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On the structure of one-generated Leibniz rings

Abstract. The paper is devoted to the study of one-generated Leibniz rings. We begin with a short discussion of the relation between Lie rings, Lie algebras, Leibniz algebras, and Leibniz rings, and recall the basic notions needed in the sequel. A special construction of Leibniz rings generated by one element is introduced and investigated. It is shown that these rings play the role of free objects in the class of one-generated Leibniz rings. In particular, every one-generated Leibniz ring is an epimorphic image of the Leibniz ring $FL(1,\mathbb{Z})$, and the finite-exponent case is treated by means of the corresponding rings $FL(1,\mathbb{Z}_k)$. We also present an equivalent description of $FL(1,\mathbb{Z})$ in terms of the polynomial ring $\mathbb{Z}[X]$. Further, the structure of one-generated Leibniz rings is considered in several important cases, including torsion-free additive groups of finite 0-rank and additive groups which are elementary abelian p-groups.

Key words: Leibniz ring, Lie ring, Leibniz kernel, lower central series

Анотація. Статтю присвячено вивченню кілець Лейбніца, породжених одним елементом. Спочатку обговорюється зв'язок між кільцями Лі, алгебрами Лі, алгебрами Лейбніца та кільцями Лейбніца, а також нагадуються основні поняття, потрібні для подальшого викладу. Вводиться та досліджується спеціальна конструкція кілець Лейбніца, породжених одним елементом. Показано, що ці кільця відіграють роль вільних об'єктів у класі кілець Лейбніца, породжених одним елементом. Зокрема, кожне таке кільце Лейбніца є епіморфним образом кільця Лейбніца $FL(1,\mathbb{Z})$, а випадок скінченної експоненти розглядається за допомогою відповідних кілець $FL(1,\mathbb{Z}_k)$. Також наведено еквівалентний опис кільця $FL(1,\mathbb{Z})$ у термінах кільця многочленів $\mathbb{Z}[X]$. Далі розглянуто будову кілець Лейбніца, породжених одним елементом, у кількох важливих випадках, зокрема для адитивної групи скінченного 0-рангу без скруту та для адитивних груп, що є елементарними абелевими p-групами.

Ключові слова: кільце Лейбніца, кільце Лі, ядро Лейбніца, нижній центральний ряд

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1. Introduction

The theory of Lie rings is one of the most developed among the theories of non-associative rings. However, it is considerably poorer than the theory of Lie algebras, which are the linear analogue of Lie rings. Lie algebras are one of the most studied types of non-associative algebras. The theory of Lie algebras has more than a century-long history and is one of the most developed algebraic theories, rich in important and interesting results, many of which have already become classical.

However, there is a very substantial difference between Lie rings and Lie algebras. The additive groups of Lie algebras have a very simple structure: they are either elementary abelian p-groups for some prime p, or torsion-free divisible abelian groups. At the same time, the structure of the additive groups of arbitrary Lie rings can be much more complicated. This explains the fact that the theory of Lie rings is not as well developed as the theory of Lie algebras.

In turn, Lie algebras are a special case of Leibniz algebras. More precisely, Lie algebras are exactly the anticommutative Leibniz algebras.

Leibniz algebras first appeared in the papers of A. Blokh [2–4], where they were called D-algebras. However, at that time these works did not attract much attention. Only two decades later did a real interest in Leibniz algebras arise. This happened thanks to the work of J.-L. Loday [8]; see also [9, Section 10.6]. He "rediscovered" these algebras and used the term Leibniz algebras, since it was Leibniz who discovered and proved the Leibniz rule for the differentiation of functions. The main motivation for the introduction of Leibniz algebras was the study of periodicity phenomena in algebraic K-theory.

Leibniz algebras turned out to be naturally related to several areas, such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, and so on. They have also found some applications in physics. Nowadays, the theory of Leibniz algebras is one of the actively developing areas of modern algebra. We note that in recent years two monographs [1, 7] devoted to the exposition of some results of this theory have been published.

A natural extension of the concept of a Leibniz algebra is the concept of a Leibniz ring. In contrast to the theory of Lie rings, one can hardly speak of a developed theory of Leibniz rings. The number of papers devoted to Leibniz rings can be counted on the fingers of one hand. Therefore, the question of a systematic and consistent development of the theory of Leibniz rings naturally arises.

A set L endowed with two binary operations $+$ and $[,]$ is called a *Leibniz ring* (more precisely, a *left Leibniz ring*) if it satisfies the following properties:

(i) L is an abelian group with respect to addition;

(ii) $[a, b + c] = [a, b] + [a, c]$, $[a + b, c] = [a, c] + [b, c]$;

(iii) $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$

for all elements $a, b, c \in L$.

Dually, if instead of the last identity L satisfies

$$[a,[b,c]] = [[a,b],c] - [[a,c],b],$$

then L is called a *right Leibniz ring*.

We note at once that the classes of left Leibniz rings and right Leibniz rings are different. The corresponding example is given in [6]. Nevertheless, there is a very close connection between right and left Leibniz rings. If R is a right Leibniz ring, then, defining on R a new operation by

$$[\![a,b]\!] = [b,a],$$

we obtain on R the structure of a left Leibniz ring (see [6]). Similarly, one can pass from left Leibniz rings to right Leibniz rings.

In this paper, we will work with left Leibniz rings. Therefore, in what follows, by a *Leibniz ring* we mean a left Leibniz ring.

Let L be a Leibniz ring. As usual, a subset A of L is called a *subring* of L if A is closed under both operations $+$ and $[,]$ and is a Leibniz ring with respect to the restrictions of these operations.

If A and B are subgroups of the additive group of L, then $[A,B]$ denotes the subgroup of the additive group of L generated by all elements $[a,b]$, where $a \in A$ and $b \in B$. In particular, a subgroup A of the additive group of L is a subring of L if and only if $[A,A] \leq A$.

A subring A of a Leibniz ring L is called a *left ideal* (respectively, a *right ideal*) of L if $[y,x] \in A$ (respectively, $[x,y] \in A$) for every $x \in A$ and $y \in L$. In other words, if A is a left ideal (respectively, a right ideal), then $[L,A] \leq A$ (respectively, $[A,L] \leq A$).

A subring A of L is called an *ideal* (more precisely, a *two-sided ideal*) of L if it is both a left ideal and a right ideal, that is, if $[x,y],[y,x] \in A$ for every $x \in A$ and $y \in L$.

If A is an ideal of L, then we can consider the factor-ring, or quotient ring, L/A. It is not hard to see that this factor-ring is also a Leibniz ring.

A Leibniz ring L is called *abelian* (or *trivial*) if $[a,b] = 0$ for all elements $a, b \in L$. In particular, an abelian Leibniz ring is a Lie ring.

For many algebraic structures, one of the first natural steps is the study of those objects which are generated by one element. Lie rings generated by one element have a very simple structure. The situation with Leibniz rings generated by one element is much more complicated. This is already visible from the results of [5], where one-generated Leibniz algebras were described. The aim of this paper is to obtain a description of one-generated Leibniz rings.

2. Key construction and necessary definitions

An important role here is played by the following construction of Leibniz rings.

Let $C = \mathbb{Z}/k\mathbb{Z}$, where $k = 0$ or k is a positive integer. Put $A = \bigoplus_{n\in\mathbb{N}} C_n$, where C_n is isomorphic to C as an additive group for each $n \in \mathbb{N}$. Every element of A has the form $x = (\alpha_n)_{n\in\mathbb{N}}$, where $\alpha_n \in \mathbb{Z}/k\mathbb{Z}$ for every $n \in \mathbb{N}$ and $\alpha_n \neq 0$ only for finitely many indices n.

Define an operation $[,]$ on A by the following rule. Let $x = (\alpha_n)_{n\in\mathbb{N}}$ and $y = (\beta_n)_{n\in\mathbb{N}}$. Put $[x, y] = (\gamma_n)_{n\in\mathbb{N}}$, where

$$\gamma_1 = 0, \quad \gamma_2 = \alpha_1\beta_1, \quad \gamma_3 = \alpha_1\beta_2, \quad \gamma_{n+1} = \alpha_1\beta_n, \quad n \in \mathbb{N}.$$

Let $z = (\sigma_n)_{n\in\mathbb{N}}$, $[y, z] = (\lambda_n)_{n\in\mathbb{N}}$, $[x, z] = (\mu_n)_{n\in\mathbb{N}}$, $[x, [y, z]] = (\xi_n)_{n\in\mathbb{N}}$, and $[y, [x, z]] = (\eta_n)_{n\in\mathbb{N}}$.

By the definition of the operation $[,]$, we obtain that $[[x, y], z] = 0$, so that we only have to prove that $[x, [y, z]] = [y, [x, z]]$. We have

$$\xi_1 = 0, \quad \xi_2 = \alpha_1\lambda_1 = 0, \quad \xi_3 = \alpha_1\lambda_2 = \alpha_1(\beta_1\sigma_1),$$

and, for $n \geq 2$,

$$\xi_{n+1} = \alpha_1\lambda_n = \alpha_1(\beta_1\sigma_{n-1}).$$

Similarly,

$$\eta_1 = 0, \quad \eta_2 = \beta_1\mu_1 = 0, \quad \eta_3 = \beta_1\mu_2 = \beta_1(\alpha_1\sigma_1),$$

and, for $n \geq 2$,

$$\eta_{n+1} = \beta_1\mu_n = \beta_1(\alpha_1\sigma_{n-1}).$$

Thus we see that

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

It is not hard to prove that $[x+y, z] = [x, z]+[y, z]$ and $[x, y+z] = [x, y]+[x, z]$ for all elements $x, y, z \in A$. Hence A is a Leibniz ring.

Put $a_j = (\delta_{jn})_{n\in\mathbb{N}}$, $j \in \mathbb{N}$, where δ_{jn} is the Kronecker symbol. Then $[a_1, a_1] = a_2$, $[a_1, a_2] = a_3$, and $[a_1, a_n] = a_{n+1}$, $n \in \mathbb{N}$. If $x = (\alpha_n)_{n\in\mathbb{N}}$ is an arbitrary element of A, then there is a number s such that $x = (\alpha_1, \alpha_2, \ldots, \alpha_s, 0, 0, \ldots)$. Then $x = \alpha_1 a_1 + \cdots + \alpha_s a_s$. It follows that A is generated as a Leibniz ring by the element a_1.

Put now $A_j = \bigoplus_{n\geq j} C_n$. Clearly, $A_j = \gamma_j(A)$, so that the descending series

$$A = A_1 \geq A_2 \geq \cdots \geq A_n \geq A_{n+1} \geq \cdots A_\omega = \bigcap_{n\in\mathbb{N}} A_n = \langle 0\rangle$$

is the lower central series of the Leibniz ring A. It is not hard to see that A_n is generated as a subring by the element a_n, $n \in \mathbb{N}$.

If $C \cong \mathbb{Z}$, then we denote A by $FL(1, \mathbb{Z})$. If $C \cong \mathbb{Z}/k\mathbb{Z} = \mathbb{Z}_k$, where k is a positive integer, then we denote A by $FL(1, \mathbb{Z}_k)$.

Now we show that the Leibniz rings constructed above play the role of free objects for one-generated Leibniz rings. But first we recall some necessary notions and results.

Let L be a Leibniz ring. Define the *lower central series* of L,

$$L = \gamma_1(L) \geq \gamma_2(L) \geq \cdots \gamma_\alpha(L) \geq \gamma_{\alpha+1}(L) \geq \cdots \gamma_\delta(L),$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α, and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for limit ordinals λ. The last term $\gamma_\delta(L)$ is called the *lower hypocenter* of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then

$$\gamma_k(L) = [L, [L, [L, \ldots, L] \ldots]].$$

Note that $[L, L]$ is an ideal of a Leibniz ring L [6, Proposition 4]; this ideal is called the *derived ideal* of the ring L.

Every Leibniz ring has one specific ideal. Denote by $\mathrm{Leib}(L)$ the subgroup of the additive group of L generated by the elements $[a, a]$, $a \in L$. From this definition we can see that the derived ideal $[L, L]$ contains $\mathrm{Leib}(L)$. We note that $\mathrm{Leib}(L)$ is an ideal of L such that $L/\mathrm{Leib}(L)$ is a Lie ring. Moreover, if H is an ideal of L such that L/H is a Lie ring, then $\mathrm{Leib}(L) \leq H$ [6, Proposition 5]. The ideal $\mathrm{Leib}(L)$ is called the *Leibniz kernel* of the ring L.

Let L be a Leibniz ring. Put

$$\zeta^{\mathrm{left}}(L) = \{x \in L \mid [x, y] = 0 \text{ for each element } y \in L\}$$

and

$$\zeta^{\mathrm{right}}(L) = \{x \in L \mid [y, x] = 0 \text{ for each element } y \in L\}.$$

The subset $\zeta^{\mathrm{left}}(L)$ is called the *left center* of L, and the subset $\zeta^{\mathrm{right}}(L)$ is called the *right center* of L. Of course, if L is a Lie ring, then the left and right centers of L coincide.

3. Main results

Consider now some important properties.

Proposition 1. Let L be a Leibniz ring. Then $\zeta^{\mathrm{right}}(L)$ is a subring of L and $\zeta^{\mathrm{left}}(L)$ is an ideal of L.

Proof. The fact that $\zeta^{\mathrm{left}}(L)$ and $\zeta^{\mathrm{right}}(L)$ are subrings of L is almost obvious. Let x, y be arbitrary elements of L and let $z \in \zeta^{\mathrm{left}}(L)$. We have

$$[[z, x], y] = [0, y] = 0$$

and

$$[[x, z], y] = [x, [z, y]] - [z, [x, y]] = [x, 0] - 0 = 0.$$

These equalities show that $\zeta^{\mathrm{left}}(L)$ is an ideal of L.

We note that, in general, the left and right centers are different; they may even have different dimensions. This already occurs for Leibniz algebras; the corresponding examples are given in [7, Chapter 1].

Proposition 2. Let L be a Leibniz ring. Then $\zeta^{\mathrm{left}}(L)$ contains the Leibniz kernel of L. In particular, $\mathrm{Leib}(L)$ is an abelian ideal.

Proof. Indeed,

$$[[a,a],x] = [a,[a,x]] - [a,[a,x]] = 0.$$

Hence $\mathrm{Leib}(L) \leq \zeta^{\mathrm{left}}(L)$.

Let L be a Leibniz ring and let M be a subset of L. Denote by $\langle M\rangle$ the subgroup of the additive group of L generated by M, and denote by $\mathrm{lr}(M)$ the subring generated by M, that is, the intersection of all subrings of L containing M. As usual, if $L = \mathrm{lr}(M)$, then we say that the subset M *generates the Leibniz ring* L.

Let L be a Leibniz ring and let a be an element of L. Put $\gamma_1(a) = a$, $\gamma_2(a) = [a,a]$, and recursively $\gamma_{n+1}(a) = [a,\gamma_n(a)]$ for all positive integers n.

Let L be a Leibniz ring generated by an element a, and let $K = \mathrm{Leib}(L)$. Then L/K is a Lie ring generated by the coset $a+K$. As noted above, L/K is abelian and coincides with the additive group $\langle a+K\rangle$. In particular, $L = \langle a\rangle + K$. Here we have two possibilities: the additive cyclic group $\langle a+K\rangle$ is infinite, or $\langle a+K\rangle$ is finite.

Proposition 3. Let L be a Leibniz ring, let a be an element of L, and let B_j be the additive subgroup generated by the subset $\{\gamma_n(a) \mid n \geq j\}$. Then $[\langle a\rangle, B_j] = B_{j+1}$ for every $j \in \mathbb{N}$.

Proof. Every element x of B_j has the form

$$x = k_j\gamma_j(a) + k_{j+1}\gamma_{j+1}(a) + \cdots + k_{j+n}\gamma_{j+n}(a)$$

for some integers $k_j, k_{j+1}, \ldots, k_{j+n}$. Let m be an integer. We have

$$[ma,x] = \left[ma, \sum_{s=j}^{j+n} k_s\gamma_s(a)\right] = \sum_{s=j}^{j+n} mk_s\gamma_{s+1}(a) \in B_{j+1}.$$

Hence $[\langle a\rangle, B_j] \leq B_{j+1}$.

Conversely, for every element $y \in B_{j+1}$ we have

$$\begin{aligned} y &= r_{j+1}\gamma_{j+1}(a) + r_{j+2}\gamma_{j+2}(a) + \cdots + r_{j+1+n}\gamma_{j+1+n}(a) \\ &= r_{j+1}[a,\gamma_j(a)] + r_{j+2}[a,\gamma_{j+1}(a)] + \cdots + r_{j+1+n}[a,\gamma_{j+n}(a)] \in [\langle a\rangle, B_j]. \end{aligned}$$

Therefore $B_{j+1} \leq [\langle a\rangle, B_j]$, and hence $[\langle a\rangle, B_j] = B_{j+1}$.

Corollary 1. *Let L be a Leibniz ring, let a be an element of L, and let B_j be the additive subgroup generated by the subset $\{\gamma_n(a) \mid n \geq j\}$. Then*

$$[B_1, B_j] = [\langle a \rangle, B_j] = [a, B_j] = B_{j+1}$$

for every $j \in \mathbb{N}$.

Proof. Indeed, $B_1 = \langle a \rangle + B_2$. Proposition 2 implies that $[\gamma_j(a), \gamma_s(a)] = 0$ whenever $j \geq 2$, and therefore $[B_2, B_j] = \langle 0 \rangle$. It follows that

$$[B_1, B_j] = [\langle a \rangle, B_j] = B_{j+1}.$$

Note that $[a, M] = [\langle a \rangle, M]$ for every element a and every subset M of a Leibniz ring L. Hence

$$[B_1, B_j] = [\langle a \rangle, B_j] = [a, B_j] = B_{j+1}.$$

Corollary 2. *Let L be a Leibniz ring and let a be an element of L. Then the subring of L generated by the element a coincides with the additive subgroup generated by the subset $\{\gamma_n(a) \mid n \in \mathbb{N}\}$.*

Corollary 3. *Let L be a Leibniz ring, let a be an element of L, and let B be the subring of L generated by the element a. If $na \in \mathrm{Leib}(L)$ for some positive integer n, then $nB = \langle 0 \rangle$.*

Proof. Indeed, for $j \geq 2$ we have

$$n\gamma_j(a) = n[a, \gamma_{j-1}(a)] = [na, \gamma_{j-1}(a)].$$

By Proposition 2, $\mathrm{Leib}(L) \leq \zeta^{\mathrm{left}}(L)$, and therefore

$$[na, \gamma_{j-1}(a)] = 0.$$

Hence $n\gamma_j(a) = 0$ for every $j \geq 2$.

Theorem 1. *Let L be a Leibniz ring generated by an element a. If the additive group $L/\mathrm{Leib}(L)$ is infinite cyclic, then L is an epimorphic image of the Leibniz ring $FL(1, \mathbb{Z})$. If the additive group $L/\mathrm{Leib}(L)$ is finite cyclic of order k, then L is an epimorphic image of the Leibniz ring $FL(1, \mathbb{Z}_k)$.*

Proof. Suppose first that the additive group $L/\mathrm{Leib}(L)$ is infinite cyclic. Put $\gamma_1(a) = a_1 = a$, $\gamma_2(a) = [a_1, a_1] = [a_1, \gamma_1(a)]$, and $a_{n+1} = \gamma_{n+1}(a) = [a, \gamma_n(a)]$ for all positive integers n.

Define now a mapping $f : FL(1, \mathbb{Z}) \longrightarrow L$ by the following rule. Let $x = (\alpha_n)_{n \in \mathbb{N}}$. Then there is a number s such that $x = (\alpha_1, \alpha_2, \ldots, \alpha_s, 0, 0, \ldots)$. Put

$$f(x) = \sum_{1 \leq j \leq s} \alpha_j a_j.$$

By Corollary 2, L is generated as an additive group by the subset $\{a_n \mid n \in \mathbb{N}\}$. It follows that the mapping f is surjective.

Let $y = (\beta_n)_{n\in\mathbb{N}}$ be another element of $FL(1,\mathbb{Z})$. Then there is a number t such that $y = (\beta_1, \beta_2, \ldots, \beta_t, 0, 0, \ldots)$. Without loss of generality, we can assume that $s = t$. Let $[x, y] = (\gamma_n)_{n\in\mathbb{N}}$. Then

$$\gamma_1 = 0, \quad \gamma_2 = \alpha_1\beta_1, \quad \gamma_3 = \alpha_1\beta_2, \quad \ldots, \quad \gamma_s = \alpha_1\beta_{s-1}, \quad \gamma_{s+1} = \alpha_1\beta_s,$$

and $\gamma_j = 0$ whenever $j > s + 1$.

On the other hand,

$$\begin{aligned}[f(x), f(y)] &= \left[\sum_{1\le j\le s} \alpha_j a_j, \sum_{1\le j\le s} \beta_j a_j\right] \\ &= \sum_{1\le j\le s} [\alpha_1 a_1, \beta_j a_j] = \sum_{1\le j\le s} \alpha_1\beta_j [a_1, a_j] \\ &= \sum_{1\le j\le s} \alpha_1\beta_j a_{j+1} = f([x, y]).\end{aligned}$$

Hence f is an epimorphism from $FL(1,\mathbb{Z})$ onto L.

Suppose now that the additive group $L/\operatorname{Leib}(L)$ is finite cyclic of order k. Then Corollary 3 implies that $kL = \langle 0\rangle$. Now we may repeat the previous proof for the Leibniz ring $FL(1,\mathbb{Z}_k)$.

Corollary 4. *Let L be a Leibniz ring generated by an element a. Then L is an epimorphic image of the Leibniz ring $FL(1,\mathbb{Z})$.*

Proof. Indeed, if the additive group $L/\operatorname{Leib}(L)$ is infinite cyclic, then the result follows from Theorem 1. If the additive group $L/\operatorname{Leib}(L)$ is finite cyclic of order k, then, using Theorem 1 again, we obtain that L is an epimorphic image of the Leibniz ring $FL(1,\mathbb{Z}_k)$. But clearly the Leibniz ring $FL(1,\mathbb{Z}_k)$ is an epimorphic image of $FL(1,\mathbb{Z})$.

The construction given above can be transformed as follows.

Let $R = \mathbb{Z}[X]$ be the polynomial ring over the ring $\mathbb{Z}$ of integers. Let $FL(\mathbb{Z}[X])$ be the additive abelian group which is the direct sum of an infinite additive cyclic group $\langle a\rangle$ and the additive group of the ring R. Define the operation $[,]$ on $FL(\mathbb{Z}[X])$ in the following way.

Let u, v be arbitrary elements of $FL(\mathbb{Z}[X])$,

$$u = \alpha_0 a + f(X), \quad f(X) = \lambda_0 + \lambda_1 X + \lambda_2 X^2 + \cdots + \lambda_n X^n,$$

where $\alpha_0, \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{Z}$, and

$$v = \beta_0 a + g(X), \quad g(X) = \mu_0 + \mu_1 X + \mu_2 X^2 + \cdots + \mu_k X^k,$$

where $\beta_0, \mu_0, \mu_1, \mu_2, \ldots, \mu_k \in \mathbb{Z}$. Put

$$[u, v] = \alpha_0\beta_0 + \alpha_0\mu_0 X + \alpha_0\mu_1 X^2 + \alpha_0\mu_2 X^3 + \cdots + \alpha_0\mu_k X^{k+1}.$$

By this definition, $[f(X), u] = 0$ for every element $u \in FL(\mathbb{Z}[X])$ and every polynomial $f(X) \in \mathbb{Z}[X]$. Let $w = \gamma_0 a + h(X)$ be another arbitrary element of $FL(\mathbb{Z}[X])$, where

$$h(X) = \nu_0 + \nu_1 X + \nu_2 X^2 + \cdots + \nu_m X^m,$$

and $\gamma_0, \nu_0, \nu_1, \nu_2, \ldots, \nu_m \in \mathbb{Z}$. We have

$$[u, v] = \alpha_0\beta_0 + \alpha_0\mu_0 X + \alpha_0\mu_1 X^2 + \alpha_0\mu_2 X^3 + \cdots + \alpha_0\mu_k X^{k+1},$$

$$[u, w] = \alpha_0\gamma_0 + \alpha_0\nu_0 X + \alpha_0\nu_1 X^2 + \alpha_0\nu_2 X^3 + \cdots + \alpha_0\nu_m X^{m+1},$$

$$[v, w] = \beta_0\gamma_0 + \beta_0\nu_0 X + \beta_0\nu_1 X^2 + \beta_0\nu_2 X^3 + \cdots + \beta_0\nu_m X^{m+1},$$

$$[u, [v, w]] = \alpha_0\beta_0\gamma_0 X + \alpha_0\beta_0\nu_0 X^2 + \alpha_0\beta_0\nu_1 X^3 + \alpha_0\beta_0\nu_2 X^4 + \cdots + \alpha_0\beta_0\nu_m X^{m+2},$$

$$[[u, v], w] = 0,$$

and

$$[v, [u, w]] = \beta_0\alpha_0\gamma_0 X + \beta_0\alpha_0\nu_0 X^2 + \beta_0\alpha_0\nu_1 X^3 + \beta_0\alpha_0\nu_2 X^4 + \cdots + \beta_0\alpha_0\nu_m X^{m+2}.$$

Thus we can see that

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]],$$

so that $FL(\mathbb{Z}[X])$ is a Leibniz ring.

Theorem 2. *The Leibniz rings $FL(1, \mathbb{Z})$ and $FL(\mathbb{Z}[X])$ are isomorphic.*

Proof. Define the mapping $\phi : FL(\mathbb{Z}[X]) \longrightarrow FL(1, \mathbb{Z})$ by the following rule. For every element $u = \alpha_0 a + f(X)$, where

$$f(X) = \lambda_0 + \lambda_1 X + \lambda_2 X^2 + \cdots + \lambda_n X^n,$$

we put $\phi(u) = (\xi_j)_{j\in\mathbb{N}}$, where

$$\xi_1 = \alpha_0, \quad \xi_2 = \lambda_0, \quad \xi_3 = \lambda_1, \quad \ldots, \quad \xi_{n+2} = \lambda_n,$$

and $\xi_j = 0$ whenever $j > n + 2$.

Let $v = \beta_0 a + g(X)$, where

$$g(X) = \mu_0 + \mu_1 X + \mu_2 X^2 + \cdots + \mu_k X^k,$$

be another arbitrary element of $FL(\mathbb{Z}[X])$. Then $\phi(v) = (\eta_j)_{j\in\mathbb{N}}$, where

$$\eta_1 = \beta_0, \quad \eta_2 = \mu_0, \quad \eta_3 = \mu_1, \quad \ldots, \quad \eta_{k+2} = \mu_k,$$

and $\eta_j = 0$ whenever $j > k+2$.

Clearly, $\phi(u+v) = \phi(u) + \phi(v)$. We have

$$[u,v] = \alpha_0\beta_0 + \alpha_0\mu_0 X + \alpha_0\mu_1 X^2 + \alpha_0\mu_2 X^3 + \cdots + \alpha_0\mu_k X^{k+1}.$$

Hence $\phi([u,v]) = (\sigma_j)_{j\in\mathbb{N}}$, where

$$\sigma_1 = 0, \quad \sigma_2 = \alpha_0\beta_0, \quad \sigma_3 = \alpha_0\mu_0, \quad \sigma_4 = \alpha_0\mu_1, \quad \ldots, \quad \sigma_{k+2} = \alpha_0\mu_k,$$

and $\sigma_j = 0$ whenever $j > k+2$.

On the other hand,

$$[\phi(u), \phi(v)] = [(\xi_j)_{j\in\mathbb{N}}, (\eta_j)_{j\in\mathbb{N}}] = (\gamma_j)_{j\in\mathbb{N}},$$

where

$$\gamma_1 = 0, \quad \gamma_2 = \alpha_0\beta_0, \quad \gamma_3 = \alpha_0\mu_0, \quad \gamma_4 = \alpha_0\mu_1, \quad \ldots, \quad \gamma_{k+2} = \alpha_0\mu_k,$$

and $\gamma_j = 0$ whenever $j > k+2$.

Thus we can see that $\phi([u,v]) = [\phi(u), \phi(v)]$, and hence the mapping ϕ is a homomorphism. Clearly, it is surjective and $\mathrm{Ker}(\phi) = \langle 0\rangle$. Hence ϕ is an isomorphism.

For one-generated Leibniz rings of finite exponent k, there is an analogous construction; one only has to consider the ring $\mathbb{Z}_k[X]$ instead of the polynomial ring $\mathbb{Z}[X]$.

Theorem 3. *Let L be a Leibniz ring generated by an element a, and suppose that the additive group of L is torsion-free. Then either L is isomorphic to $FL(1,\mathbb{Z})$ or L has finite 0-rank.*

Proof. Let L be generated by an element a. Corollary 2 implies that L is generated as an additive group by the subset $\{\gamma_n(a) \mid n \in \mathbb{N}\}$. Note that the Leibniz kernel $\mathrm{Leib}(L)$ is generated as an additive subgroup by the subset $\{\gamma_n(a) \mid n > 1\}$.

Denote by B_j the additive subgroup generated by $\{\gamma_n(a) \mid n \geq j\}$, and by C_j the additive subgroup generated by the subset $\{\gamma_n(a) \mid 2 \leq n \leq j\}$. Then Corollary 1 shows that $B_2 = \mathrm{Leib}(L)$ and $B_{j+1} = [a, B_j]$, $j \in \mathbb{N}$. Corollary 3 implies that $L/\mathrm{Leib}(L)$ is an infinite cyclic group. It follows that $\langle a\rangle \cap \mathrm{Leib}(L) = \langle 0\rangle$.

Put $a_n = \gamma_n(a)$, $n \in \mathbb{N}$. Naturally, there appear the following two possibilities:

$$\langle a_{j+1}\rangle \cap C_j = \langle 0\rangle \quad \text{for every positive integer } j,$$

or

$$\langle a_{m+1}\rangle \cap C_m \neq \langle 0\rangle \quad \text{for some positive integer } m.$$

It is not hard to show that in the first case L is isomorphic to $FL(1,\mathbb{Z})$.

Consider the second situation. We have

$$ka_{m+1} = \sum_{2\leq j\leq m} t_j a_j$$

for some integer $k \neq 0$ and some integers $t_2, \ldots, t_m$. The equality $a_{m+2} = [a, a_{m+1}]$ implies that

$$\begin{aligned} ka_{m+2} = k[a, a_{m+1}] = [a, ka_{m+1}] &= \left[a, \sum_{2\leq j\leq m} t_j a_j\right] \\ &= \sum_{2\leq j\leq m} t_j [a, a_j] = t_2 a_3 + \cdots + t_{m-1} a_m + t_m a_{m+1}. \end{aligned}$$

It follows that

$$\begin{aligned} k^2 a_{m+2} &= kt_2 a_3 + \cdots + kt_{m-1} a_m + t_m k a_{m+1} \\ &= kt_2 a_3 + \cdots + kt_{m-1} a_m + t_m \left(\sum_{2\leq j\leq m} t_j a_j\right) \\ &= t_m t_2 a_2 + (t_m t_3 + kt_2) a_3 + \cdots + (t_m^2 + kt_{m-1}) a_m. \end{aligned}$$

Using the above arguments and ordinary induction, we obtain that for every $s \in \mathbb{N}$,

$$k^s a_{m+s} \in C_m.$$

In particular, it follows that the additive factor group L/C_m is periodic. Since C_m is finitely generated, this means that the additive group of L has finite 0-rank.

For studying the case where the additive group of L is torsion-free and has finite 0-rank, the following approach will be very useful.

Let L be a Leibniz ring. Consider the mapping $\ell_a : L \longrightarrow L$ defined by the rule $\ell_a(x) = [a, x]$, $x \in L$. For every $x, y \in L$ we have

$$\ell_a(x + y) = [a, x + y] = [a, x] + [a, y] = \ell_a(x) + \ell_a(y),$$

and

$$\ell_a([x, y]) = [a, [x, y]] = [[a, x], y] + [x, [a, y]] = [\ell_a(x), y] + [x, \ell_a(y)].$$

Further,

$$(\ell_a + \ell_b)(x) = \ell_a(x) + \ell_b(x) = [a, x] + [b, x] = [a + b, x] = \ell_{a+b}(x),$$

which implies that $\ell_a + \ell_b = \ell_{a+b}$. Moreover,

$$\begin{aligned}[\ell_a, \ell_b](x) &= (\ell_a \circ \ell_b - \ell_b \circ \ell_a)(x) \\ &= (\ell_a \circ \ell_b)(x) - (\ell_b \circ \ell_a)(x) \\ &= \ell_a(\ell_b(x)) - \ell_b(\ell_a(x)) \\ &= \ell_a([b, x]) - \ell_b([a, x]) \\ &= [a, [b, x]] - [b, [a, x]] \\ &= [[a, b], x] = \ell_{[a,b]}(x),\end{aligned}$$

which implies that $[\ell_a, \ell_b] = \ell_{[a,b]}$. This shows that the set $\{\ell_a \mid a \in L\}$ is a subalgebra of the Lie ring associated with the associative ring $\mathrm{End}(L)$.

The mapping ℓ_a is called an inner derivation of L.

Let now L be a Leibniz ring generated by an element a, and let $A = \mathrm{Leib}(L)$. As we have seen above, the mapping ℓ_a is an endomorphism of the additive group of L. We have

$$c = \gamma_2(a) = [a, a] = \ell_a(a),$$

$$\gamma_3(a) = [a, \gamma_2(a)] = \ell_a(\gamma_2(a)) = \ell_a(\ell_a(a)) = \ell_a^2(a),$$

and recursively $\gamma_{n+1}(a) = \ell_a^n(a)$ for all positive integers n. Corollary 2 implies that A is generated as an additive group by the elements $\gamma_n(a)$, $n \geq 2$. In other words, A is generated as an additive group by the elements $\ell_a^n(a)$, $n \geq 1$.

Conversely, let $\langle a \rangle$ be an infinite additive cyclic group, let A be a torsion-free abelian group, and let ϕ be an endomorphism of A. Put $L(\phi) = L = \langle a \rangle \oplus A$ and define the operation $[,]$ on $L(\phi)$ by the following rule.

We fix some element $c \in A$. Let u, v be arbitrary elements of L,

$$u = na + x, \qquad v = ka + y,$$

where n, k are integers and $x, y \in A$. Put

$$[u, v] = nkc + n\phi(y).$$

In particular, by this definition we see that $[a, a] = c$ and $[A, L] = \langle 0 \rangle$.

Let w be another element of L, $w = sa + z$, where $s \in \mathbb{Z}$ and $z \in A$. We have

$$[v, w] = [ka + y, sa + z] = ksc + k\phi(z),$$

$$[u, [v, w]] = [na + x, ksc + k\phi(z)] = n\phi(ksc + k\phi(z)) = nks\phi(c) + nk\phi^2(z),$$

$$[u, w] = [na + x, sa + z] = nsc + n\phi(z),$$

$$[v, [u, w]] = [ka + y, nsc + n\phi(z)] = k\phi(nsc + n\phi(z)) = nks\phi(c) + nk\phi^2(z),$$

and

$$[[u, v], w] = 0.$$

Therefore

$$[u,[v,w]] = [[u,v],w] + [v,[u,w]].$$

Thus we can see that L is a Leibniz ring.

If we suppose that A is generated as a group by the elements $c, \phi(c), \phi^2(c), \ldots$, then we obtain that the Leibniz ring $L(\phi)$ is generated by the element a.

This approach makes it possible to use the theory of linear operators over finite-dimensional vector spaces, which is very well developed, in the study of one-generated Leibniz rings.

Let now L be a one-generated Leibniz ring whose additive group is periodic. Then Corollary 3 shows that there is a positive integer n such that $nL = \langle 0 \rangle$. If p is a prime, then denote by L_p the maximal p-subgroup of the additive group of L. Then L_p is an ideal of L [6, Corollary 2 of Proposition 7] and

$$L = \bigoplus_{p \in \Pi(L)} L_p$$

[6, Corollary 5 of Proposition 7]. Put

$$Q_p = \bigoplus_{q \neq p} L_q.$$

Then Q_p is an ideal of L for every prime p. The isomorphism $L_p \cong L/Q_p$ shows that L_p is a one-generated Leibniz ring for every prime p.

Theorem 4. *Let L be a Leibniz ring generated by an element a, and suppose that the additive group of L is an elementary abelian p-group for some prime p. Then either L is isomorphic to $FL(1, \mathbb{Z}_p)$ or L is finite.*

Proof. Let L be generated by an element a. Corollary 2 implies that L is generated as an additive group by the subset $\{\gamma_n(a) \mid n \in \mathbb{N}\}$. Note that the Leibniz kernel $\mathrm{Leib}(L)$ is generated as an additive subgroup by the subset $\{\gamma_n(a) \mid n > 1\}$.

Denote by B_j the additive subgroup generated by $\{\gamma_n(a) \mid n \geq j\}$, and by C_j the additive subgroup generated by the subset $\{\gamma_n(a) \mid 2 \leq n \leq j\}$. Then Corollary 1 shows that $B_2 = \mathrm{Leib}(L)$ and $B_{j+1} = [a, B_j]$, $j \in \mathbb{N}$.

Put $a_n = \gamma_n(a)$, $n \in \mathbb{N}$. If $a_j \neq 0$, then the fact that $|\langle a_j \rangle|$ is prime shows that there appear the following two possibilities:

$$\langle a_{j+1} \rangle \cap C_j = \langle 0 \rangle \quad \text{for every positive integer } j,$$

or

$$\langle a_{m+1} \rangle \cap C_m \neq \langle 0 \rangle \quad \text{for some positive integer } m.$$

It is not hard to show that in the first case L is isomorphic to $FL(1, \mathbb{Z}_p)$.

Consider the second situation. Since $\langle a_{m+1}\rangle$ has prime order, it follows that $a_{m+1} \in C_m$. Hence

$$a_{m+1} = \sum_{2\le j\le m} t_j a_j.$$

The equality $a_{m+2} = [a, a_{m+1}]$ implies that

$$\begin{aligned} a_{m+2} = [a, a_{m+1}] &= \left[a, \sum_{2\le j\le m} t_j a_j\right] = \sum_{2\le j\le m} t_j [a, a_j] \\ &= t_2 a_3 + \cdots + t_{m-1} a_m + t_m a_{m+1} \\ &= t_2 a_3 + \cdots + t_{m-1} a_m + t_m \left(\sum_{2\le j\le m} t_j a_j\right) \\ &= t_m t_2 a_2 + (t_m t_3 + t_2) a_3 + \cdots + (t_m^2 + t_{m-1}) a_m. \end{aligned}$$

Thus we can see that $a_{m+2} \in C_m$. Using the above arguments and ordinary induction, we obtain that $a_{m+s} \in C_m$ for every positive integer s. In particular, it follows that $B_j \le C_m$ for all $j \ge 2$. Hence $\mathrm{Leib}(L) = B_2$ is finite, and therefore L is finite.

Corollary 5. *Let L be a Leibniz ring whose additive group is a p-group for some prime p. If L is infinite, then L/pL is isomorphic to $FL(1, \mathbb{Z}_p)$.*

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