Res. Math. **34**(1), 2026, p. 102–122 ISSN (Print) 2664-4991
doi:10.15421/242607 ISSN (Online) 2664-5009

UDK 517.9

O. V. Perehuda*, **O. A. Kapustian***, **V. V. Mogylova****

* Taras Shevchenko National University of Kyiv, Kyiv 01601.
** National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Kyiv 03056.
E-mails: perehudao@knu.ua, olenakapustian@knu.ua, mogylova.viktoria@gmail.com

Approximation of optimal control for functional-differential equations by optimal control of equations without delay in infinite-dimensional spaces

Abstract. The problem of optimal control for a functional-differential equation of parabolic type in a Banach space is studied. An approximation procedure is proposed, which consists of replacing the original equation by the system of parabolic equations without delay. It is proved that the optimal control for such a system weakly converges to the optimal control of the original problem

Key words: C_0-semigroup, functional-differential equation, delay, weak convergence, mild solution, optimal control, approximation system

Анотація. Вивчається задача оптимального керування функціонально-диференціальним рівнянням параболічного типу в банаховому просторі, шляхом його заміни апроксимаційною системою параболічних рівнянь без запізнення. Доведено, що оптимальне керування для такої системи слабко збігається до оптимального керування початкового рівняння.

Ключові слова: C_0-напівгрупа, слабка збіжність, мінімізатор, м'який розв'язок, оптимальне керування, апроксимаційна система

MSC2020: Pri 34K05, Sec 34K35, 49K21

1. Introduction

Let X be a Banach space with the norm $\|\cdot\|_X$, which will be denoted by $\|\cdot\|$ in the sequel. By $C_h = C([-h, 0]; X)$ we denote the space of X-valued continuous functions $\varphi : [-h, 0] \mapsto X$ with the norm

$$\|\varphi\|_C = \sup_{t\in[-h,0]} \|\varphi(t)\|,$$

© *O. V. PEREHUDA, O. A. KAPUSTIAN, V. V. MOGILOVA, 2026*

where $h > 0$ is the delay value. Let $L(X, X)$ be the space of linear bounded operators acting in X.

Let $A : X \mapsto X$ be an unbounded closed linear operator, $D(A)$ be its domain which is dense in X. We assume that A is the infinitesimal generator of C_0-semigroup $S(t) = e^{tA}$, $t \geq 0$ of bounded operators, and $S(t)$ is a compact operator for $t > 0$.

We consider the optimal control problem for infinite-dimensional functional-differential equation

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + f_1(t, x(t), \int_{-h}^{0} x(t+\theta)d\theta) \\ \qquad\quad + f_2\big(t, x(t), \int_{-h}^{0} x(t+\theta)d\theta\big)u(t), \quad t \in [0, T], \\ x(t) = \varphi(t), \quad t \in [-h, 0] \end{cases} \tag{1.1}$$

with the cost functional

$$J[u] = \int_0^T G(t, x(t), u(t))dt \to \inf. \tag{1.2}$$

Problem (1.1)-(1.2) is put into correspondence an optimal control problem for specially constructed system of equations without delay (2.4)-(2.5) (see below), which is called an approximation system.

The main result of the paper is to establish convergence of optimal solutions of the approximation system (2.4)-(2.5) to the optimal solutions of the original problem (1.1)-(1.2). More precisely , if m is the number of equations in system (2.4), and (x^*, u^*, J^*) and (z_{0m}^*, u_m^*, J_m^*) are solutions of (1.1)-(1.2) and (2.4)-(2.5), then

1) $J_m^* \to J^*, \quad m \to \infty$,

2) on some subsequence $u_{m_k}^* \overset{w}{\to} u^*$ in $L^p(0, T; X)$, $z_{0m}^* \to x_*$ in $C([0, T]; X)$.

Optimal control problems of functional-differential equations are of great interest both from a theoretical point of view and due to a wide range of applications. The theory of optimal control of such equations in finite-dimensional spaces is considered in [10]. Regarding such problems in infinite-dimensional spaces, we refer to the monograph [12] where classical issues are presented: the maximum principle, dynamic programming, etc. The viscosity solutions for functional-differential equations in Banach spaces was studied in the works [7], [6]. In [1], an approach related to the Hamilton-Jacobi equation is developed. In the work [11], sufficient coefficient conditions for the existence of optimal controls for functional-differential equations of parabolic type were obtained, and for equations of neutral type it was done in [15].

In this paper, a new approach is applied to the study of optimal control problems, which allows reducing the study of the control problem of functional-differential equations to the control problem of the specially constructed system of equations without delay.

It should be noted that the method of studying functional-differential equations by constructing an approximation system of equations without delay

was used in the works [4], [13] when studying stability and boundary value problems in the finite-dimensional case. Such an approach has not been applied to optimal control problems before.

The paper consists of five sections. The first section is the introduction. In section 2, we give a rigorous statement of the problem and the formulation of the main result. The third section contains some auxiliary results. The proof of the main result is given in section 4. In section 5, we demonstrate the application of the obtained theoretical results to the optimal control problem of functional-differential equations in partial derivatives.

2. Setting of the problem and the main result

Assume that the following conditions hold:

H1) Semigroup $S(t)$ is compact for $t > 0$.

This assumption is not restrictive. It holds for standard differential operators (see Example).

A1) $f_i : [0,T] \times X \times X \mapsto X$ are continuous functons, $i = 1, 2$;

A2) (linear growth condition) there exists $K > 0$ such that

$$\|f_1(t,x,y)\| + \|f_2(t,x,y)\| \le K(1 + \|x\| + \|y\|)$$

for every $t \in [0,T],\ x, y \in X$;

A3) (Lipshitz condition)

$$\|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)\| + \|f_2(t,x_1,y_1) - f_2(t,x_2,y_2)\| \le K(\|x_1 - x_2\| + \|y_1 - y_2\|),$$

for every $t \in [0,T]$, $x_1, x_2, y_1, y_2 \in X$;

A4) initial function $\varphi : [-h, 0] \mapsto X$ is continuous;

A5) control function $u(t)$ is assumed to be admissible if $u \in L^p(0,T;X)$ $(p > 1)$ and $u(t) \in F$ for a.a. $t \in (0,T)$, where F is a closed convex subset of X.

The set of admissible controls we denote by U, and

$$\|u\|_p = \Big(\int_0^T \|u\|^p dt \Big)^{\frac{1}{p}};$$

A6) $G : [0,T] \times X \times X \mapsto [0,\infty)$ is continuous, and the function $u \mapsto G(t,x,u)$ is convex;

A7) there exist constants $C_0 > 0$, $C > 0$, $\nu > 0$ such that for arbitrary x_1, x_2, $u \in X$, $t \in [0,T]$ we have

$$|G(t,x_1,u) - G(t,x_2,u)| \le C_0\|x_1 - x_2\|(1 + \|x_1\|^\nu + \|x_2\|^\nu + \|u\|^p); \quad (2.1)$$

$$G(t,x,u) \ge C\|u\|^p; \quad (2.2)$$

A8) the mapping $u \mapsto G(t,x,u)$ is Fréchet differentiable, the corresponding Fréchet derivative L_u is continuous, and there exist positive constants C_1, α such that

$$\|L_u(t,x,u)\|_* \le C_1(1+\|x\|^\alpha + \|u\|^{p-1}),$$

where $\|\cdot\|_*$ is the norm in the dual space X^*.

The initial-value problem (1.1) we will understand in the mild sense.

Definition 1. A continuous function $[0,T] \ni t \mapsto x(t) \in X$ is called a mild solution of (1.1) on $[0,T]$ if

1) $x(t) = \varphi(t)$, $t \in [-h,0]$;

2) $x \in C([0,T],X)$;

3) $x(t)$ satisfies the integral equation

$$x(t) = S(t)\varphi(0) + \int_0^t S(t-s) f_1\Big(s, x(s), \int_{-h}^0 x(s+\theta)d\theta\Big)ds + \int_0^t S(t-s) f_2\Big(s, x(s), \int_{-h}^0 x(s+\theta)d\theta\Big)u(s)ds. \tag{2.3}$$

From [17] we deduce that such a solution exists and it is unique.

Now we construct an approximation system of equations without delay.

Let us fix $m \in \mathbb{N}$ and split the interval $[-h,0]$ by points $-\frac{h}{m}j$, $j = \overline{0,m}$ into m parts.

We define functions $z_j(t) \in X$ as solutions of the following initial-value problems:

$$\begin{cases} dz_0 = \Big(Az_0 + f_1\Big(t, z_0(t), \frac{h}{m}\sum\limits_{j=1}^m z_j(t)\Big) + f_2\Big(t, z_0(t), \frac{h}{m}\sum\limits_{j=1}^m z_j(t)\Big)u(t)\Big)dt, \\ dz_j(t) = (\frac{m}{h}(z_{j-1}(t) - z_j(t)))dt, \quad t \in [0,T], \\ z_j(0) = \varphi(-\frac{hj}{m}), \quad j = \overline{0,m}. \end{cases} \tag{2.4}$$

Here $z_0(t)$ we understand in the mild sense, and others - in the usual ODE sense.

It is easy to verify that problem (2.4) has a unique solution.

Definition 2. We say that system (2.4) is an approximation problem for (1.1), if

$$\sup_{t\in[0,T]} \|x(t - \frac{h}{m}j, u) - z_j(t,u)\| \to 0, \quad m \to \infty, \quad j = \overline{0,m}.$$

For the system (2.4) we consider the cost functional

$$J_m[u] = \int_0^T G(t, z_0(t), u(t))dt \to \inf. \tag{2.5}$$

Let us denote

$$J^* = \inf_{u\in U} J[u], \ J_m^* = \inf_{u\in U} J_m[u].$$

The main result of the paper is the following theorem.

Theorem 1. *Assume that conditions H1), A1)-A8) hold. Then optimal control problems* (1.1)- (1.2) *and* (2.4)-(2.5) *have solutions* $(x^*(t), u^*(t))$ *and* $(z^*_{0m}(t), u^*_m(t))$. *Moreover,*

1)

$$J^*_m \to J^*, \quad m \to \infty; \tag{2.6}$$

2) for any $\eta > 0$ *there exists* m_0 *such that for* $m > m_0$ *we have*

$$\left|J^* - J[u^*_m]\right| < \eta, \tag{2.7}$$

i.e., the optimal control of the approximation problem is "almost optimal"for the original one;

3) there exists a sequence $m_k \to \infty$, $k \to \infty$ *such that*

$$z^*_{om_k}(t) \to x^*(t) \text{ in } C([0,T];X); \tag{2.8}$$
$$u^*_m \xrightarrow{w} u^* \text{ weakly in } L^p(0,T).$$

If, additionally, problem (1.1)-(1.2) *has a unique solution then convergences* (2.8) *take place for* $m \to \infty$.

3. Auxiliary results

Let us denote by W the set of all admissible controls u such that $\|u\|_p \le B$ for some $B > 0$. For $u \in W$ we denote by $x(t,u)$ and $z_j(t,u)$ the corresponding solutions of (1.1) and (2.4).

Theorem 2. *Assume that conditions H1), A1)-A5) hold. Then* (2.4) *is the approximation problem for* (1.1) *uniformly with respect to* $u \in W$, *i.e.,*

$$\sup_{u\in W}\sup_{t\in[0,T]} \|x(t-\frac{h}{m}j,u) - z_j(t,u)\| \to 0, \quad m \to \infty. \tag{3.1}$$

To prove this theorem we need the following result.

Lemma 1. *Under conditions of Theorem 2 the solution of* (1.1) *with* $u \in W$ *satisfies the inequality*

$$\sup_{\substack{|t_1-t_2|\le l,\\ t_1,t_2\in[-h,T]}} \|x(t_2,u) - x(t_1,u)\| \le C(T, \|\varphi\|_C, h, l, B) \to 0, \quad l \to 0. \tag{3.2}$$

Proof. In the proof, we will denote $x(t,u)$ and $z_j(t,u)$ by $x(t)$ and $z_j(t)$, respectively. We will also omit the dependence on u of all the latter functions, observing the uniformity of all estimates for $u \in W$.

According to the definition of solution of (2.3) we have

$$\|x(t)| \le \|S(t)\varphi(0)\| + \int_0^t \|S(t-s)f_1\Big(s, x(s), \int_{-h}^0 x(s+\theta)d\theta\Big)\|ds +$$

$$+\int_0^t \|S(t-s)f_2\Big(s,x(s),\int_{-h}^0 x(s+\theta)d\theta\Big)\|\|u(s)\|ds. \tag{3.3}$$

Since $S(t)$ is a C_0-semigroup, so $\|S(t)\| \le M$, $t \in [0,T]$.

It is easy to see that

$$\int_{-h}^0 \|x(s+\theta)\|d\theta \le \|\varphi\|_C + \int_0^s \|x(s_1)\|ds_1.$$

Then from A2) and *Hölder's* inequality, we get

$$\begin{aligned}\|x(t)\|^q &\le 3^{q-1}\Big(M^q\|\varphi(0)\|^q + M^qK^q4^{q-1}\int_0^t(1+\|x(s)\|^q + \|\varphi\|_C^q \\ &+ \int_0^s \|x(s_1)\|^q ds_1)ds + M^qK^q4^{q-1}\int_0^t(1+\|x(s)\|^q \\ &+ \|\varphi\|_C^q + \int_0^s \|x(s_1)\|^q ds_1)ds\cdot\|u\|_p^q)\Big),\end{aligned} \tag{3.4}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

So,

$$\sup_{s\in[0,t]}\|x(s)\|^q \le C_1 + C_2(B)\int_0^t \sup_{s_1\in[0,s]}\|x(s_1)\|^q ds_1,$$

and from the Gronwall inequality we obtain

$$\sup_{s\in[0,T]}\|x(s)\|^q \le C_3(T,\|\varphi\|_C,h,B) \tag{3.5}$$

uniformly with respect to $u \in W$.

Let $t_2 = t_1 + r,\ r \le l$.

If $t_1 \ge 0$ then

$$\begin{aligned}x(t_2)-x(t_1) &= (S(t_2)-S(t_1))\varphi(0) \\ &+ \int_0^{t_1}((S(t_2-s)-S(t_1-s))f_1(s,x(s),\int_{-h}^0 x(s+\theta)d\theta)ds \\ &+ \int_{t_1}^{t_2} S(t_2-s)f_1(s,x(s),\int_{-h}^0 x(s+\theta)d\theta)ds \\ &+ \int_0^{t_1}((S(t_2-s)-S(t_1-s))f_2(s,x(s),\int_{-h}^0 x(s+\theta)d\theta)u(s)ds \\ &+ \int_{t_1}^{t_2} S(t_2-s)f_2(s,x(s),\int_{-h}^0 x(s+\theta)d\theta)u(s)ds.\end{aligned}$$

Thus, in this case we deduce that

$$\sup_{|t_1-t_2|<l}\|x(t_2)-x(t_1)\|^q = \sup_{r\in[0,l]}\|x(t_1+r)-x(t_1)\|^q) \le$$

$$\le 5^{q-1}\Big(\sup_{r\in[0,l]} \|S(t_1+r)\varphi(0) - S(t_1)\varphi(0)\|^q$$
$$+ \sup_{r\in[0,l]} \int_0^{t_1} \|S(t_1+r-s) - S(t_1-s)\|^q \|f_1(s,x(s), \int_{-h}^0 x(s+\theta)d\theta)\|^q ds$$
$$+ \sup_{r\in[0,l]} \int_{t_1}^{t_1+l} \|S(t_1+r-s)\|^q \|f_1(s,x(s), \int_{-h}^0 x(s+\theta)d\theta)\|^q ds$$
$$+ \sup_{r\in[0,l]} \int_0^{t_1} \|S(t_1+r-s) - S(t_1-s)\|^q \|f_2(s,x, \int_{-h}^0 x(s+\theta)d\theta)\|^q \|u\|^q ds$$
$$+ \sup_{r\in[0,l]} \int_{t_1}^{t_1+l} \|S(t_1+r-s)\|^q \|f_2(s,x, \int_{-h}^0 x(s+\theta)d\theta)(s)\|^q \|u\|^q ds\Big)$$
$$= I_1 + I_2 + I_3 + I_4 + I_5. \tag{3.6}$$

Let us estimate each summand in (3.6).

Due to the uniform continuity of C_0-semigroup we obtain that I_1 tends to zero as $l \to 0$ uniformly on t_1.

Applying A2) and (3.4), we have

$$I_2 \le T \sup_{r\in[0,l]} \int_0^{t_1} \|S(t_1+r-s) - S(t_1-s)\|^q ds \cdot$$
$$\sup_{s\in[0,T]} \|f_1(s,x(s), \int_{-h}^0 x(s+\theta)d\theta)\|^q \le$$
$$\le \int_0^T \|S(t+r) - S(t)\|^q dt K^q (1 + 2C_3(T, \|\varphi\|_C, h))^q, \tag{3.7}$$

due to (3.5).

Compactness of $S(t)$ for $t > 0$ implies continuity in the uniform operator topology for $t > 0$. Therefore, due to the Lebesgue dominated convergence theorem the right-hand part of (3.7) tends to zero as $l \to 0$.

Further, due to (3.4) we have

$$I_3 \le K^q (1 + 2C_3(T, \|\varphi\|_C, h))^q \int_{t_1}^{t_2} \|S(t_1+r-s)\|^q ds \le$$
$$\le K^q M^q (1 + 2C_3(T, \|\varphi\|_C, h)^q \cdot l \to 0, \ as\ l \to 0. \tag{3.8}$$

Using conditions A2), A5) and *Hölder's* inequality, we get

$$I_4 \le T \int_0^T \|S(t+r) - S(t)\|^q dt K^q (1 + 2C_3(T, \|\varphi\|_C, h))^q \Big(\int_0^t \|u(s)\|^p \Big)^{\frac{q}{p}}$$
$$\le T \int_0^T \|S(t+r) - S(t)\|^q dt (BK)^q (1 + 2C_3(T, \|\varphi\|_C, h))^q \to 0, \ as\ l \to 0. \tag{3.9}$$

Finally,

$$I_5 \le \sup_{r\in[0,l]} \int_{t_1}^{t_1+l} \|S(t+r-s)\|^q \|f_2(s,x(s), \int_{-h}^0 x(s+\theta)d\theta)\|^q ds \int_{t_1}^{t_1+l} \|u\|^p \Big)^{\frac{q}{p}} \le$$

$$\leq (BK)^q(1+2C_3(T,\|\varphi\|_C,h))^q M^q l \to 0,\ l\to 0. \tag{3.10}$$

If t_1 and t_1+l belong to $[-h,0]$, then from the definition of solution we have

$$\sup_{|t_2-t_1|\leq l}\|x(t_2,u)-x(t_1,u)\| = \sup_{|t_2-t_1|\leq l}\|\varphi(t_2)-\varphi(t_1)\|\to 0,\ l\to 0$$

due to uniform continuity of $\varphi(t)$ on $[-h,0]$.

If $t_1\in[-h,0]$ and $t_2>0$, then

$$\|x(t_2,u)-x(t_1,u)\|\leq \|x(t_2,u)-\varphi(0)\|+\|\varphi(0)-x(t_1,u)\|. \tag{3.11}$$

In this case we see that if $l\to 0$ then $t_1\to 0$ and $t_2\to 0$. In particular, the second summand in (3.11) tends to zero uniformly on $u\in W$. For the first summand we have

$$\|x(t_2)-\varphi(0)\|\leq \|S(t_2)\varphi(0)-\varphi(0)\|+l\sup_{t\in[0,T]}\|f_1(t,x(t),\int_{-h}^{0}x(t+\theta)d\theta\|$$

$$+\,l\sup_{t\in[0,T]}\|f_2(t,x(t),\int_{-h}^{0}x(t+\theta)d\theta\|B^p.$$

Thus, using assumtions on f_1, f_2 and estimates (3.5) we conclude that the first summmand in (3.11) also tends to zero uniformly on $u\in W$.

The Lemma is proved.

Unlike the finite-dimensional case, the lemma does not establish the rate of convergence in formula (3.1). Here, the rate depends on the behavior of the semigroup $S(t)$ as $t\to 0$, which in turn depends on the form of the operator A.

Let's move on to the proof of Theorem 2.

Proof. For sufficiently small $\mu>0$ we put

$$x_\mu(t,u)=\frac{1}{\mu}\int_t^{t+\mu}x(s)ds,\quad t\in[-h,T], \tag{3.12}$$

where for values $s\geq T$ we assume $x(s)$ to be a constant by continuity.

It is clear that the function $x_\mu(t)=x_\mu(t,u)$ is strictly differentiable and

$$\dot{x}_\mu(t)=\frac{1}{\mu}(x(t+\mu)-x(t)). \tag{3.13}$$

Moreover, due to Lemma 1

$$\sup_{t\in[-h,T]}\|x(t)-x_\mu(t)\|=\frac{1}{\mu}\sup_{t\in[-h,T]}\|\int_t^{t+\mu}(x(t)-x(s))ds\|\leq$$

$$\leq\frac{1}{\mu}\int_t^{t+\mu}\sup_{t,s\in[-h,T]}\|x(t)-x(s)\|ds\leq C(T,\|\varphi\|_C,\mu,B)\to 0,\quad \mu\to 0. \tag{3.14}$$

Let us denote $y_j(t) = x(t - \frac{h}{m}j)$ and

$$N_j(t) = \|y_j(t) - z_j(t)\|, \quad j = \overline{0, m}.$$

We split system (2.4) into two subsystems, i.e.,

$$z_j(t) = z_j^{(1)}(t) + z_j^{(2)}(t),$$

where $z_j^{(1)}(t)$ is the solution of the system

$$\begin{cases} \frac{h}{m}\dot{z}_1^{(1)} = x(t) - z_1^{(1)}(t), \\ \frac{h}{m}\dot{z}_j^{(1)} = z_{j-1}^{(1)}(t) - z_j^{(1)}(t), \quad j = \overline{1, m}, \\ z_j^{(1)}(0) = x(-\frac{hj}{m}), \end{cases} \tag{3.15}$$

and $z_j^{(2)}(t)$ is the solution of the system

$$\begin{cases} \frac{h}{m}\dot{z}_1^{(2)} = -z_1^{(2)}(t) + (z_0(t) - x(t)), \\ \frac{h}{m}\dot{z}_j^{(2)} = z_{j-1}^{(2)}(t) - z_j^{(2)}(t), \quad j = \overline{1, m}, \\ z_j^{(2)}(0) = 0. \end{cases} \tag{3.16}$$

Then

$$\sup_{t\in[0,T]} \|x(t - \frac{h}{m}j) - z_j(t)\| \le \sup_{t\in[0,T]} \|y_j(t) - z_j^{(1)}(t)\| + \sup_{t\in[0,T]} \|z_j^{(2)}(t)\|. \tag{3.17}$$

Let us prove that

$$\sup_{t\in[0,T]} \|y_j(t) - z_j^{(1)}(t)\| \le C_1(T, \|\varphi\|, h, \frac{h}{m}, B) \to 0, m \to 0. \tag{3.18}$$

We write $x(t)$ in the form

$$x(t) = x_\mu(t) + x(t) - x_\mu(t) = x_\mu(t) - x_1(t)$$

and put

$$y_j(t) = y_j^{(1)}(t) + y_j^{(2)}(t),$$

where $y_j^{(1)}(t) = x_\mu(t - \frac{h}{m}j)$, $y_j^{(2)}(t) = x_1(t - \frac{h}{m}j)$.

Then the solution of (3.15) can be written in the form

$$z_j^{(1)}(t) = \varphi_j(t) + v_j(t),$$

where $\varphi_j(t) = \varphi_j(t, u)$ is the solution of the system

$$\begin{cases} \frac{h}{m}\dot{\varphi}_1 = -\varphi_1 + x_\mu, \\ \frac{h}{m}\dot{\varphi}_j = -\varphi_{j-1} - \varphi_j, \quad j = \overline{2, m}, \\ \varphi_j(0) = y_j^{(1)}(0) = x_\mu(-\frac{h}{m}j), \end{cases} \tag{3.19}$$

and $v_j(t) = v_j(t,u)$ is the solution of the system

$$\begin{cases} \frac{h}{m}\dot{v}_1 = -v_1 + x_1, \\ \frac{h}{m}\dot{v}_j = v_{j-1} - v_j, \quad j = \overline{2,m}, \\ v_j(0) = y_j^{(2)}(0) = x_1(-\frac{hj}{m}) - x_\mu(-\frac{hj}{m}). \end{cases} \tag{3.20}$$

Then

$$\sup_{t\in[0,T]} \|y_j(t) - z_j^{(1)}(t)\| \le \sup_{t\in[0,T]} \|y_j^{(1)}(t) + y_j^{(2)}(t) - \varphi_j(t) - v_j(t)\|$$
$$\le \sup_{t\in[0,T]} \|y_j^{(1)}(t) - \varphi_j(t)\| + \sup_{t\in[0,T]} \|y_j^{(2)}(t)\| + \sup_{t\in[0,T]} \|v_j(t)\|. \tag{3.21}$$

Let us estimate each summand in the right-hand part of (3.21).

We start with $j = 1$. Let us denote $\varepsilon_1(t) = \varphi_1(t) - y_1^{(1)}(t)$.

Then $\varepsilon_1(0) = 0$ and

$$\dot{\varepsilon}_1 = \dot{\varphi}_1 - \dot{y}_1^{(1)} = \frac{m}{h}(x_\mu - x_1) - \dot{y}_1^{(1)} =$$
$$= \frac{m}{h}(-x_1 + \dot{y}_1^{(1)}) + \frac{m}{h}(x_\mu - \dot{y}_1^{(1)}) - \dot{y}_1^{(1)} =$$
$$= -\frac{m}{h}\varepsilon_1 + \frac{m}{h}(x_\mu - \dot{y}_1^{(1)}) - y_1^{(1)} = -\frac{m}{h}\varepsilon_1 + \psi(t),$$

where $\psi(t) = \frac{h}{m}(x_\mu - y_1^{(1)}) - \dot{y}_1^{(1)}$.

Therefore, $\varepsilon_1(t)$ satisfies the initial-value problem

$$\begin{cases} \dot{\varepsilon}_1 = -\frac{m}{h}\varepsilon_1 + \psi(t), \\ \varepsilon_1(0) = 0. \end{cases} \tag{3.22}$$

For $\psi(t)$ due to the strict differentiability of $x_\mu(t)$ we get

$$\|\psi(t,u_m)\| = \|\frac{m}{h}(x_\mu(t) - y_1^{(1)}(t)) - \dot{y}_1^{(1)}(t)\|$$
$$= \|\frac{m}{h}(x_\mu(t) - x_\mu(t - \frac{h}{m})) - \dot{x}_\mu(t - \frac{h}{m})\|$$
$$= \frac{m}{h}\|(x_\mu(t) - x_\mu(t - \frac{h}{m})) - \dot{x}_\mu(t - \frac{h}{m})\frac{h}{m}\|$$
$$\le \frac{1}{\mu}\frac{m}{h} \sup_{s\in[t-\frac{h}{m},t]} \|\dot{x}_\mu(s) - \dot{x}_\mu(t - \frac{h}{m})\|\frac{h}{m}$$
$$\le \frac{1}{\mu}(\sup_{s\in[t-\frac{h}{m},t]} (\|x(s+\mu) - x(t - \frac{h}{m} + \mu)\| + \|x(s) - x(t - \frac{h}{m})\|)) \to 0,$$

$m \to \infty$.

The last convergence is uniform with respect to $u \in W$ according to Lemma 1. So,

$$\sup_{t\in[0,T]} \|\psi(t,u_m)\| \to 0, \quad m \to \infty.$$

The variation of constants formula implies

$$\varepsilon_1(t) = \int_0^t \exp(-\frac{m}{h}(t-s))\psi(s)ds.$$

Then from the estimate

$$\begin{aligned}
&\sup_{t\in[0,T]} \| \int_0^t \exp(-\frac{m}{h}(t-s)) \cdot \psi(s)ds\| \\
&\leq \sup_{t\in[0,T]} \| \int_0^t \exp(-\frac{m}{h}(t-s))\|ds \sup_{t\in[0,T]} \|\psi(t)\| \\
&\leq \frac{h}{m} C(T, \|\varphi\|_C, \frac{h}{m}, B)
\end{aligned} \tag{3.23}$$

we deduce that $\varepsilon_1(t)$ satisfies the inequality

$$\sup_{t\in[0,T]} \|\varepsilon_1(t)\| \leq \frac{h}{m} C(T, \|\varphi\|_C, \frac{h}{m}, B).$$

Further, for $y_1^{(2)}(t)$ from (3.14) we have

$$\sup_{t\in[0,T]} \|y_1^{(2)}(t)\| = \sup_{t\in[0,T]} \|x(t-\frac{h}{m}) - x_\mu(t-\frac{h}{m})\| \leq C(T, \|\varphi\|_C, \mu, B|).$$

For $v_1(t)$ we apply the variation of constants formula to (3.20):

$$v_1(t) = \exp(-\frac{m}{h}t)v_1(0) + \int_0^t \exp(-\frac{m}{h}(t-s))x_1(s)ds.$$

So,

$$\sup_{t\in[0,T]} \|v_1(t)\| \leq \|v_1(0)\| + \sup_{t\in[0,T]} \| \int_0^t e^{-\frac{m}{h}(t-s)} x_1(s)ds\|.$$

But $v_1(0) = x_1(-\frac{h}{m})$. So, $\|v_1(0)\| \leq C(T, \|\varphi\|, \mu)$ and due to (3.14)

$$\sup_{t\in[0,T]} \| \int_0^t e^{-\frac{m}{h}(t-s)} x_1(s)ds\| \cdot \frac{h}{m} \leq \frac{h}{m} C(T, \|\varphi\|, \mu, B). \tag{3.24}$$

From the estimates obtained above we conclude that

$$\sup_{t\in[0,T]} \|v_1(t)\| \leq C(T, \|\varphi\|, \mu, B).$$

Therefore,

$$\sup_{t\in[0,T]} \|y_1(t) - z_1^{(1)}(t)\| \leq \frac{h}{m\mu} C(T, \|\varphi\|, \frac{h}{m}, B) + C(T, \|\varphi\|, \mu, B).$$

Having made similar estimates for $j = 2$, we obtain

$$\sup_{t\in[0,T]} \|y_j(t) - z_j^{(1)}(t)\| \le C_1(T, \|\varphi\|, h, \frac{h}{m}, B) \to 0, \quad m \to \infty.$$

Thus, we have proved inequality (3.18).

To estimate the second summand in (3.17) we apply the variation of constant method. After that we have

$$\|z_j^{(2)}(t)\| \le \sup_{t\in[0,T]} N_0(t).$$

Further, let us estimate $z_0(t) - x(t)$. From assumption A3) we get

$$\begin{aligned}
&\|z_0(t) - x(t)\| \\
&= KM \int_0^t \Big(\|z_0(s) - x(s)\| + \|\frac{h}{m}\sum_{j=1}^m z_j(s) - \int_{-h}^0 x(s+\theta)d\theta)\| \Big) ds \\
&+ KM \int_0^t \Big(\|z_0(s) - x(s)\| + \|\frac{h}{m}\sum_{j=1}^m z_j(s) - \int_{-h}^0 x(s+\theta)d\theta)\| \Big) \|u(s)\| ds.
\end{aligned}$$

Taking into account the equality

$$\begin{aligned}
&\int_{-h}^0 x(s+\theta)d\theta - \frac{h}{m}\sum_{j=1}^m z_j(s) = \sum_{j=1}^m \int_{-\frac{h}{m}j}^{-\frac{h}{m}(j-1)} x(s+\theta)d\theta \\
&- \frac{h}{m}\sum_{j=1}^m z_j(s) = \sum_{j=1}^m \int_{-\frac{h}{m}j}^{-\frac{h}{m}(j-1)} x(s+\theta)d\theta \\
&- \sum_{j=1}^m \int_{-\frac{h}{m}j}^{-\frac{h}{m}(j-1)} z_j(s)d\theta = \sum_{j=1}^m \int_{-\frac{h}{m}j}^{-\frac{h}{m}(j-1)} (x(s+\theta) - z_j(s)))d\theta,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]} \| \int_{-h}^0 x(s+\theta)d\theta - \frac{h}{m}\sum_{j=1}^m z_j(s)\| \\
&\le \sup_{t\in[0,T]} \sum_{j=1}^m \int_{-\frac{h}{m}j}^{-\frac{h}{m}(j-1)} \|x(s+\theta) - z_j(s)\| d\theta \\
&\le C(T, \|\varphi\|_C, \frac{h}{m}, B)
\end{aligned}$$

as a consequence of Lemma 1.

Combining estimates obtained above we conclude that

$$\|z_0(t)-x(t)\|^q \le 2^{q-1}(KM)^q\Big(\int_0^t(\|z_0(s)-x(s)\| + \sum_{j=1}^{m}\int_{-\frac{h}{m}j}^{-\frac{h}{m}(j-1)}\|x(s+\theta)-z_j(s)\|)ds\Big)^q + (KM)^q\Big(\int_0^t(\|z_0(s)-x(s)\| + \sum_{j=1}^{m}\int_{-\frac{h}{m}j}^{-\frac{h}{m}(j-1)}\|x(s+\theta)-z_j(s))\|\|u\|)ds\Big)^q = 2^{q-1}(I_1+I_2),$$

where

$$I_1 \le (MK)^q 2^{q-1}\Big(\int_0^t\|z_0(s)-x(s)\|^q ds + TC^q(T,\|\varphi\|_C,\frac{h}{m},B)\Big),$$

$$I_2 \le (MK)^q\|u\|_p^q\Big(\int_0^t\|z_0(s)-x(s)\|^q ds + TC^q(T,\|\varphi\|_C,\frac{h}{m},B)\Big).$$

So,

$$\|z_0(t)-x(t)\|^q \le (MK)^q 2^{2q-1}(1+\|u\|_p^q)\int_0^t\|z_0(s)-x(s)\|^q ds + (MK)^q 2^{q-1}TC^q(T,\|\varphi\|_C,\frac{h}{m},B) = C_1^*\int_0^t\|z_0(s)-x(s)\|^q ds + C_2^*(\frac{h}{m}).$$

Applying Gronwall's lemma we have

$$\sup_{t\in[0,T]}\|z_0(t)-x(t)\|^q \le C_2^*(\frac{h}{m})\exp(C_1^*T)\to 0, \quad m\to\infty,$$

which completes the proof.

Lemma 2. *Assume that conditions H1), A1) - A5) hold. If* $u_m \overset{w}{\to} u_0$ *in* $L^p(0,T;X)$ *as* $m\to\infty$, *then the solution of initial-value problem* (1.1) *with* $u(t)=u_m(t)$ *converges uniformly on* $[0,T]$ *to the solution of* (1.1) *with control* $u(t)=u_0(t)$, *i.e.,*

$$x(t,u_m)\rightrightarrows x(t,u_0), \quad m\to\infty. \tag{3.25}$$

Proof of Lemma 2.

We note that the weak convergence of $\{u_m\}$ implies boundness. So, inequality (3.5) guarantees uniform boundness of $\{x(t,u_m)\}$ on $[0,T]$. Lemma 1 guarantees equicontinuity of $\{x(t,u_m)\}$ on $[0,T]$. Let us show precompactness of $\{x(t,u_m)\}$ in X for every $t\in[0,T]$. To simplify notations we put $x_m(t)=x(t,u_m)$.

According to the definition

$$x_m(t) = S(t)\varphi(0) + \int_0^t S(t-s) f_1(s, x_m(s), \int_{-h}^0 x_m(s+\theta)d\theta)ds$$
$$+ \int_0^t S(t-s) f_2(s, x_m(s), \int_{-h}^0 x_m(s+\theta)d\theta)u_m(s)ds. \tag{3.26}$$

It is clear that the required compactness takes place for $t = 0$. The set $S(t)\varphi(0)$ is compact for $t \in (0, T]$.

To prove compactness of sets which are described by the second and the third summand in (3.26) we need the following result.

Proposition. [16], Prop. 8.1] *If $S(t)$, $t > 0$ is a compact operator and $p \geq 1$, then the operator*

$$F(f)(t) = \int_0^t S(t-s) f(s)ds, \quad f \in L^p(0, T; X), \quad t \in [0, T]$$

is a compact operator from $L^p(0, T; X)$ into $C([0, T], X)$.

Boundness of $\{x_m(t)\}$ and condition A2) imply boundness of $f_1(t, x_m(t), \int_{-h}^0 x_m(s+\theta)d\theta)$ in $L^1(0, T; X)$. So, Proposition guarantees the compactness of the second term in (3.26). In the same way, using A5), boundness of $\{x_m(t)\}$, and boundness of $\{u_m\}$ we can show the compactness of the third summand.

So, due to the Ascoli Theorem, the sequence $\{x_m(t)\}$ is precompact in $C([0, T], X)$ and we can extract a subsequence $x_{m_k}(t)$, which converges to $x_0(t)$ in $C([0, T], X)$. Let us show that $x_0(t)$ is a solution of initial-value problem (1.1) with control $u = u_0$.

Indeed, we have

$$x_{m_k}(t) = S(t)\varphi(0) + \int_0^t S(t-s) f_1(s, x_{m_k}(s), \int_{-h}^0 x_{m_k}(s+\theta)d\theta)ds$$
$$+ \int_0^t S(t-s) f_2(s, x_{m_k}(s), \int_{-h}^0 x_{m_k}(s+\theta)d\theta)u_{m_k}(s)ds$$
$$- \int_0^t S(t-s) f_2(s, x_0(s), \int_{-h}^0 x_0(s+\theta)d\theta)u_{m_k}(s)ds$$
$$+ \int_0^t S(t-s) f_2(s, x_0(s), \int_{-h}^0 x_0(s+\theta)d\theta)u_{m_k}(s)ds$$
$$= I_1 + I_2 + I_3 + I_4. \tag{3.27}$$

The second term I_2 tends to $\int_0^t S(t-s) f_1(s, x_0(s), \int_{-h}^0 x_0(s+\theta)d\theta)ds$ due to convergence of x_{m_k}, condition A2), and the Lebesgue dominated convergence theorem.

For I_3 we have

$$|I_3| \leq MN(1+h) \sup_{[0,T]} |x_m(t) - x_0(t)| \|u_m\|_p^p \to 0, \ m_k \to \infty,$$

as a consequence of boundness of u_m in $L^p(0,T;X)$.

For I_4 weak convergence of u_{m_k} gives

$$I_4 \to \int_0^t S(t-s)f_2(s,x_0(s),\int_{-h}^0 x_0(s+\theta)d\theta)u_0(s)ds,\ m_k \to \infty$$

So, passing to the limit in (3.27) as $m_k \to \infty$, we get

$$x_0(t) = S(t)\varphi(0) + \int_0^t S(t-s)f_1(s,x_0(s),\int_{-h}^0 x_0(s+\theta)d\theta)ds$$
$$+ \int_0^t S(t-s)f_2(s,x_0(s),\int_{-h}^0 x_0(s+\theta)d\theta)u_0(s)ds \tag{3.28}$$

Due to the uniqueness of solution of (3.28) we can claim that the whole sequence x_m tends to x_0. Lemma is proved.

As a direct consequence of Theorem 2, Lemma 2, and the inequality

$$\|z_0(t,u_m) - x(t,u_0)\| \le \|z_0(t,u_m) - x(t,u_m)\|$$
$$+ \|x(t,u_m) - x(t,u_0)\|$$

we obtain the following result.

Lemma 3. *Assume that conditions H1), A1)-A5) hold. If* $u_m \overset{w}{\to} u_0$ *in* $L^p(0,T;X)$ *as* $m \to \infty$, *then the solution of initial-value problem* (2.4) $z_0(t,u_m)$ *with* $u(t) = u_m(t)$ *converges uniformly on* $[0,T]$ *to the solution* $x(t,u_0)$, *i.e.,*

$$z_0(t,u_m) \rightrightarrows x(t,u_0), \quad m \to \infty.$$

4. Proof of the main rezult

Existence of the optimal pair (z_{0m}^*, u_m^*) for every m can be proved in a standard way by extracting a weakly convergent to $u_m^*(t)$ minimizing sequence $u_m^{(n)}(t)$ with passing to the limit as $n \to \infty$. In this case, conditions A6)-A8) are used. Moreover, due convexity of F and Mazur's theorem [18] we obtain that $u_m^* \in U$. Existence of the optimal pair $(x^*(t), u^*(t))$ of the problem (1.1)-(1.2) follows from results of [11].

So, let us analyze

$$J_m^* = J_m(u_m^*) = \int_0^T G(t, z_{0m}^*(t), u_m^*(t))dt.$$

Let $\overline{u}$ be an arbitrary constant vector from U. Then

$$J_m^* = J_m(u_m^*) \le J_m(\overline{u}). \tag{4.1}$$

Thus, for every m we have

$$\int_0^T G(t, z_{0m}^*(t), u_m^*(t))dt \le \int_0^T G(t, z_{0m}(t, \overline{u}), \overline{u})dt,$$

where $z_{0m}(t, \overline{u})$ is a solution of the first equation in (2.4) with control $\overline{u}$.

From Lemma 3 we have

$$\sup_{t\in[0,T]} |z_{0m}(t, \overline{u}) - x(t, \overline{u})| \to 0, \ m \to \infty.$$

Then condition (2.1) implies

$$\sup_{t\in[0,T]} \Big|G(t, z_{0m}(t, \overline{u}), \overline{u}) - G(t, x(t, \overline{u}), \overline{u})\Big| \to 0, \ m \to \infty.$$

So, there exists a constant $\gamma > 0$, which does not depend on m such that

$$\int_0^T G(t, z_{0m}^*(t), u_m^*)dt \le \gamma.$$

Then condition (2.1) implies

$$\|u_m^*\|_p^p \le \frac{\gamma}{C}. \tag{4.2}$$

So, the sequence $\{u_m^*\}$ is weakly compact in $L^p(0, T, X)$ and there exists a subsequence of optimal controls (still denoted by u_m^*) which converges weakly to some $u_0 \in U$. Let $x_0(t)$ be a solution of (1.1)-(1.2) with control $u(t) = u_0(t)$. Lemma 3 implies that $z_{0m}^*(t)$ uniformly on $[0, T]$ converges to $x_0(t)$ as $m \to \infty$.

Let us prove that

$$J_m^* \to J^*, \ m \to \infty.$$

From (2.1) we have

$$\Big|J_m(u^*) - J(u^*)\Big| \le \int_0^T \Big|G(t, z_{0m}(t, u^*), u^*) - G(t, x^*(t, u^*), u^*))\Big|dt \to 0,$$

$m \to \infty$.

So, for any $\eta > 0$ there exists m_0 such that for $m > m_0$

$$\Big|J_m(u^*) - J(u^*)\Big| < \eta. \tag{4.3}$$

In particular,

$$J_m(u^*) < J(u^*) + \eta \tag{4.4}$$

for $m > m_0$.

From the other hand,

$$J^* \le J(u_m^*) = J_m^* + J(u_m^*) - J_m(u_m^*). \tag{4.5}$$

Due to Theorem 2 and conditions (2.1), (4.2) we obtain

$$\begin{aligned}&\Big|J_m(u_m^*) - J(u_m^*)\Big| \\ &= \Big|\int_0^T G(t, z_{0m}^*(t, u_m^*), u_m^*)dt - \int_0^T G(t, x(t, u_m^*), u_m^*)dt\Big| \\ &\le \int_0^T \Big|G(t, z_{0m}^*(t, u_m^*), u_m^*)dt - G(t, x(t, u_m^*), u_m^*)\Big|dt \to 0,\ m \to \infty \end{aligned} \tag{4.6}$$

Therefore, for any $\eta > 0$ there exists m_0 such that for $m > m_0$

$$\Big|J_m(u_m^*) - J(u_m^*)\Big| < \eta \tag{4.7}$$

Then $J^* \le J_m^* + \eta$ and from inequality (4.4) for $m > m_0$ we get

$$\Big|J_m^* - J^*\Big| < \eta. \tag{4.8}$$

It means that

$$J_m^* \to J^*, \qquad m \to \infty, \tag{4.9}$$

and the first statement of Theorem 1 is proved.

From the inequality

$$\Big|J^* - J(u_m^*)\Big| \le \Big|J_m^* - J^*\Big| + \Big|J_m(u_m^*) - J(u_m^*)\Big|,$$

and (4.7), (4.9) we obtain statement 2) of Theorem 1.

Let us prove statement 3). First we prove that $(x_0(t), u_0(t))$ is an optimal pair in (1.1) - (1.2).

Since $J_m^* = \int_0^T G(t, z_{0m}^*(t), u_m^*(t))dt$, then due to (4.9) we have

$$J^* = \lim_{m\to\infty} \int_0^T G(t, z_{0m}^*(t), u_m^*)dt. \tag{4.10}$$

Now we prove integrability of $G(t, x_0(t), u_m^*(t))$. Let $a \in U$ be an arbitrary element. Using condition A8), we get

$$\begin{aligned}&|G(t, x_0(t), u_m^*(t)) - G(t, x_0(t), a)| \\ &\le \sup_{\lambda\in(0,1)} \|L_u(t, x_0(t), a + \lambda(u_m^*(t) - a))\|_* \|u_m^*(t) - a\|\end{aligned}$$

Then, using A8) again, we have

$$\begin{aligned}&G(t, x_0(t), u_m^*(t)) \le G(t, x_0(t), a) \\ &C_1(1 + \|x_0(t)\|^\alpha + \|u_m^*(t) - a\|^{p-1})\|u_m^*(t) - a\|.\end{aligned}$$

The last inequality implies integrability of $G(t, x_0(t), u_m^*)$.

Let χ_R be an indicator function of the set $\{t \in [0, T] : \|u_0(t)\| \le R\}$.

From the convexity of $u \mapsto G(t,x,u)$ we obtain

$$\begin{aligned}
&\int_0^T G(t,x_0(t),u_m^*(t))\chi_R(t)dt \\
&\geq \int_0^T G(t,x_0(t),u_0(t))\chi_R(t)dt \\
&+ \int_0^T L_u(t,x_0(t),u_0(t))(u_m^*(t)-u_0(t))\chi_R(t)dt. \qquad (4.11)
\end{aligned}$$

The second term in (4.11) tends to zero as $m \to \infty$, because of the inequality

$$\|L_u(t,x_0(t),u_0(t)\|_*\chi_R \leq C_1(1+\|x_0(t)\|^\alpha + R^{p-1}).$$

Then

$$\lim_{n\to\infty} inf \int_0^T G(t,x_0(t),u_m^*(t))\chi_R(t)dt \geq \int_0^T G(t,x_0(t),u_0(t))\chi_R(t)dt. \quad (4.12)$$

Using $G \geq 0$, $\chi_R \leq 1$, $\chi_R(t) \to 1$, $R \to \infty$ and (4.12), we deduce that

$$\int_0^T G(t,x_0(t),u_0(t))dt \leq \lim_{m\to\infty} \inf \int_0^T G(t,x_0(t),u_m^*(t))dt. \qquad (4.13)$$

Further, we have

$$\begin{aligned}
J^* &= \lim_{m\to\infty} inf \int_0^T G(t,z_{0m}^*(t),u_m^*(t))dt \\
&\geq \lim_{m\to\infty} inf \int_0^T \Big[G(t,z_{0m}^*(t),u_m^*(t)) - G(t,x_0(t),u_m^*(t))\Big]dt \\
&+ \lim_{m\to\infty} inf \int_0^T \Big[G(t,x_0(t),u_m^*(t)) - G(t,x_0(t),u_0(t))\Big]dt \\
&+ \int_0^T G(t,x_0(t),u_0(t)). \qquad (4.14)
\end{aligned}$$

The first summand in (4.14) equals zero due to (2.1) and (4.2). The second summand is non-negative due to (4.12). So, the pair $(x_0(t),u_0(t)$ is optimal and we have proved the required property 3) of Theorem 1.

Finally, if we assume that the problem (1.1) - (1.2) has a unique solution, then according to previous results we conclude that every convergent sequence $(z_{0m}^*(t),u_m^*(t))$ tends to one and the same limit, which means that the convergence takes place for $m \to \infty$. Theorem is proved.

5. Application to semilinear partial functional-differential equation

Let Q be bounded domain in R^d with ∂Q satisfying the Lyapunov condition. In this domain we consider the elliptic operator A

$$Ax = A(y)x = \sum_{i,j=1}^{d} \partial_{y_j}(a_{ij}(y))\partial_{y_i}(x(y)) = div(a(y)\nabla x(y)). \tag{5.1}$$

The coefficients $a_{ij}(y)$ of this operator are Hölder continuous with the exponent $\beta \in (0,1)$, symmetric, and satisfy the ellipticity condition

$$\sum_{i,j=1}^{d} a_{ij}(y)\eta_i\eta_j \geq C_0|\eta|^2, \eta \in R^d, \tag{5.2}$$

for some $C_0 > 0$, here $|\cdot|$ the Euclidean norm in R^d.

We denote $X = L^2(D) = H$ with the norm $\|x\|^2 = \int_Q x^2(y)dy$, $D(A) = H^2(Q)\bigcap H_0^1(Q)$.

In this space we consider the next optimal control problem

$$\begin{aligned}
&\frac{\partial x(t,y)}{\partial t} = \sum_{i,j=1}^{d} \partial_{y_j}((a_{ij})\partial_{y_i}x(t,y)) + f_1(t,x(t,y),\int_{-h}^{0} x(t+\theta,y)d\theta)\\
&+ f_2(t,x(t,y),\int_{-h}^{0} x(t+\theta,y)d\theta)u(t,y),\\
&x(t,y) = \varphi(t,y), t \in [-h,0], y \in Q,\\
&x(t,y) = 0, y \in \partial Q, t \in [0,T).
\end{aligned} \tag{5.3}$$

$$J[u] = \int_0^T \Big[\Big(\int_Q x^2(t,y)dy\Big)^{\frac{p}{2}} + \Big(\int_Q u^2(t,y)dy\Big)^{\frac{p}{2}}\Big]dt \to inf, \tag{5.4}$$

$p \geq 2$, $p \in Z$ and even number. It will consider admissible controls to be the functions $u \in L^2(Q_T)$, $Q_T = (0,T)\times Q$, such that $u(t,\cdot) \in F$, for a.e. $t \in (0,T)$, F is a convex and closed set in $L^2(Q)$.

For example, F may be ball $B_R(Q) = \{x \in L^2(Q) : \int_Q x^2(y)dy \leq R^2\}$, for same $R > 0$. Similar problems for equations with delays were considered, for example, in the works of [3], [2], [14].

The functions

$$f_1 : [0,T] \times R^1 \times R^1 \mapsto R^1, f_2 : [0,T] \times R^1 \times R^1 \mapsto R^1$$

are continuous functions with respect to all variables.

Suppose also that ther exist constant $K > 0$ such that

$$1) \quad |f_1(t,v,z)| + |f_2(t,v,z)| \le K(1 + |v| + |z|);$$
$$2) \quad |f_1(t,v_1,z_1) - f_1(t,v_2,z_2)| + |f_1(t,v_1,z_1) - f_1(t,v_2,z_2)| \le K(|v_1 - v_2| + |z_1 - z_2|)$$

for all $t, v, z, v_1, z_1, v_2, z_2$ in the domain.

It's well known [8], that A^{-1} is compact operator, then from [9] Sec 1.4 the semigroup $S(t)$ is compact for $t > 0$. Thus the condition about the operator A holds. Using the Nemytskii operator for the functions f_1 and f_2, problem (5.3)-(5.4) can be rewriten in abstract form (1.1)-(1.2). In this case, the functional $G(t,x,u) = \|x\|^p + \|u\|^p$.

Let's check the conditions A1)-A8).

Condition A1).

From 1) it follows that $f_1, f_2 : [0,T] \times H \times H \mapsto H$. The satisfaction of conditions A2) and A3) clearly follows from conditions 1) and 2). It also follows that condition A1) holds.

Condition A6) follows from the continuity of the norm.

Let us verify condition A7).

$$|G(t,x,u) - G(t,x_1,u)| = \|x\|^p - \|x_1\|^p \le \|x - x_1\| C(1 + \|x\|^{p-1} + \|x_1\|^{p-1}).$$

The Fréchet derivative is $G'_u = \frac{p}{2}\|u\|^{p-2}u$, which follows from the Riesz representation.

Therefore, all conditions of the Theorem 1 hold. In this case, the system

$$\frac{\partial z_0(t,y)}{\partial t} = \sum_{i,j=1}^{d} \partial_{y_j}((a_{ij})\partial_{y_i} z_0(t,y)) + f_1(t, z_0(t,y), \frac{h}{m}\sum_{j=1}^{m} z_j(t,y))$$
$$+ f_2(t, z_0(t,y), \frac{h}{m}\sum_{j=1}^{m} z_j(t,y))u(t,y),$$
$$\frac{\partial z_j(t,y)}{\partial t} = \frac{m}{h}(z_{j-1}(t,y) - z_j(t,y));$$
$$z_j(0,y) = \varphi(\frac{-ht}{m}, y), \quad y \in Q;$$
$$z_j(t,y) = 0, \quad y \in \partial Q, \quad j = \overline{0,m}$$

will be a delay-free approximation for the system (5.3).

References

1. *Bayractar E., Keller C.*: Part-dependent Hamilton-Jacobi equations in infinite dimentionals. J. Funct. Anal. 2018; 275(8): pp. 2096–2161. doi:10.1016/j.jfa.2018.07.010
2. *Casas E., Wachsmuth D.*: A note on existence of solunion to control problems of semilinaer partial differential equations. SIAM J. Control Optim. 2023; 61(3): pp. 1095–1112. doi:10.1137/22M1486418

3. *Casas E., Tröltzch F.*:Sparce optimal control for a semilinear heat with mixed control-state constraints-regularity of Lagrange multipliers. ESAIM Control Optim. Calc. Var. 2021; 27. doi:10.1051/cocv/2020084
4. *Cherevko I., Piddubna L.*: Approximations of differential difference equations and calculation of nonasymptotic roots of quasipolinomial . Revue d'Analise numèrique et de thèorie de L'approximation. 1999; 28(1): pp.15–21.
5. *Lark J., Misiats O., Stanzhytskiy O., Mogylova V.*: Asomptotic behavior of stochastic functional differential evolution equation. Electron. J. Differential Equations. 2023; 2023(35): pp. 1–21. doi:10.58997/ejde.2023.35
6. *Cosso A., Federico S., Gozzi F., Rosestolato M., Touzi N.*: Part-dependent equations and viscosity solutions in infinite dimension. Ann. Probab. 2018; 46(1): pp. 126–174. doi:10.1214/17-AOP1183
7. *Ekren I., Touzi N., Zhang J.*: Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part 1. Ann. Probab. 2016; 44(2): pp. 1212–1253. doi:10.1214/15-AOP1004
8. *Evans L.C.*: Partical Differential Equations. American Mathematical Society. Providence, 2010.
9. *Henry D.*: Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics. Springer-Verlag Berlin-Heidelberg-New York, 1981.
10. *Kolmanovskii V., Shaikhet L.*: Control of systems with after-effect. Amer.Math.Society. Providence, 1992.
11. *Latysh A., Kichmarenko O.*: Optimal control of functional-differential equations of parabolic type in Banach spaces. Journal of Math.Scien. 2024; 278: pp. 1034–1054. doi:10.1007/s10958-024-06983-8
12. *Li X., Yong J.*: Optimal control theory for infinite dimentional systems. Birkhäuser, 2012.
13. *Matvii O.V., Cherevko I.M.*: On approximation of systems with delay and their stability. Nonlinear Oscilations. 2004; 7(2): pp.207–215. doi:10.1007/s11072-005-0006-x
14. *Neitsel I., Vexler V.*: A priori error estimates for space-time finite element discretization of semilinear parabolic optimal control problems. Numer. Math. 2012; 120: pp.345–386. doi:10.1007/s00211-011-0414-y
15. *Perehuda O.V., Stanzhytskiy O.M., Klimchuk T.V., Ospanov M.N.*: Optimal Control for Functional-Differential Equation of Neutral Type in Banach Spaces. Journal of Optimization, Differential Equations and Their Applications 2025; 33(1): pp. 182–207. doi:10.15421/142509
16. *Da Prato G., Zabczyk J.*: Stochastic Equations in Infinite Dimensions. Cambridge University Press, 1992.
17. *Stanzhytskyi A., Misiats O., Stanzhytskyi O.*: Invariant measure for neutral stochastic functional differential equations with non-lipschts coefficients. Evolution Equations and Control Theory 2022; 11(5): pp. 1929–1953. doi:10.3934/eect.2022005
18. *Yosida K.*: Functional analysis. Springer-Verlag Berlin-Heidelberg-New York, 1980.

Received: 30.11.2025. *Accepted:* 14.06.2026