdot + jh)\|_X < \infty$$

Proof. If $p < \infty$, then applying lemma 1 we obtain the convergence of the following sequence:

$$\sum_{j\in\mathbb{Z}} \|s_j(\cdot + jh)\|_X^p$$

It is necessary for convergence that all elements of the sequence are bounded, therefore

$$\sup_{j\in\mathbb{Z}} \|s_j(\cdot + jh)\|_X < \infty$$

Consider the case $p = \infty$. Since $P_m(\Delta_0)$ is a finite-dimensional space, then every two norms in it are equivalent. So, there exist a number $C_X > 0$, depending on the norm X, such that for every $x \in P_m(\Delta_0)$:

$$\|x\|_{L_\infty(\Delta_0)} \geq C_X \|x\|_X$$

Then:

$$\infty > \|s\|_{L_\infty} = \sup_{j\in\mathbb{Z}} \|s_j\|_{L_\infty(\Delta_j)} = \sup_{j\in\mathbb{Z}} \|s_j(\cdot + jh)\|_{L_\infty(\Delta_0)} \geq C_X \sup_{j\in\mathbb{Z}} \|s_j(\cdot + jh)\|_X$$

which concludes the proof.

Lemma 3. *If* a_j *is a bi-infinite sequence of non-negative numbers, and* $\sum\limits_{j\in\mathbb{Z}} a_j$ *converges, then* $\sum\limits_{j\in\mathbb{Z}} a_j^\alpha$ *also converges for* $\alpha \geq 1$.

Proof. Since $A = \sum\limits_{j\in\mathbb{Z}} a_j$ converges, then $\lim\limits_{j\to+\infty} a_j = \lim\limits_{j\to-\infty} a_j = 0$. Therefore, there exist a number $N_0 > 0$, such that for all $j \in \mathbb{Z}$, $|j| > N_0$ it is true that $a_j < 1$.

So,

$$\sum_{j\in\mathbb{Z}} a_i^\alpha = \sum_{|j|\leq N_0} a_i^\alpha + \sum_{|j|>N_0} a_i^\alpha$$

First addend is a finite number. Second addend is less than A, since $a_i^\alpha \leq a_i$, when $a_i < 1$ and $\alpha \geq 1$.

Therefore $\sum\limits_{j\in\mathbb{Z}} a_i^\alpha$ is finite.

Now we can proof the provided properites of polynomial splines on the real line:

Proof of Theorem 1. If $p = \infty$ the theorem statement is obviously true. Consider $p < \infty$.

Using lemma 1 with norm $L_q(\Delta_0)$ we get that sequence:

$$\sum_{j\in\mathbb{Z}} \left(\|s_j(\cdot + jh)\|_{L_q(\Delta_0)} \right)^p = \sum_{j\in\mathbb{Z}} a_j$$

converges. Then by lemma 3 $\sum\limits_{j\in\mathbb{Z}} a_j^{\frac{q}{p}}$ converges when $p \leq q < \infty$. So,

$$\infty > \sum_{j\in\mathbb{Z}} a_j^{\frac{q}{p}} = \sum_{j\in\mathbb{Z}} \left(\|s_j(\cdot + jh)\|_{L_q(\Delta_0)} \right)^q = \sum_{j\in\mathbb{Z}} \int\limits_{\Delta_0} |s_j(t+jh)|^q \, dt =$$

$$= \sum_{j\in\mathbb{Z}} \int\limits_{\Delta_j} |s_j(t)|^q \, dt = \int\limits_{\mathbb{R}} |s(t)|^q dt = \|s\|_{L_q}$$

therefore, $s \in L_q$, for $p \leq q < \infty$.

It is left to prove the case $q = \infty$. Using lemma 2 with norm $L_\infty(\Delta_0)$, we get that:

$$\infty > \sup_{j\in\mathbb{Z}} \|s_j(\cdot + jh)\|_{L_\infty(\Delta_0)} = \sup_{j\in\mathbb{Z}} \|s_j\|_{L_\infty(\Delta_j)} = \|s\|_{L_\infty}$$

which concludes the proof.

Proof of Theorem 2. It is clear that $s^{(k)}Q \in S_{m-k+r}(\Delta(\theta, h))$, so it is left to prove that $s^{(k)}Q \in L_p$.

Let us define a norm $\|\cdot\|_{A(I)}$ on a space $P_m(I)$. Let $p(x) \in P_m(I)$, then $p(x) = \sum\limits_{i=0}^{m} a_i x^i$. Then norm is defined as $\|p\|_{A(I)} = \sup\limits_{x\in I} \sum\limits_{i=0}^{m} |a_i| \, |x|^i$. We can verify that it satisfies norm axioms:

1. $\|p+q\|_{A(I)} = \sup\limits_{x\in I} \sum\limits_{i=0}^{m} |a_i + b_i| \, |x|^i \leq \sup\limits_{x\in I} \sum\limits_{i=0}^{m} \left(|a_i| \, |x|^i + |b_i| \, |x|^i \right) \leq$
   $\leq \sup\limits_{x\in I} \sum\limits_{i=0}^{m} |a_i| \, |x|^i + \sup\limits_{x\in I} \sum\limits_{i=0}^{m} |b_i| \, |x|^i = \|p\|_{A(I)} + \|q\|_{A(I)}$
2. $\|\lambda p\|_{A(I)} = |\lambda| \, \|p\|_{A(I)}$

3. $\|p\|_{A(I)} = 0$ implies $\sup\limits_{x\in I} \sum\limits_{i=0}^{m} |a_i|\,|x|^i = 0$, which is only possible if all a_i are zero, which means $p \equiv 0$.

Let us denote $s_j(t) = \sum\limits_{i=0}^{m} a_{ji}t^i$, and $Q(t) = \sum\limits_{i=0}^{k} b_i t^i$ (if $r < k$, then $b_k = \dots = b_{r+1} = 0$).

Directly computing k-th derivate of $s_j(t)$, obtain for $t \in \Delta_j$:

$$\left|s_j^{(k)}(t)\right| = \left|\sum_{i=k}^{m} a_{ji}\frac{i!}{(i-k)!}t^{i-k}\right| \leq \sum_{i=k}^{m} |a_{ji}|\,\frac{i!}{(i-k)!}\,|t|^{i-k} \leq m!\sum_{i=k}^{m} |a_{ji}|\,|t|^{i-k} =$$

$$= \frac{m!}{|t|^k}\sum_{i=k}^{m} |a_{ji}|\,|t|^i \leq \frac{m!}{|t|^k}\sum_{i=0}^{m} |a_{ji}|\,|t|^i \leq \frac{m!}{|t|^k}\,\|s_j\|_{A(\Delta_j)}$$

Since $\lim\limits_{t\to\pm\infty}\left|\frac{Q(t)}{t^k}\right| = |b_k|$, then for $\varepsilon = 1$, there exists a number $N_0 \in \mathbb{N}$, such that for all $|t| > \theta + N_0 h$, $|Q(t)| \leq (|b_k| + 1)\,|t|^k = C_Q\,|t|^k$, where $C_Q > 0$ and depends only on Q.

So, when $|j| > N_0$, $t \in \Delta_j$:

$$\left|s_j^{(k)}(t)Q(t)\right| \leq \frac{m!}{|t|^k}\,\|s_j\|_{A(\Delta_j)}\,C_Q\,|t|^k = C_Q m!\,\|s_j\|_{A(\Delta_j)} \tag{1}$$

Consider the case $p < \infty$.

Let us denote

$$I_j = \int\limits_{\Delta_j} \left|s_j^{(k)}(t)Q(t)\right|^p dt$$

Since both $s_j^{(k)}(t)$ and $Q(t)$ are polynomials and Δ_j is a segment of a finite length, it implies $I_j \in \mathbb{R}$.

Then

$$\left\|s^{(k)}Q\right\|_{L_p}^p = \int\limits_{\mathbb{R}} \left|s^{(k)}(t)Q(t)\right|^p dt = \sum_{j\in\mathbb{Z}} I_j = \sum_{|j|\leq N_0} I_j + \sum_{|j|>N_0} I_j$$

Clearly $\sum\limits_{|j|\leq N_0} I_j$ is a finite number. So, for $s^{(k)}Q$ to be in L_p it is sufficient that $\sum\limits_{|j|>N_0} I_j$ is finite.

But when $|j| > N_0$ we can use equation 1, so:

$$\sum_{|j|>N_0} I_j \leq \sum_{|j|>N_0}\int\limits_{\Delta_j} \left|C_Q m!\,\|s_j\|_{A(\Delta_j)}\right|^p dt \leq M\sum_{|j|>N_0} \|s_j\|_{A(\Delta_j)}^p \leq$$

$$\leq M\sum_{j\in\mathbb{Z}} \|s_j(\cdot + jh)\|_{A(\Delta_0)}^p$$

Applying lemma 1 with norm $\|\cdot\|_{A(\Delta_0)}$, we get that

$$\sum_{j\in\mathbb{Z}} \|s_j(\cdot+jh)\|^p_{A(\Delta_0)}$$

converges, which means $\sum\limits_{|j|>N_0} I_j$ is finite, showing that $s^{(k)}Q \in L_p$.

Consider the case $p=\infty$. Then

$$\left\|s^{(k)}Q\right\|_{L_\infty} = \sup_{j\in\mathbb{Z}} \left\|s_j^{(k)}Q\right\|_{L_\infty}$$

Since both $s_j^{(k)}(t)$ and $Q(t)$ are polynomials, it is clear that $\sup\limits_{|j|\le N_o} \left\|s_j^{(k)}Q\right\|_{L_\infty}$ is finite, therefore for $s^{(k)}Q$ to be in L_∞, it is enough to show that $\sup\limits_{|j|>N_o} \left\|s_j^{(k)}Q\right\|_{L_\infty}$ is finite.

But when $|j|>N_0$, we can use equation 1, so:

$$\sup_{|j|>N_o} \left\|s_j^{(k)}Q\right\|_{L_\infty} \le \sup_{|j|>N_0} C_Q m! \|s_j\|_{A(\Delta_j)} \le M \sup_{|j|>N_0} \|s_j\|_{A(\Delta_j)} \le$$

$$\le M \sum_{j\in\mathbb{Z}} \|s_j(\cdot+jh)\|_{A(\Delta_0)}$$

By lemma 2, the right hand side is finite, which concludes the proof.

Proof of Theorem 3. Firstly, we prove the statement for the spline s itself. Using lemma 1 with norm $\|x\|_X = \sup\limits_{t\in\Delta_0} |x(t)|$, we get the convergence of the following sequence:

$$\sum_{j\in\mathbb{Z}} \left(\sup_{t\in\Delta_j} |s_j(t)|\right)^p = \sum_{j\in\mathbb{Z}} a_j$$

It is necessary for convergence that $\lim\limits_{j\to\pm\infty} a_j = 0$. And since $|s(t)| \le \sup\limits_{t\in\Delta_j} |s_j(t)|$ for all $t\in\Delta_j$, it implies $\lim\limits_{t\to\pm\infty} |s(t)| = 0$.

So it is established that $s\in S_m(\Delta(\theta,h))\cap L_p$ implies $\lim\limits_{t\to\pm\infty} |s(t)| = 0$.

Now by corollary 1 $s\in S_m(\Delta(\theta,h))\cap L_p$ implies $s^{(k)}\in S_{m-k}(\Delta(\theta,h))\cap L_p$ for $k=0,1,\ldots,m$, which in turn implies $\lim\limits_{t\to\pm\infty} \left|s^{(k)}(t)\right| = 0$.

5. Set orthogonal to $S_m^k \cap L_p$

The main result obtained in this section is the following theorem:

Theorem 4. *Let $\theta \in \mathbb{R}, h > 0$, $\Delta(\theta, h)$ - an equidistant partition of $\mathbb{R}$, $1 \le p < \infty$, $m \in \mathbb{Z}$, $m \ge 0$, $1 \le k \le m+1$ and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$.*

Then $f(x) \perp S_m^k(\Delta(\theta, h)) \cap L_p$ if and only if $f(x) = g^{(m+1)}(x)$ for some $g(x) \in L_{p'}^{m+1}$, such that $g^{(j)}(x)$ vanishes at $\Delta(\theta, h)$ for $j = 0, \ldots, k-1$.

To get this result additional statements need to be proven:

Lemma 4. *Let $\theta \in \mathbb{R}, h \in \mathbb{R}_+$, $\Delta(\theta, h)$ - an equidistant partition of $\mathbb{R}$, $1 \le p \le \infty$, $m \in \mathbb{Z}$, $m \ge 0$ and $g(x) \in L_p^{m+1}$. Let $B_{i,m}$ be a B-spline defined over points $\tau_i, \ldots, \tau_{i+m+1}$. Then*

$$\int\limits_{\mathbb{R}} g^{(m+1)}(x) B_{i,m} dx = \frac{(-1)^{m+1}}{h^{m+1}} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} g(\tau_{i+j})$$

Proof. Can be easily seen integrating by parts m times.

Lemma 5. *Let $\theta \in \mathbb{R}, h \in \mathbb{R}_+, m \in N$, $1 \le p \le \infty$. Given an equidistant partition $\Delta(\theta, h)$ and function $f \in L_p^m$, such that f vanishes at $\Delta(\theta, h)$, it is true that $f^{(k)}$ is bounded for $k = 0, \ldots, m-1$.*

Proof. Since $f(\tau_j) = 0, j \in \mathbb{Z}$, then by Rolle's theorem $f^{(k)}$ has at least one zero on a segment $[\tau_j; \tau_{j+k+1}]$ for all $j \in \mathbb{Z}$.

For each $t \in \mathbb{R}$ we can find a number j, such that $t \in [\tau_j; \tau_{j+k+1}]$ and as stated above, there is a point $\alpha \in [\tau_j; \tau_{j+k+1}]$ such that $f^{(k)}(\alpha) = 0$.

Therefore:

$$\left| f^{(k)}(t) \right| = \left| \int\limits_{\alpha}^{t} f^{(k+1)}(\xi) d\xi \right| \le \int\limits_{\tau_j}^{\tau_{j+k+1}} \left| f^{(k+1)}(\xi) \right| d\xi$$

For $k = r-1$, by Hölder's inequality $\left(\frac{1}{p'} + \frac{1}{p} = 1\right)$:

$$\int\limits_{\tau_j}^{\tau_{j+r}} \left| f^{(r)}(\xi) \right| d\xi \le \left\| 1 \cdot f^{(r)} \right\|_{L_1((\tau_j, \tau_{j+r}))} \le$$

$$\le \|1\|_{L_{p'}((\tau_j, \tau_{j+r}))} \left\| f^{(r)} \right\|_{L_p((\tau_j, \tau_{j+r}))} \le (rh)^{\frac{1}{p}} \left\| f^{(r)} \right\|_{L_p}$$

So, $\left| f^{(r-1)}(t) \right| \le (rh)^{\frac{1}{p}} \left\| f^{(r)} \right\|_{L_p}$ and since $f^{(r)} \in L_p$, then $f^{(r-1)}$ is bounded.

Then, assuming that $f^{(k)}$ are bounded for $k = r-1, r-2, \ldots, m$, $m \ge 1$, we can prove that $f^{(m-1)}$ is also bounded.

Again,

$$\left| f^{(m-1)}(t) \right| \le \int\limits_{\tau_j}^{\tau_{j+m}} \left| f^{(m)}(\xi) \right| d\xi \le mh \sup_{t \in \mathbb{R}} \left| f^{(m)}(t) \right|$$

Since $f^{(m)}(t)$ is bounded by assumption, we have $f^{(m-1)}(t)$ is also bounded.

Proof of Theorem 4. We will prove the theorem by nested induction, first by m, then by k.

Basis: $m = 0, k = 1$.

Firstly we prove the necessity. Let $F(x)$ be some antiderivative of $f(x)$. It is clear that $g(x) = F(x) + C$ is also an antiderivative of $f(x)$. Therefore, it is possible to choose C in such way that $g(\tau_0) = 0$. If $f(x) \perp S_0^1(\Delta(\theta, h)) \cap L_p$ then for B-spline $B_{i,0}$ defined over points τ_i, τ_{i+1}:

$$0 = \int\limits_{\mathbb{R}} f(x)B_{i,0}dx = \int\limits_{\tau_i}^{\tau_{i+1}} g^{(1)}(x)B_{i,0}(x)dx = g(\tau_{i+1}) - g(\tau_i)$$

So $g(\tau_{i+1}) = g(\tau_i)$ for all $i \in \mathbb{Z}$, which implies $g(\tau_j) = g(\tau_0) = 0$ for all $j \in \mathbb{Z}$, so $g(x)$ vanishes at partition $\Delta(\theta, h)$.

Now we prove the sufficiency. Let $g(x) \in L_{p'}^{m+1}$, such that $g(x)$ vanishes at partition $\Delta(\theta, h)$ and $s \in S_0^1(\Delta(\theta, h)) \cap L_p$. Since s has degree 0, it is equal to some constant c_j on each interval $(\tau_j; \tau_{j+1})$. Then:

$$\int\limits_{\mathbb{R}} g^{(1)}(x)s(x)dx = \sum_{j\in\mathbb{Z}} c_j(g(\tau_{j+1}) - g(\tau_j)) = 0$$

So $g^{(1)} \perp s$ and since s is chosen arbitrarily, $g^{(1)} \perp S_0^1(\Delta(\theta, h)) \cap L_p$.

Now, assume the theorem is proven for all $m \leq n-1$ for all viable k.

Let us now prove the theorem for $m = n$.

Here we start induction by k. Basis: $k = 1$.

First we prove the necessity. Let $F_{n+1}(x)$ be some (n+1)-th antiderivative of $f(x)$. It is clear that $F_{n+1}(x) + P_n(x)$ is also an (n+1)-th antiderivative of $f(x)$, where P_n is any polynomial of degree at most n.

Therefore there exist a function $g(x)$, which is (n+1)-th antiderivative of $f(x)$ and $g(\tau_j) = 0$, $j = 0, \dots, n$.

If $f(x) \perp (S_n^1(\Delta(\theta, h)) \cap L_p)$, then for B-spline $B_{0,n}$ defined over points $\tau_0, \tau_1, \dots, \tau_{n+1}$:

$$0 = \int\limits_{\mathbb{R}} f(x)B_{0,n}(x)dx = \int\limits_{\tau_0}^{\tau_{n+1}} g^{(n+1)}(x)B_{0,n}(x)dx$$

Using lemma 4 we get:

$$0 = \frac{(-1)^{n+1}}{h^{n+1}} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} g(\tau_j)$$

Since $g(\tau_j) = 0$ for $j = 0, \dots, n$, then $g(\tau_{n+1})$ must also be zero.

Applying the same considerations with $B_{1,n}(x)$ get that $g(\tau_{n+2}) = 0$ and so on. The same way we get $g(\tau_{-1}) = 0$ and so on. Therefore $g(\tau_j) = 0$, $j \in \mathbb{Z}$, which means g vanishes at partition $\Delta(\theta, h)$.

Now we prove the sufficiency. Let $f(x) = g^{(n+1)}(x)$, where $g(x) \in L_{p'}^{n+1}$, such that $g(\tau_j) = 0$ for $j \in \mathbb{Z}$. Let $s(x) \in S_n^1(\Delta(\theta, h)) \cap L_p$

Then, taking into account lemma 5 and theorem 3

$$\int\limits_{\mathbb{R}} f(x)s(x)dx = \int\limits_{\mathbb{R}} g^{(n+1)}(x)s(x)dx = -\int\limits_{\mathbb{R}} g^{(n)}(x)s^{(1)}(x)dx$$

Since $s \in S_n^1(\Delta(\theta, h)) \cap L_p$, then by corollary 1 $s^{(1)} \in S_{n-1}^1(\Delta(\theta, h)) \cap L_p$, therefore we can apply induction assumption to $g^{(n)}$ and $s^{(1)}$ to get that $\int\limits_{\mathbb{R}} g^{(n)}(x)s^{(1)}(x)dx = 0$, which concludes the proof of the case $k = 1$.

Now we assume that the theorem is correct for all $k \leq l - 1$ and prove it for $k = l$.

First we prove the necessity. Let $f(x) \perp S_n^l(\Delta(\theta, h)) \cap L_p$. Then $f(x) \perp S_n^{l-1}(\Delta(\theta, h)) \cap L_p$, since $S_n^{l-1}(\Delta(\theta, h)) \subset S_n^l(\Delta(\theta, h))$. By induction assumption, we get that there exist $g(x) \in L_{p'}^{n+1}$, such that $f(x) = g^{(n+1)}(x)$ and $g^{(j)}(x)$ vanishes at partition $\Delta(\theta, h)$ for $j = 0, \ldots, l - 2$.

Consider B-splines $B_{i,n}$ defined over points $t_0, \ldots, t_{n+1}$, which are defined as $t_0 = t_1 = \cdots = t_{l-1} = \tau_0$, $t_j = \tau_{j-l+1}, j = l, \ldots, n + 1$. It is clear that $B_{i,n} \in (S_n^l(\Delta(\theta, h)) \cap L_p)$. Let us denote $I_k = \int\limits_{\mathbb{R}} g^{(k+1)}(x)B_{0,k}(x)dx$ for $k = l - 1, l, \ldots, n$. So:

$$I_k = \int\limits_{t_0}^{t_{k+1}} g^{(k+1)}(x)B_{0,k}(x)dx =$$

$$= g^{(k)}(t_{k+1})B_{0,k}(t_{k+1}) - g^{(k)}(t_0)B_{0,k}(t_0) -$$

$$-k \int\limits_{t_0}^{t_{k+1}} g^{(k)}(x) \left(d_{0,k}B_{0,k-1}(x) - d_{1,k}B_{1,k-1}(x)\right) dx =$$

$$= X_k - Y_k - kZ_k(x) + kW_k(x) \tag{2}$$

where

$$X_k = g^{(k)}(t_{k+1})B_{0,k}(t_{k+1})$$

$$Y_k = g^{(k)}(t_0)B_{0,k}(t_0)$$

$$Z_k(x) = \int\limits_{t_0}^{t_k} g^{(k)}(x)d_{0,k}B_{0,k-1}(x)dx$$

$$W_k(x) = \int\limits_{t_1}^{t_{k+1}} g^{(k)}(x) d_{1,k} B_{1,k-1}(x) dx$$

The knot t_{k+1} has multiplicity 1 (since $l \leq k+1$), so spline $B_{0,k}$ is continuous at this knot, therefore $B_{0,k}(t_{n+1}) = 0$ and X_k in equation 2 is zero.

The knot t_0 has multiplicity l, so spline $B_{0,k}$ is continuous at this knot if $l \leq k$ and discontinuous if $l = k+1$, so $B_{0,k}(t_0) = 0$ if $l \leq k$ and $B_{0,k}(t_0) \neq 0$ if $l = k+1$.

The knot t_1 has degree $l-1$, so spline $B_{1,k-1} \in S_{k-1}^{l-1}(\Delta(\theta, h)) \cap L_p$. Therefore $W_k(x) = 0$ by induction assumption.

If $l = k+1$, then $d_{0,k} = 0$ and therefore $Z_k(x) = 0$. If $l \leq k$, then $B_{0,k-1} \in S_{k-1}^{l}(\Delta(\theta, h)) \cap L_p$, so

$$Z_k(x) = d_{0,k} I_{k-1}$$

Therefore continuing on 2:

$$I_k = X_k - Y_k - kZ_k(x) + kW_k(x) = \begin{cases} -kd_{0,k} I_{k-1} & k \geq l \\ -g^{(k)}(t_0) B_{0,k}(t_0) & k = l-1 \end{cases}$$

Therefore

$$I_n = \begin{cases} (-1)^{(n-l)} n d_{0,n} \times (n-1) d_{0,n-1} \times \cdots \times l d_{0,l} \times g^{(l-1)}(t_0) B_{0,l-1}(t_0) & n \geq l \\ -g^{(n)}(t_0) B_{0,n}(t_0) & n = l-1 \end{cases}$$

On the other hand $I_n = 0$ because $f(x)$ is orthogonal to $B_{0,n}$. Since knot t_0 has multiplicity l, it means $B_{0,l-1}(t_0) \neq 0$, so $I_n = 0$ implies $g^{(l-1)}(\tau_0) = 0$.

Exactly the same considerations may be applied by building B-spline over points $t_i = t_{i+1} = \cdots = t_{i+l-1} = \tau_i$, $t_j = \tau_{j-l+1}, j = i+l, \ldots, i+n+1$, which will give us $g^{(l-1)}(\tau_i) = 0$ concluding the proof of necessity.

Now we prove the sufficiency. Let $g(x) \in L_{p'}^{n+1}$ such that $g^{(j)}(x)$ vanishes at $\Delta(\theta, h)$ for $j = 0, \ldots, k-1$ and $s(x) \in S_n^k(\Delta(\theta, h)) \cap L_p$.

If $k \leq n$, then using lemma 5 and theorem 3 we have:

$$\int\limits_{\mathbb{R}} g^{(n+1)}(x) s(x) dx = -\int\limits_{\mathbb{R}} g^{(n)}(x) s^{(1)}(x) dx$$

Since $s^{(1)} \in S_{n-1}^k \Delta(\theta, h)) \cap L_p$, then by induction assumption

$$\int\limits_{\mathbb{R}} g^{(n)}(x) s^{(1)}(x) dx = 0$$

.

If $k = n+1$, then

$$\int\limits_{\mathbb{R}} g^{(n+1)}(x) s(x) dx = \sum_{j \in \mathbb{Z}} \int\limits_{\tau_j}^{\tau_{j+1}} g^{(n+1)}(x) s_j(x) dx =$$

$$= \sum_{j\in\mathbb{Z}} \left(g^{(n)}(\tau_{j+1})s_j^{(1)}(\tau_{j+1}) - g^{(n)}(\tau_j)s_j^{(1)}(\tau_j) - \int\limits_{\tau_j}^{\tau_{j+1}} g^{(n)}(x)s_j^{(1)}(x)dx \right)$$

Since $g^{(k-1)}(x) = g^{(n)}(x)$ is vanishing at partition $\Delta(\theta, h)$, we have

$$\int\limits_{\mathbb{R}} g^{(n+1)}(x)s(x)dx = -\sum_{j\in\mathbb{Z}} \int\limits_{\tau_j}^{\tau_{j+1}} g^{(n)}(x)s_j^{(1)}(x)dx$$

Continuing integration by parts, we have:

$$\int\limits_{\mathbb{R}} g^{(n+1)}(x)s(x)dx = (-1)^n \sum_{j\in\mathbb{Z}} \int\limits_{\tau_j}^{\tau_{j+1}} g(x)s_j^{(n)}(x)dx = (-1)^n \sum_{j\in\mathbb{Z}} c_j(g(\tau_{j+1}) - g(\tau_j)) = 0$$

which concludes the proof of the sufficiency.

References

1. *Babenko V.F., Ligun A.A., Korneichuk N.P.*: Extremal Properties of Polynomials and Splines. Nova Science Publishers, Inc., 1996.
2. *Babenko V.F., Parfinovich N.V.*: Exact values of best approximations for classes of periodic functions by splines of deficiency 2. Math Notes 2009; 85(3-4): pp. 515–527. `doi:10.1134/S0001434609030237`
3. *Babenko V.F., Parfinovich N.V.*: On the exact values of the best approximations of classes of differentiable periodic functions by splines. Math. Notes 2010; 87(5-6): pp 623–635. `doi:10.1134/S0001434610050032`
4. *Korneichuk N.P.*: Exact constants in approximation theory. Cambridge University Press, 1991.
5. *Ligun A.A.*: Inequalities for upper bounds of functionals. Analysis Math. 1976; 2(1): pp. 11–40.
6. *Magaril-Il'yaev G.G.*: On the best approximation of functional classes by splines on the real line. MIRAS Papers 1992; 194(1): pp. 148–159.
7. *Nikol'skii S.M.*: Approximation of functions in the mean by trigonometrical polynomials. Izv. Akad. Nauk SSSR Ser. Mat. 1946; 10(3): pp. 207–256.
8. *Parfinovych N.V.*: On extremal subspaces for the widths of classes of convolutions. Researchers in Mathematics 2017; 22(8): pp. 68-79. `doi:10.15421/241708`
9. *Parfinovych N.V.*: Exact values of the best (α, β)-approximations for the classes of convolutions with kernels that do not increase the number of sign changes. Ukr. Math. J. 2018; 69(8): pp. 1248–1261. `doi:10.1007/s11253-017-1428-1`
10. *Parfinovych N.*: Non-symmetric approximations of functional classes by splines on the real line. Carpathian Math. Publ. 2021; 13(2): pp. 831–837. `doi:10.15330/cmp.13.3.831-837`
11. *Schumaker L.*: Spline Functions: Basic Theory. 3rd edn. Cambridge University Press (Cambridge Mathematical Library), 2007.

Received: 12.06.2025. *Accepted:* 25.12.2025