Res. Math. **34**(1), 2026, p. 34–48
doi:10.15421/242603
ISSN (Print) 2664-4991
ISSN (Online) 2664-5009

UDK 517.5

Inna V. Kal'chuk, Yurii I. Kharkevych
Lesya Ukrainka Volyn National University, Lutsk 43025.
E-mails: k.inna.80@gmail.com, kharkevich.juriy@gmail.com

Approximation properties of some singular integral on the classes of functions defined by the modulus of continuity

Abstract. The paper investigates the asymptotic behavior of the exact upper bounds of deviations in the uniform metric of the Weierstrass integrals of functions belonging to the classes $W^r H_\omega$, generated by the modulus of continuity $\omega(t)$.

Key words: modulus of continuity, asymptotic equality, Weierstrass integral, functional classes

Анотація. В роботі досліджено асимптотичну поведінку точних верхніх меж відхилень в рівномірній метриці інтегралів Вейєрштрасса від функцій з класів $W^r H_\omega$, породжених модулем неперервності $\omega(t)$.

Ключові слова: модуль неперервності, асимптотична рівність, інтеграл Вейєрштрасса, функціональні класи

MSC2020: 41A35, 41A60

1. Introduction

Let C be the space of 2π–periodic continuous functions, where the norm is defined by the equality $\|f\|_C = \max\limits_t |f(t)|$ and L be the space of 2π–periodic integrable functions over one period, where the norm is defined by the equality $\|f\|_L = \|f\|_1 = \int\limits_{-\pi}^{\pi} |f(t)|dt$.

Definition 1. [28, p. 113] The modulus of continuity of the function $f(x)$, continuous on the interval $[a, b]$, is called the function $\omega(t) = \omega(f, t)$, defined for $t \in [0, b-a]$ by the equality

$$\omega(t) = \omega(f,t) = \sup_{0\le h\le t} \max_{a\le x\le b-h} |f(x+h) - f(x)| = \sup_{\substack{|x'-x''|\le t,\\ x',x''\in[a,b]}} |f(x') - f(x'')|.$$

© *I. V. KAL'CHUK, YU. I. KHARKEVYCH, 2026*

Let $\omega(t)$ be an arbitrary fixed modulus of continuity. We say that the function $f(x) \in C$ belongs to the class H_ω if $\omega(f,t) \le \omega(t)$, or, equivalently, if the inequality

$$|f(t_1) - f(t_2)| \le \omega\left(|t_1 - t_2|\right)$$

holds for any $t_1, t_2 \in \mathbb{R}$.

Let W^r, $r \in \mathbb{N}$, denote the set of 2π-periodic functions whose derivatives up to order $r-1$ are absolutely continuous and whose r-th derivative satisfies $|f^{(r)}(t)| \le 1$ for almost all t.

Then $W^r H_\omega$ is the class of functions $f(t)$ from C such that $f^{(r)} \in H_\omega$, i.e.

$$W^r H_\omega = \{f \in W^r : \ f^{(r)} \in H_\omega\}, \quad r \in \mathbb{N}.$$

Let us consider the boundary value problem in the unit disk for the equation

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} - \frac{1}{\rho^2}\frac{\partial^4 u}{\partial x^4} = 0. \tag{1.1}$$

The solution of equation (1.1) satisfying the boundary condition

$$u\left(\rho, x\right)\big|_{\rho=1} = f(x), \quad -\pi \le x \le \pi, \tag{1.2}$$

where $f(x)$ is an integrable 2π–periodic function, will be denoted by $W(\rho; f; x)$. Then the solution of the boundary value problem (1.1)–(1.2) can be written as

$$W(\rho; f; x) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t+x)\Big\{\frac{1}{2} + \sum_{k=1}^{\infty} \rho^{k^2}\cos kt\Big\}dt, \quad 0 \le \rho < 1. \tag{1.3}$$

The quantity (1.3) is called the Weierstrass integral of the function f (see, for example, [1, p. 150]). Similarly to [31], setting $\rho = e^{-\frac{1}{\delta}}$, we can rewrite the Weierstrass integral as

$$W_\delta(f; x) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x+t)\Big\{\frac{1}{2} + \sum_{k=1}^{\infty} e^{-\frac{k^2}{\delta}}\cos kt\Big\}dt, \quad \delta > 0. \tag{1.4}$$

The Weierstrass integral (1.4), like the polyharmonic integrals [17, 20, 21], belongs to the class of integrals with delta-like kernels. Therefore, studying its approximation properties in certain functional spaces is highly relevant from the standpoint of modern applied mathematics [2, 4, 10, 11, 24, 27].

The problem of finding asymptotic equalities for the quantity

$$\mathcal{E}\left(\mathfrak{N}; W_\delta\right)_C = \sup_{f\in\mathfrak{N}} \left\|f\left(\cdot\right) - W_\delta\left(f; \cdot\right)\right\|_C, \tag{1.5}$$

where $\mathfrak{N} \subseteq C$ is the given class of functions and $W_\delta(f; x)$ is the Weierstrass integral, is called, following O. I. Stepanets [28], the Kolmogorov–Nikol'skii problem.

If there exists an explicit function $\varphi(\delta)$ such that as $\delta \to \infty$

$$\mathcal{E}(\mathfrak{N}; W_\delta)_C = \varphi(\delta) + o(\varphi(\delta)),$$

then the Kolmogorov–Nikol'skii problem is said to be solved for the Weierstrass integral $W_\delta(f; x)$ on the class $\mathfrak{N}$ in the metric of the space C.

In the uniform metric, the Kolmogorov–Nikol'skii problem for the Weierstrass integral on the Zygmund classes Z_α was studied by Korovkin [15], Bausov [8], and Falaleev [13]; for the Sobolev classes W^r_∞ — by Baskakov [9]. Asymptotic equalities for approximation of Stepanets classes [28] by means of the Weierstrass integrals were obtained in [16, 18].

At present, the study of approximation characteristics of differentiable function classes defined by modulus of continuity and linear summation methods of Fourier series determined by a set of functions of a natural argument and depending on a real parameter δ, still involves a number of open problems. Among such methods, in particular, is the Weierstrass integral. It is worth mentioning the work [19], where the similar problem was considered. At the same time, the approximation properties of triangular linear summation methods of Fourier series on the mentioned classes of functions are quite thoroughly studied [25, 26, 29].

Therefore, the main objective of this paper is to investigate the asymptotic behavior of the quantity

$$\mathcal{E}(W^r H_\omega; W_\delta)_C = \sup_{f \in W^r H_\omega} \|f(\cdot) - W_\delta(f; \cdot)\|_C, \ r = 1, 2, \ \delta \to \infty. \tag{1.6}$$

2. Approximation of the class $W^1 H_\omega$ by Weierstrass integrals

Theorem 1. *For any modulus of continuity $\omega(t)$, as $\delta \to \infty$, the following asymptotic inequality holds:*

$$\mathcal{E}\left(W^1 H_\omega; W_\delta\right)_C \le \frac{1}{2\sqrt{\delta}} \int_0^{\pi\sqrt{\delta}} \omega\left(\frac{2t}{\sqrt{\delta}}\right)\left[\Phi\left(\frac{\pi\sqrt{\delta}}{2}\right) - \Phi\left(\frac{t}{2}\right)\right] dt + \\ + O\left(e^{-\frac{\pi^2\delta}{4}} \omega\left(\frac{1}{\sqrt{\delta}}\right)\right), \tag{2.1}$$

where $\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$ is the probability integral. If $\omega(t)$ is a convex modulus of continuity, then the inequality (2.1) turns into an equality.

Proof. To prove the theorem, we first rewrite the Weierstrass integral in the form of the well-known singular integral. To this end, the kernel of the Weierstrass integral can be represented as follows:

$$K_\delta(t) = \frac{1}{2} + \sum_{k=1}^{\infty} e^{-\frac{k^2}{\delta}} \cos kt = \frac{1}{2}\varphi_\delta(0) + \sum_{k=1}^{\infty} \varphi_\delta(k),$$

where $\varphi_\delta(u) = e^{-\frac{u^2}{\delta}} \cos ut$.

Applying the Poisson summation formula (see, e.g., [30, p. 60]), we obtain

$$K_\delta(t) = \sqrt{2\pi} \left\{ \frac{1}{2}\Phi_\delta(0) + \sum_{k=1}^{\infty} \Phi_\delta(2\pi k) \right\}, \tag{2.2}$$

where $\Phi_\delta(u)$ is the cosine Fourier transform of the function $\varphi_\delta(u)$, that is,

$$\Phi_\delta(u) = \sqrt{\frac{2}{\pi}} \int_0^\infty \varphi_\delta(z) \cos zu dz = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{z^2}{\delta}} \cos zt \cos zu dz =$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{z^2}{\delta}} \cos z(u+t) dz + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{z^2}{\delta}} \cos z(u-t) dz = I_1 + I_2.$$

Using the formula (3.896.4) from [14]

$$\int_0^\infty e^{-\beta x^2} \cos bx dx = \frac{1}{2}\sqrt{\frac{\pi}{\beta}} \exp\left(-\frac{b^2}{4\beta}\right), \quad \mathrm{Re}\beta > 0,$$

we find the values of the integrals I_1 and I_2

$$I_1 := \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{z^2}{\delta}} \cos z(u+t) dz = \frac{\sqrt{\delta}}{2\sqrt{2}} e^{-\frac{\delta(u+t)^2}{4}}, \tag{2.3}$$

$$I_2 := \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{z^2}{\delta}} \cos z(u-t) dz = \frac{\sqrt{\delta}}{2\sqrt{2}} e^{-\frac{\delta(u-t)^2}{4}}. \tag{2.4}$$

Therefore, taking into account (2.2)–(2.4), we obtain

$$\Phi_\delta(u) = \frac{\sqrt{\delta}}{2\sqrt{2}} \left(e^{-\frac{\delta(u+t)^2}{4}} + e^{-\frac{\delta(u-t)^2}{4}} \right).$$

Hence,

$$K_\delta(t) = \sqrt{2\pi} \left\{ \frac{\sqrt{\delta}}{2\sqrt{2}} e^{-\frac{\delta t^2}{4}} + \frac{\sqrt{\delta}}{2\sqrt{2}} \sum_{k=1}^{\infty} \left(e^{-\frac{\delta(2\pi k+t)^2}{4}} + e^{-\frac{\delta(2\pi k-t)^2}{4}} \right) \right\} =$$

$$= \frac{\sqrt{\pi\delta}}{2} \sum_{k=-\infty}^{\infty} e^{-\frac{\delta(2\pi k+t)^2}{4}},$$

and, consequently,

$$W_\delta(f;x) = \frac{1}{\pi} \frac{\sqrt{\pi\delta}}{2} \int_{-\pi}^{\pi} f(x+t) \sum_{k=-\infty}^{\infty} e^{-\frac{\delta(2\pi k+t)^2}{4}} dt =$$

$$= \frac{\sqrt{\delta}}{2\sqrt{\pi}} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} f(x+t) e^{-\frac{\delta(2\pi k+t)^2}{4}} dt = \frac{\sqrt{\delta}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} (f(x+t))_{2\pi} e^{-\frac{\delta t^2}{4}} dt. \quad (2.5)$$

Along with the function $f(t)$, consider the shifted function $f_1(t) = f(t+x)$. Since the class $W^r H_\omega$ is invariant under translations, we have $f_1 \in W^r H_\omega$ whenever $f \in W^r H_\omega$. Moreover, $f_1(0) - W_\delta(f_1; 0) = f(x) - W_\delta(f; x)$. Therefore,

$$\mathcal{E}\left(W^r H_\omega; W_\delta\right)_C = \sup_{f \in W^r H_\omega} \left| f(0) - W_\delta(f; 0) \right|. \quad (2.6)$$

Using representation (2.5), we can write the difference $f(0) - W_\delta(f; 0)$ as

$$f(0) - W_\delta(f; 0) = 2f(0) \frac{\sqrt{\delta}}{2\sqrt{\pi}} \int_0^{\infty} e^{-\frac{\delta t^2}{4}} dt - \frac{\sqrt{\delta}}{2\sqrt{\pi}} \int_{-\infty}^{0} (f(t))_{2\pi} e^{-\frac{\delta t^2}{4}} dt -$$

$$- \frac{\sqrt{\delta}}{2\sqrt{\pi}} \int_0^{\infty} (f(t))_{2\pi} e^{-\frac{\delta t^2}{4}} dt = \frac{\sqrt{\delta}}{2\sqrt{\pi}} \int_0^{\infty} [2f(0) - f(t) - f(-t)]_{2\pi} e^{-\frac{\delta t^2}{4}} dt =$$

$$= \frac{1}{2\sqrt{\pi}} \int_0^{\infty} \left[2f(0) - f\left(\frac{t}{\sqrt{\delta}}\right) - f\left(-\frac{t}{\sqrt{\delta}}\right) \right]_{2\pi\sqrt{\delta}} e^{-\frac{t^2}{4}} dt. \quad (2.7)$$

Consequently,

$$\mathcal{E}\left(W^1 H_\omega; W_\delta\right)_C = \sup_{f \in W^1 H_\omega} \left| \frac{1}{2\sqrt{\pi}} \int_0^{\infty} \left(\varphi\left(\frac{t}{\sqrt{\delta}}\right) \right)_{2\pi\sqrt{\delta}} e^{-\frac{t^2}{4}} dt \right|, \quad (2.8)$$

where

$$\varphi(t) := f(t) + f(-t) - 2f(0).$$

Then

$$\frac{1}{2\sqrt{\pi}} \int_0^{\infty} \left(\varphi\left(\frac{t}{\sqrt{\delta}}\right) \right)_{2\pi\sqrt{\delta}} e^{-\frac{t^2}{4}} dt = \frac{1}{2\sqrt{\pi}} \int_0^{\pi\sqrt{\delta}} \varphi\left(\frac{t}{\sqrt{\delta}}\right) e^{-\frac{t^2}{4}} dt +$$

$$+ \frac{1}{2\sqrt{\pi}} \int_{\pi\sqrt{\delta}}^{\infty} \left(\varphi\left(\frac{t}{\sqrt{\delta}}\right) \right)_{2\pi\sqrt{\delta}} e^{-\frac{t^2}{4}} dt = I(f) + R(f), \quad (2.9)$$

where

$$I(f) := \frac{1}{2\sqrt{\pi}} \int_0^{\pi\sqrt{\delta}} \left(\varphi\left(\frac{t}{\sqrt{\delta}}\right) \right) e^{-\frac{t^2}{4}} dt,$$

$$R(f) := \frac{1}{2\sqrt{\pi}} \int_{\pi\sqrt{\delta}}^{\infty} \left(\varphi\left(\frac{t}{\sqrt{\delta}}\right) \right)_{2\pi\sqrt{\delta}} e^{-\frac{t^2}{4}} dt.$$

Taking into account that $\varphi(0)=0$ and integrating by parts, we obtain an estimate for $I(f)$:

$$|I(f)|=\frac{1}{2\sqrt{\pi}}\left|\int\limits_0^{\pi\sqrt{\delta}}\varphi\left(\frac{t}{\sqrt{\delta}}\right)e^{-\frac{t^2}{4}}dt\right|=\frac{1}{2\sqrt{\pi}}\left|\int\limits_0^{\pi\sqrt{\delta}}\varphi\left(\frac{t}{\sqrt{\delta}}\right)d\left(\int\limits_{\pi\sqrt{\delta}}^{t}e^{-\frac{\tau^2}{4}}d\tau\right)\right|=$$

$$=\frac{1}{2\sqrt{\pi\delta}}\left|\int\limits_0^{\pi\sqrt{\delta}}\varphi'\left(\frac{t}{\sqrt{\delta}}\right)\int\limits_{\pi\sqrt{\delta}}^{t}e^{-\frac{\tau^2}{4}}d\tau dt\right|.$$

Since $\varphi'(t)=f'(t)-f'(-t)$ and $f\in W^1H_\omega$, we have $\left|\varphi'\left(\frac{t}{\sqrt{\delta}}\right)\right|\le\omega\left(\frac{2t}{\sqrt{\delta}}\right)$. Therefore,

$$\sup_{f\in W^1H_\omega}|I(f)|\le\frac{1}{2\sqrt{\pi\delta}}\int\limits_0^{\pi\sqrt{\delta}}\omega\left(\frac{2t}{\sqrt{\delta}}\right)\left|\int\limits_{\pi\sqrt{\delta}}^{t}e^{-\frac{\tau^2}{4}}\,d\tau\right|dt.$$

Since

$$\left|\int\limits_{\pi\sqrt{\delta}}^{t}e^{-\frac{\tau^2}{4}}\,d\tau\right|=\sqrt{\pi}\left[\Phi\left(\frac{\pi\sqrt{\delta}}{2}\right)-\Phi\left(\frac{t}{2}\right)\right],$$

where $\Phi(x)=\frac{2}{\sqrt{\pi}}\int\limits_0^x e^{-u^2}du$ is the probability integral, we obtain

$$\sup_{f\in W^1H_\omega}|I(f)|\le\frac{1}{2\sqrt{\delta}}\int\limits_0^{\pi\sqrt{\delta}}\omega\left(\frac{2t}{\sqrt{\delta}}\right)\left[\Phi\left(\frac{\pi\sqrt{\delta}}{2}\right)-\Phi\left(\frac{t}{2}\right)\right]dt. \tag{2.10}$$

Now let us estimate the value of $R(f)$.

Using the Lagrange mean value theorem, we obtain

$$\left|\varphi\left(\frac{t}{\sqrt{\delta}}\right)\right|=\left|\left(f\left(\frac{t}{\sqrt{\delta}}\right)-f(0)\right)-\left(f(0)-f\left(-\frac{t}{\sqrt{\delta}}\right)\right)\right|=$$

$$=\left|\frac{t}{\sqrt{\delta}}f'(\xi_1)-\frac{t}{\sqrt{\delta}}f'(\xi_2)\right|=\frac{t}{\sqrt{\delta}}|f'(\xi_1)-f'(\xi_2)|,$$

where ξ_1 and ξ_2 are some points belonging respectively to the intervals $\left(0,\frac{t}{\sqrt{\delta}}\right)$ and $\left(-\frac{t}{\sqrt{\delta}},0\right)$. Hence, taking into account that $f\in W^1H_\omega$ and using the properties of the modulus of continuity, we obtain

$$\left|\varphi\left(\frac{t}{\sqrt{\delta}}\right)\right|=\frac{t}{\sqrt{\delta}}|f'(\xi_1)-f'(\xi_2)|\le\frac{t}{\sqrt{\delta}}\omega\left(\frac{2t}{\sqrt{\delta}}\right)\le\frac{(2t+1)t}{\sqrt{\delta}}\omega\left(\frac{1}{\sqrt{\delta}}\right). \tag{2.11}$$

Using the following formula (see, for example, [22, p. 33]):

$$1-\frac{2}{\sqrt{\pi}}\int\limits_0^x e^{-z^2}dz=\frac{2}{\sqrt{\pi}}\int\limits_x^\infty e^{-z^2}dz=$$
$$=\frac{e^{-x^2}}{\sqrt{\pi}x}\left[1+\sum_{k=1}^{\infty}(-1)^k\frac{1\cdot 3\cdot\ldots\cdot(2k-1)}{(2x^2)^k}\right], \tag{2.12}$$

from (2.11) we have

$$|R(f)|=\left|\frac{1}{2\sqrt{\pi}}\int\limits_{\pi\sqrt{\delta}}^{\infty}\left(\varphi\left(\frac{t}{\sqrt{\delta}}\right)\right)_{2\pi\sqrt{\delta}}e^{-\frac{t^2}{4}}dt\right|\le$$

$$\le\frac{K_1}{\sqrt{\delta}}\omega\left(\frac{1}{\sqrt{\delta}}\right)\left|\int\limits_{\pi\sqrt{\delta}}^{\infty}((2t+1)t)_{2\pi\sqrt{\delta}}\,e^{-\frac{t^2}{4}}dt\right|\le K_2\sqrt{\delta}\omega\left(\frac{1}{\sqrt{\delta}}\right)\int\limits_{\pi\sqrt{\delta}}^{\infty}e^{-\frac{t^2}{4}}dt=$$
$$=K_3e^{-\frac{\pi^2\delta}{4}}\omega\left(\frac{1}{\sqrt{\delta}}\right)\left[1+\sum_{k=1}^{\infty}(-1)^k\frac{1\cdot 3\cdot\ldots\cdot(2k-1)}{(\frac{\pi^2\delta}{2})^k}\right].$$

Therefore, we obtain

$$|R(f)|=O\left(e^{-\frac{\pi^2\delta}{4}}\omega\left(\frac{1}{\sqrt{\delta}}\right)\right). \tag{2.13}$$

Combining (2.8)–(2.10), (2.13), we obtain relation (2.1).

Let now ω be a convex modulus of continuity. To show the sharpness of the obtained estimate, we choose a function $f_0\in W^1H_\omega$ such that its derivative f_0' is an odd 2π-periodic function and, for $0\le t\le\pi$, is defined by

$$f_0'(t)=\begin{cases}\dfrac{1}{2}\omega(2t), & 0\le t\le\dfrac{\pi}{2},\\[2mm] \dfrac{1}{2}\omega(2(\pi-t)), & \dfrac{\pi}{2}\le t\le\pi.\end{cases}$$

Then $f_0'\in H_\omega$ and hence $f_0\in W^1H_\omega$. Moreover, for $0\le t\le\dfrac{\pi}{2}$ we have

$$\varphi'(t)=f_0'(t)-f_0'(-t)=2f_0'(t)=\omega(2t).$$

Thus, for the chosen function f_0 relation (2.10) becomes an equality, which proves the theorem.

Corollary 1. *Substituting $\omega(t)=t$ into equation (2.1) of Theorem 1, we obtain the asymptotic equality as $\delta\to\infty$*

$$\mathcal{E}\left(W^2;W_\delta\right)_C=\frac{1}{\delta}+O\left(e^{-\frac{\pi^2\delta}{4}}\frac{1}{\sqrt{\delta}}\right).$$

Proof. Indeed, substituting $\omega(t) = t$ into equation (2.1), we obtain

$$\mathcal{E}\left(W^2; W_\delta\right)_C = \frac{1}{2\sqrt{\delta}} \int\limits_0^{\pi\sqrt{\delta}} \frac{2t}{\sqrt{\delta}} \left[\Phi\left(\frac{\pi\sqrt{\delta}}{2}\right) - \Phi\left(\frac{t}{2}\right)\right] dt + O\left(e^{-\frac{\pi^2\delta}{4}} \frac{1}{\sqrt{\delta}}\right).$$

Applying integration by parts twice and using formula (2.12), we estimate the integral in the last equality

$$\frac{1}{2\sqrt{\delta}} \int\limits_0^{\pi\sqrt{\delta}} \frac{2t}{\sqrt{\delta}} \left[\Phi\left(\frac{\pi\sqrt{\delta}}{2}\right) - \Phi\left(\frac{t}{2}\right)\right] dt = \frac{1}{2\delta} \int\limits_0^{\pi\sqrt{\delta}} t^2 \left(\Phi\left(\frac{t}{2}\right)\right)' dt =$$

$$= \frac{1}{2\sqrt{\pi}\delta} \int\limits_0^{\pi\sqrt{\delta}} t^2 e^{-\frac{t^2}{4}} dt = \frac{1}{2\sqrt{\pi}\delta} \left(-2\pi\sqrt{\delta} e^{-\frac{\pi^2\delta}{4}} + 2 \int\limits_0^{\pi\sqrt{\delta}} e^{-\frac{t^2}{4}} dt\right) =$$

$$= \frac{1}{\sqrt{\pi}\delta} \int\limits_0^{\pi\sqrt{\delta}} e^{-\frac{t^2}{4}} dt + O\left(e^{-\frac{\pi^2\delta}{4}} \frac{1}{\sqrt{\delta}}\right) = \frac{1}{\delta}\Phi\left(\frac{\pi\sqrt{\delta}}{2}\right) + O\left(e^{-\frac{\pi^2\delta}{4}} \frac{1}{\sqrt{\delta}}\right) =$$

$$= \frac{1}{\delta} - \frac{1}{\delta} \frac{e^{-\frac{\pi^2\delta}{4}}}{\sqrt{\pi}\frac{\pi\sqrt{\delta}}{2}} \left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{(\frac{\pi^2\delta}{2})^k}\right] + O\left(e^{-\frac{\pi^2\delta}{4}} \frac{1}{\sqrt{\delta}}\right) =$$

$$= \frac{1}{\delta} + O\left(e^{-\frac{\pi^2\delta}{4}} \frac{1}{\sqrt{\delta}}\right).$$

The corollary is proved.

Corollary 2. *Substituting $\omega(t) = t^\alpha$, $0 < \alpha < 1$, into equation (2.1) of Theorem 1, we obtain the asymptotic equality as $\delta \to \infty$*

$$\mathcal{E}\left(W^1 H^\alpha; W_\delta\right)_C = \frac{2^{2\alpha}\Gamma\left(\frac{\alpha+2}{2}\right)}{(1+\alpha)\sqrt{\pi}} \frac{1}{\delta^{\frac{1+\alpha}{2}}} + O\left(e^{-\frac{\pi^2\delta}{4}} \frac{1}{\delta^{\frac{\alpha}{2}}}\right).$$

Proof. Indeed, substituting $\omega(t) = t^\alpha$ into equation (2.1) and integrating by parts, we get

$$\mathcal{E}\left(W^1 H^\alpha; W_\delta\right)_C = \frac{2^{\alpha-1}}{\delta^{\frac{1+\alpha}{2}}} \int\limits_0^{\pi\sqrt{\delta}} t^\alpha \left[\Phi\left(\frac{\pi\sqrt{\delta}}{2}\right) - \Phi\left(\frac{t}{2}\right)\right] dt + O\left(e^{-\frac{\pi^2\delta}{4}} \frac{1}{\delta^{\frac{\alpha}{2}}}\right) =$$

$$= \frac{2^{\alpha-1}}{(1+\alpha)\sqrt{\pi}\delta^{\frac{1+\alpha}{2}}} \int\limits_0^{\pi\sqrt{\delta}} t^{1+\alpha} e^{-\frac{t^2}{4}} dt + O\left(e^{-\frac{\pi^2\delta}{4}} \frac{1}{\delta^{\frac{\alpha}{2}}}\right). \tag{2.14}$$

Using formula (860.17) from [12]

$$\int_0^\infty t^\alpha e^{-\frac{t^2}{4}}dt = 2^\alpha \Gamma\left(\frac{\alpha+1}{2}\right), \quad \alpha > -1, \tag{2.15}$$

we obtain

$$\int_0^{\pi\sqrt{\delta}} t^{1+\alpha} e^{-\frac{t^2}{4}}dt = \int_0^\infty t^{1+\alpha} e^{-\frac{t^2}{4}}dt - \int_{\pi\sqrt{\delta}}^\infty t^{1+\alpha} e^{-\frac{t^2}{4}}dt =$$

$$= 2^{1+\alpha}\Gamma\left(\frac{\alpha+2}{2}\right) - \int_{\pi\sqrt{\delta}}^\infty t^{1+\alpha} e^{-\frac{t^2}{4}}dt. \tag{2.16}$$

Integrating by parts and using formula (2.12), we have

$$\int_{\pi\sqrt{\delta}}^\infty t^{1+\alpha} e^{-\frac{t^2}{4}}dt \le K_1\delta^{\frac{\alpha}{2}} e^{-\frac{\pi^2\delta}{4}} + K_2\int_{\pi\sqrt{\delta}}^\infty t^{\alpha-1} e^{-\frac{t^2}{4}}dt \le$$

$$\le K_1\delta^{\frac{\alpha}{2}} e^{-\frac{\pi^2\delta}{4}} + \frac{K_3}{\sqrt{\delta}} e^{-\frac{\pi^2\delta}{4}} \le K_4\delta^{\frac{\alpha}{2}} e^{-\frac{\pi^2\delta}{4}}. \tag{2.17}$$

From formulas (2.14)–(2.17), we obtain the statement of the corollary.

3. Approximation of the class W^2H_ω by Weierstrass integrals

Theorem 2. *For any modulus of continuity $\omega(t)$, as $\delta \to \infty$, the following asymptotic inequality holds:*

$$\mathcal{E}\left(W^2H_\omega; W_\delta\right)_C \le \frac{1}{\delta} + O\left(\frac{1}{\delta}\omega\left(\frac{1}{\sqrt{\delta}}\right) + \frac{1}{\sqrt{\delta}} e^{-\frac{\pi^2\delta}{4}}\right). \tag{3.1}$$

If $\omega(t)$ is a convex modulus of continuity, then inequality (3.1) turns into an equality.

Proof. Similarly to (2.8), the following equality holds:

$$\mathcal{E}\left(W^2H_\omega; W_\delta\right)_C = \sup_{f\in W^2H_\omega}\left|\frac{1}{2\sqrt{\pi}}\int_0^\infty \left(\varphi_1\left(\frac{t}{\sqrt{\delta}}\right)\right)_{2\pi\sqrt{\delta}} e^{-\frac{t^2}{4}}dt\right|,$$

where

$$\varphi_1(t) := f(t) + f(-t) - 2f(0).$$

Then

$$\frac{1}{2\sqrt{\pi}}\int\limits_0^{\infty}\left(\varphi_1\left(\frac{t}{\sqrt{\delta}}\right)\right)_{2\pi\sqrt{\delta}}e^{-\frac{t^2}{4}}dt=\frac{1}{2\sqrt{\pi}}\int\limits_0^{\pi\sqrt{\delta}}\varphi_1\left(\frac{t}{\sqrt{\delta}}\right)e^{-\frac{t^2}{4}}dt+$$

$$+\frac{1}{2\sqrt{\pi}}\int\limits_{\pi\sqrt{\delta}}^{\infty}\left(\varphi_1\left(\frac{t}{\sqrt{\delta}}\right)\right)_{2\pi\sqrt{\delta}}e^{-\frac{t^2}{4}}dt=I_3(f)+R_3(f). \tag{3.2}$$

where

$$I_3(f):=\frac{1}{2\sqrt{\pi}}\int\limits_0^{\pi\sqrt{\delta}}\varphi_1\left(\frac{t}{\sqrt{\delta}}\right)e^{-\frac{t^2}{4}}dt,$$

$$R_3(f):=\frac{1}{2\sqrt{\pi}}\int\limits_{\pi\sqrt{\delta}}^{\infty}\left(\varphi_1\left(\frac{t}{\sqrt{\delta}}\right)\right)_{2\pi\sqrt{\delta}}e^{-\frac{t^2}{4}}dt.$$

Taking into account that $\varphi_1(0)=\varphi_1'(0)=0$, and integrating by parts twice, we obtain

$$|I_3(f)|=\left|\frac{1}{2\sqrt{\pi}}\int\limits_0^{\pi\sqrt{\delta}}\varphi_1\Big(\frac{t}{\sqrt{\delta}}\Big)e^{-\frac{t^2}{4}}dt\right|=\left|\frac{1}{2\sqrt{\pi}}\int\limits_0^{\pi\sqrt{\delta}}\varphi_1\Big(\frac{t}{\sqrt{\delta}}\Big)d\int\limits_{\pi\sqrt{\delta}}^{t}e^{-\frac{\tau^2}{4}}d\tau\right|=$$

$$=\frac{1}{2\sqrt{\pi\delta}}\left|\int\limits_0^{\pi\sqrt{\delta}}\varphi_1'\Big(\frac{t}{\sqrt{\delta}}\Big)\int\limits_{\pi\sqrt{\delta}}^{t}e^{-\frac{\tau^2}{4}}d\tau\,dt\right|=$$

$$=\frac{1}{2\sqrt{\pi\delta}}\left|\int\limits_0^{\pi\sqrt{\delta}}\varphi_1''\Big(\frac{t}{\sqrt{\delta}}\Big)\int\limits_{\pi\sqrt{\delta}}^{t}\int\limits_z^{\pi\sqrt{\delta}}e^{-\frac{\tau^2}{4}}d\tau\,dz\,dt\right|=$$

$$=\frac{1}{2\sqrt{\pi\delta}}\left|\int\limits_0^{\pi\sqrt{\delta}}\left(\Big(f''\Big(\frac{t}{\sqrt{\delta}}\Big)-f''(0)\Big)+\Big(f''\Big(-\frac{t}{\sqrt{\delta}}\Big)-f''(0)\Big)\right)\times\right.$$

$$\left.\times\int\limits_{\pi\sqrt{\delta}}^{t}\int\limits_z^{\pi\sqrt{\delta}}e^{-\frac{\tau^2}{4}}d\tau\,dz\,dt+2\int\limits_0^{\pi\sqrt{\delta}}f''(0)\int\limits_{\pi\sqrt{\delta}}^{t}\int\limits_z^{\pi\sqrt{\delta}}e^{-\frac{\tau^2}{4}}d\tau\,dz\,dt\right|.$$

We estimate the second term on the right-hand side of the last equality:

$$\frac{1}{2\sqrt{\pi\delta}}\left|2\int\limits_0^{\pi\sqrt{\delta}}f''(0)\int\limits_{\pi\sqrt{\delta}}^{t}\int\limits_z^{\pi\sqrt{\delta}}e^{-\frac{\tau^2}{4}}d\tau\,dz\,dt\right|=$$

$$= \frac{|f''(0)|}{\sqrt{\pi}\delta} \left| \int\limits_0^{\pi\sqrt{\delta}} \int\limits_{\pi\sqrt{\delta}}^{t} \int\limits_z^{\pi\sqrt{\delta}} e^{-\frac{\tau^2}{4}} d\tau\, dz\, dt \right|.$$

Since

$$\left| \int\limits_0^{\pi\sqrt{\delta}} \int\limits_{\pi\sqrt{\delta}}^{t} \int\limits_z^{\pi\sqrt{\delta}} e^{-\frac{\tau^2}{4}} d\tau\, dz\, dt \right| = \frac{1}{2} \int\limits_0^{\pi\sqrt{\delta}} t^2 e^{-\frac{t^2}{4}} dt,$$

we obtain

$$\frac{1}{2\sqrt{\pi}\delta} \left| 2 \int\limits_0^{\pi\sqrt{\delta}} f''(0) \int\limits_{\pi\sqrt{\delta}}^{t} \int\limits_z^{\pi\sqrt{\delta}} e^{-\frac{\tau^2}{4}} d\tau\, dz\, dt \right| = \frac{|f''(0)|}{2\sqrt{\pi}\delta} \int\limits_0^{\pi\sqrt{\delta}} t^2 e^{-\frac{t^2}{4}} dt.$$

Taking into account that $|f''(0)| \le 1$, we get

$$\frac{|f''(0)|}{2\sqrt{\pi}\delta} \int\limits_0^{\pi\sqrt{\delta}} t^2 e^{-\frac{t^2}{4}} dt \le \frac{1}{2\sqrt{\pi}\delta} \int\limits_0^{\pi\sqrt{\delta}} t^2 e^{-\frac{t^2}{4}} dt.$$

As in the proof of Corollary 1,

$$\frac{1}{2\sqrt{\pi}\delta} \int\limits_0^{\pi\sqrt{\delta}} t^2 e^{-\frac{t^2}{4}} dt = \frac{1}{\delta} + O\left(e^{-\frac{\pi^2\delta}{4}} \frac{1}{\sqrt{\delta}} \right).$$

Therefore,

$$\frac{1}{2\sqrt{\pi}\delta} \left| 2 \int\limits_0^{\pi\sqrt{\delta}} f''(0) \int\limits_{\pi\sqrt{\delta}}^{t} \int\limits_z^{\pi\sqrt{\delta}} e^{-\frac{\tau^2}{4}} d\tau\, dz\, dt \right| \le \frac{1}{\delta} + O\left(e^{-\frac{\pi^2\delta}{4}} \frac{1}{\sqrt{\delta}} \right).$$

Hence,

$$\sup_{f \in W^2 H_\omega} |I_3(f)| \le \frac{1}{\delta} +$$

$$+ \frac{1}{2\sqrt{\pi}\delta} \sup_{f \in W^2 H_\omega} \left| \int\limits_0^{\pi\sqrt{\delta}} \left[\left(f''\left(\frac{t}{\sqrt{\delta}} \right) - f''(0) \right) + \left(f''\left(-\frac{t}{\sqrt{\delta}} \right) - f''(0) \right) \right] \right.$$

$$\left. \times \int\limits_{\pi\sqrt{\delta}}^{t} \int\limits_z^{\pi\sqrt{\delta}} e^{-\frac{\tau^2}{4}} d\tau\, dz\, dt \right| + O\left(\frac{1}{\sqrt{\delta}} e^{-\frac{\pi^2\delta}{4}} \right) \le$$

$$\le \frac{1}{\delta} + \frac{1}{\sqrt{\pi}\delta} \int\limits_0^{\pi\sqrt{\delta}} \omega\left(\frac{t}{\sqrt{\delta}} \right) \left| \int\limits_{\pi\sqrt{\delta}}^{t} \int\limits_z^{\pi\sqrt{\delta}} e^{-\frac{\tau^2}{4}} d\tau\, dz \right| dt + O\left(\frac{1}{\sqrt{\delta}} e^{-\frac{\pi^2\delta}{4}} \right). \qquad (3.3)$$

Now let us estimate $R_3(f)$.

Using the Lagrange mean value theorem, we find

$$\left|\varphi_1\left(\frac{t}{\sqrt{\delta}}\right)\right| = \left|\left(f\left(\frac{t}{\sqrt{\delta}}\right) - f(0)\right) - \left(f(0) - f\left(-\frac{t}{\sqrt{\delta}}\right)\right)\right| =$$

$$= \left|\frac{t}{\sqrt{\delta}} f'(\xi_1) - \frac{t}{\sqrt{\delta}} f'(\xi_2)\right| = \frac{t}{\sqrt{\delta}} |f'(\xi_1) - f'(\xi_2)|,$$

where ξ_1 and ξ_2 are some points from the intervals $\left(0, \frac{t}{\sqrt{\delta}}\right)$ and $\left(-\frac{t}{\sqrt{\delta}}, 0\right)$, respectively. Hence $\xi_2 < \xi_1$.

Using the Lagrange theorem again, we obtain

$$\frac{t}{\sqrt{\delta}} |f'(\xi_1) - f'(\xi_2)| = \frac{t}{\sqrt{\delta}} (\xi_1 - \xi_2) |f''(\xi_3)|,$$

where ξ_3 is some point in the interval (ξ_2, ξ_1).

Since $f \in W^2 H_\omega$, we have $|f''(\xi_3)| \le 1$, and therefore

$$\left|\varphi_1\left(\frac{t}{\sqrt{\delta}}\right)\right| \le K \frac{t^2}{\delta}.$$

Then, from the last estimate and relation (2.12), it follows that

$$|R_3(f)| = \left| \frac{1}{2\sqrt{\pi}} \int\limits_{\pi\sqrt{\delta}}^{\infty} \left(\varphi_1\left(\frac{t}{\sqrt{\delta}}\right)\right)_{2\pi\sqrt{\delta}} e^{-\frac{t^2}{4}} dt \right| \le$$

$$\le \frac{K_1}{\delta} \int\limits_{\pi\sqrt{\delta}}^{\infty} \left(t^2\right)_{2\pi\sqrt{\delta}} e^{-\frac{t^2}{4}} dt \le K_2 \int\limits_{\pi\sqrt{\delta}}^{\infty} e^{-\frac{t^2}{4}} dt = O\left(\frac{1}{\sqrt{\delta}} e^{-\frac{\pi^2\delta}{4}}\right).$$

Hence

$$|R_3(f)| = O\left(\frac{1}{\sqrt{\delta}} e^{-\frac{\pi^2\delta}{4}}\right). \tag{3.4}$$

Therefore, from formulas (3.3), (3.4), we obtain

$$\mathcal{E}\left(W^2 H_\omega; W_\delta\right)_C \le \frac{1}{\delta} + \frac{1}{\sqrt{\pi}\delta} \int\limits_0^{\pi\sqrt{\delta}} \omega\left(\frac{t}{\sqrt{\delta}}\right) \left| \int\limits_{\pi\sqrt{\delta}}^{t} \int\limits_z^{\pi\sqrt{\delta}} e^{-\frac{\tau^2}{4}} d\tau\, dz \right| dt + O\left(\frac{1}{\sqrt{\delta}} e^{-\frac{\pi^2\delta}{4}}\right).$$

Since

$$\left| \int\limits_{\pi\sqrt{\delta}}^{t} \int\limits_z^{\pi\sqrt{\delta}} e^{-\frac{\tau^2}{4}} d\tau\, dz \right| = \sqrt{\pi} \int\limits_{\pi\sqrt{\delta}}^{t} \left[\Phi\left(\frac{z}{2}\right) - \Phi\left(\frac{\pi\sqrt{\delta}}{2}\right)\right] dz,$$

we get

$$\mathcal{E}\left(W^2H_\omega;W_\delta\right)_C \le \frac{1}{\delta}+\frac{1}{\delta}\int\limits_0^{\pi\sqrt{\delta}}\omega\left(\frac{t}{\sqrt{\delta}}\right)\int\limits_{\pi\sqrt{\delta}}^{t}\left[\Phi\left(\frac{z}{2}\right)-\Phi\left(\frac{\pi\sqrt{\delta}}{2}\right)\right]dz\,dt+$$

$$+O\left(\frac{1}{\sqrt{\delta}}e^{-\frac{\pi^2\delta}{4}}\right).$$

Moreover, using the property

$$\omega\left(\frac{t}{\sqrt{\delta}}\right)\le(t+1)\omega\left(\frac{1}{\sqrt{\delta}}\right),$$

we have

$$\frac{1}{\delta}\int\limits_0^{\pi\sqrt{\delta}}\omega\left(\frac{t}{\sqrt{\delta}}\right)\int\limits_{\pi\sqrt{\delta}}^{t}\left[\Phi\left(\frac{z}{2}\right)-\Phi\left(\frac{\pi\sqrt{\delta}}{2}\right)\right]dz\,dt=O\left(\frac{1}{\delta}\omega\left(\frac{1}{\sqrt{\delta}}\right)\right).$$

Since $\omega\left(\frac{1}{\sqrt{\delta}}\right)\to 0, \delta\to\infty$, this term is not principal and has order $o\left(\frac{1}{\delta}\right)$. Thus, relation (3.1) holds.

If $\omega=\omega(t)$ is a convex modulus of continuity, then relation (1.26) turns into an equality. Indeed, in this case one can choose a function $f_0\in W^2H_\omega$ such that, in a neighbourhood of the origin, $f_0''(0)=1$ and

$$f_0''(t)-f_0''(0)=f_0''(-t)-f_0''(0)=-\omega(t),\qquad 0\le t\le\frac{\pi}{2}.$$

Then

$$\left|f_0''(t)-f_0''(0)\right|=\left|f_0''(-t)-f_0''(0)\right|=\omega(t).$$

Therefore, the estimates used in relation (1.26) become exact for this extremal function. Hence, for a convex modulus of continuity ω, relation (1.26) turns into an equality. The theorem is proved.

Note that the Weierstrass integral is, on the one hand, the solution of the partial differential equation of parabolic type, and on the other hand, the integral with the positive kernel. Therefore, the approximation properties of this integral studied in this paper on classes of differentiable functions defined by the modulus of continuity will contribute to the effective solution of certain optimization problems [3, 5–7, 23].

References

1. *Akhiezer N.I.*: Lectures on the Theory of Approximation, 1965.
2. *Babenko V.F., Babenko V.V., Kovalenko O.V., Parfinovych N.V.*: On Landau–Kolmogorov type inequalities for charges and their applications. Researches in Mathematics 2023; 31(1): pp. 3–16. doi:10.15421/242301

3. *Babenko V.F., Babenko Yu.V., Parfinovych N.V.*: Optimal recovery of mappings based on linear information with the help of T–splines in Banach spaces. Researches in Mathematics 2024; 32(3): pp. 6–11. doi:10.15421/242429
4. *Babenko V.F., Kovalenko O.V., Parfinovych N.V.*: Kolmogorov-type inequalities for hypersingular integrals with homogeneous characteristics. Researches in Mathematics 2024; 32(2): pp. 21–39. doi:10.15421/242417
5. *Babenko V.F., Parfinovych N.V., Skorokhodov D.S.*: Optimal recovery of operators in sequence spaces. Researches in Mathematics 2024; 32(3): pp. 40–52. doi:10.15421/242418
6. *Babenko V., Babenko V., Kovalenko O., Parfinovych N.*: Nagy type inequalities in metric measure spaces and some applications. Carpathian Math. Publ. 2023; 15(2): pp. 563–575. doi:10.15330/cmp.15.2.563-575
7. *Babenko V., Kolesnyk V., Kovalenko O., Parfinovych N.*: Kolmogorov-type inequalities in semilinear metric spaces. Carpathian Math. Publ. 2025; 17(2): pp. 579–590. doi:10.15330/cmp.17.2.579-590
8. *Bausov L.I.*: Approximation of functions of class Z_α by positive methods of summation of Fourier series. Uspekhi Mat. Nauk 1961; 16(3): pp. 143–149.
9. *Baskakov V.A.*: Some properties of operators of Abel–Poisson type. Math. Notes 1975; 17(2): pp. 101–107. doi:10.1007/BF01161864
10. *Bushev D.M., Kal'chuk I.V., Kharkevych Yu.I.*: Approximation of isometric function spaces by entire functions of exponential type. Carpathian Math. Publ. 2025; 17(2): pp. 679–692. doi:10.15330/cmp.17.2.679-692
11. *Chaichenko S.O., Shidlich A.L.*: Approximation of functions by linear methods in weighted Orlicz type spaces with variable exponent. Researches in Mathematics 2024; 32(2): pp. 70–87. doi:10.15421/242420
12. *Dvait G.B.*: Tables of Integrals and Other Mathematical Formulas. Nauka, 1973.
13. *Falaleev L.P.*: On approximation of functions by generalized Abel–Poisson operators. Sib. Math. J. 2001; 42(4): pp. 779–788.doi:10.1023/A:1010409901592
14. *Gradshtein I.S., Ryzhik I.M.*: Tables of Integrals, Sums, Series, and Products. Fizmatgiz, 1963.
15. *Korovkin P.P.*: On the best approximation of functions of class Z_2 by some linear operators. Dokl. Akad. Nauk 1959; 127(3): pp. 513–515.
16. *Kal'chuk I.V.*: Approximation of (ψ,β)–differentiable functions defined on the real axis by Weierstrass operators. Ukrainian Math. J. 2007; 59(9): pp. 1342–1363. doi:10.1007/s11253-007-0091-3
17. *Kharkevych Yu.I., Shutovskyi A.M.*: Approximation of functions from Hölder class by biharmonic Poisson integrals. Carpathian Math. Publ. 2024; 16(2): pp. 631–637. doi:10.15330/cmp.16.2.631-637
18. *Kharkevych Yu.I., Kal'chuk I.V.*: Approximation of (ψ,β)–differentiable functions by Weierstrass integrals. Ukrainian Math. J. 2007; 59(7): pp. 1059–1087. doi:10.1007/s11253-007-0069-1
19. *Kharkevych Y., Stepaniuk T.*: Approximate properties of Abel–Poisson integrals on classes of differentiable functions defined by moduli of continuity. Carpathian Math. Publ. 2023; 15(1): pp. 286–294. doi:10.15330/cmp.15.1.286-294
20. *Krasniqi X.Z.*: On approximation of bivariate functions by Abel–Poisson and conjugate Abel–Poisson means. The Journal of Analysis 2024; 32(3): pp. 1591–1617. doi:10.1007/s41478-023-00704-1

21. *Krasniqi X.Z.*: The degree of approximation for functions in the generalized Hölder class via the biharmonic Poisson integrals. J. Math. Sci. 2025. doi:10.1007/s10958-025-07861-7
22. *Lebedev N.N.*: Special Functions and Their Applications. Fizmatgiz, 1963.
23. *Niyozov I.E., Juraev D.A., Efendiev R.F., Fozilov D.Sh.,Elsayed E.E.*: Analytic solutions and solvability of the polyharmonic Cauchy problem in $\mathbb{R}^n$. Symmetry 2026; 18(1): 56. doi:10.3390/sym18010056
24. *Normurodov C.B., Ziyakulova S.A., Murodov S.K.O.*: On one highly accurate and efficient method for solving the biharmonic equation. International Journal of Applied Mathematics 2025; 38(4): pp. 437–453. doi:10.12732/ijam.v38i4.1
25. *Serdyuk A.S., Ovsii E.Y.*: Approximation of the classes $C^{\psi}_{\beta}H_{\omega}$ by generalized Zygmund sums. Ukrainian Math. J. 2009; 61(4): pp. 627–644. doi:10.1007/s11253-009-0229-6
26. *Serdyuk A.S., Ovsii I.Y.*: Uniform approximation of Poisson integrals of functions from the class H_{ω} by de la Vallee–Poussin sums. Anal. Math. 2012; 38(4): pp. 305–325. doi:10.1007/s10476-012-0403-1
27. *Shutovskyi A.M., Kharkevych Yu.I.*: On saturation of one linear method for summing Fourier series given by a function set. Researches in Mathematics 2025; 33(1): pp. 122–136. doi:10.15421/242508
28. *Stepanets A.I.*: Methods of approximation theory, Part I. Institute of Mathematics of the National Academy of Sciences of Ukraine, 2002.
29. *Stepanets A.I.*: Uniform Approximations by Trigonometric Polynomials. De Gruyter, 2001. doi:10.1515/9783110926033
30. *Titchmarsh E.*: Introduction to the Theory of Fourier Integrals. Oxford: Clarendon Press, 1948.
31. *Zhyhallo K.M., Kharkevych Yu.I.*: On some approximate properties of biharmonic Poisson integrals in the integral metric. Carpathian Math. Publ. 2024; 16(1): pp. 303–308. doi:10.15330/cmp.16.1.303-308

Received: 12.11.2025. *Accepted:* 26.06.2026